Möbius–Invariant Spaces and Algebras in Polydiscs

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Abstract. Let $\mathcal{M}$ be the group of biholomorphic automorphisms of the unit polydisc $\mathbb{U}^n \subset \mathbb{C}^n$ generated by rotations, coordinate permutations, and Möbius transformations. A closed subalgebra of continuous functions $\mathcal{C}(\overline{\mathbb{U}}^n)$ on the closed polydisc is called an $\mathcal{M}$-algebra if it is invariant under compositions with the members of $\mathcal{M}$. Polydisc algebra, its conjugate, constants, and $\{0\}$ are $\mathcal{M}$-algebras. For $1 \leq k \leq n$, put $T_k = \{$points in $\overline{\mathbb{U}}^n$ with at least k components of unit length$\}$, and let $\mathcal{E}_k(\mathbb{U}^n)$ consist of functions in $\mathcal{C}(\overline{\mathbb{U}}^n)$ which vanish on T_k. Each $\mathcal{E}_k(\mathbb{U}^n)$ is also an $\mathcal{M}$-algebra. We prove the following:

1. $\mathcal{E}_1(\mathbb{U}^n) = \mathcal{C}_0(\mathbb{U}^n)$ is a minimal $\mathcal{M}$-algebra, even if it is not allowed to be invariant under coordinate permutations.
2. There are three $\mathcal{M}$-algebras between each pair of $\mathcal{E}_k(\mathbb{U}^n)$ and $\mathcal{E}_{k+1}(\mathbb{U}^n)$; their members are constant, holomorphic, or conjugate-holomorphic on each $\big(n-(k+1)\big)$-torus in T_k.
3. An $\mathcal{M}$-space $\mathcal{X}$ satisfying $\mathcal{X} \cap \mathcal{E}_n(\mathbb{U}^n) = \{0\}$ or $\mathcal{E}_k(\mathbb{U}^n)$, for some k with $1 \leq k \leq n$, has the form $P[\mathcal{Y}]$ or $\mathcal{E}_k(\mathbb{U}^n) + P[\mathcal{Y}]$, respectively, where $\mathcal{Y}$ is an $\mathcal{M}$-space of $\mathcal{C}(\mathbb{T}^n)$ and P is the Poisson integral.

1. Introduction

1.1. Polydisc and Möbius-invariance. Let $\mathbb{U}$ be the open *unit disc* in $\mathbb{C}$ and $\mathbb{T}$ be the *unit circle* bounding it. We define the open *unit polydisc* in $\mathbb{C}^n$ by

$$\mathbb{U}^n = \big\{z \in \mathbb{C}^n : |z_1|, \dots, |z_n| < 1\big\},$$

and the *torus* by

$$\mathbb{T}^n = \big\{z \in \mathbb{C}^n : |z_1| = \cdots = |z_n| = 1\big\}.$$

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In words, $\mathbb{U}^n$ and $\mathbb{T}^n$ are the cartesian products of n unit discs and n unit circles, respectively. For $k = 0, 1, \ldots, n$, the *partial boundaries* of $\mathbb{U}^n$ are defined as

$$T_k = \{z \in \overline{\mathbb{U}}^n : \text{at least } k \text{ components of } z \text{ have length } 1\}.$$

With this notation, $T_0 = \overline{\mathbb{U}}^n$, the closure of $\mathbb{U}^n$; $T_1 = \partial\mathbb{U}^n$, its topological boundary; and $T_n = \mathbb{T}^n$, its *distinguished* (Shilov) *boundary*. These sets form a chain:

$$\overline{\mathbb{U}}^n \supset \partial\mathbb{U}^n \supset T_2 \supset T_3 \supset \cdots \supset T_{n-1} \supset \mathbb{T}^n. \tag{1.1}$$

Note that $\mathbb{T}^n$ is only a small part of $\partial\mathbb{U}^n$. An easy way to see this is to compare their (real) dimensions: $\dim_{\mathbb{R}}(\mathbb{T}^n) = n$ and $\dim_{\mathbb{R}}(\partial\mathbb{U}^n) = 2n - 1$.

A one-to-one biholomorphic transformation of the unit disc onto itself (an *automorphism* of $\mathbb{U}$) is a *Möbius transformation* φ_p, which is given for some $p \in \mathbb{U}$ by

$$\varphi_p(q) = \frac{p - q}{1 - \bar{p}q} \qquad (q \in \overline{\mathbb{U}}),$$

followed by a *rotation* (multiplication by an element of $\mathbb{T}$). When $n \geq 2$, the group $\mathcal{M}$ of all biholomorphic automorphisms of $\mathbb{U}^n$ (the *Möbius group*) is generated by rotations in each variable seperately

$$R_\vartheta(z) = (e^{i\vartheta_1} z_1, \ldots, e^{i\vartheta_n} z_n) \qquad (z \in \overline{\mathbb{U}}^n),$$

Möbius transformations in each variable seperately

$$\Phi_w(z) = \big(\varphi_{w_1}(z_1), \ldots, \varphi_{w_n}(z_n)\big) \qquad (z \in \overline{\mathbb{U}}^n),$$

and coordinate permutations. Here $\vartheta \in [-\pi, \pi]^n$ and $w \in \mathbb{U}^n$ are fixed, and the *coordinate permutations* are nothing but the $n!$ members of the symmetric group $\mathcal{S}_n$ on n objects. Thus an arbitrary $\Psi \in \mathcal{M}$ can be written in the form

$$\Psi(z) = \big(e^{i\vartheta_1} \varphi_{w_1}(z_{\sigma(1)}), \ldots, e^{i\vartheta_n} \varphi_{w_n}(z_{\sigma(n)})\big),$$

for some $w \in \mathbb{U}^n$ and $\vartheta \in [-\pi, \pi]^n$ (see Rudin [3, p. 167]). The subgroup of *linear* automorphisms in $\mathcal{M}$ is denoted by $\mathcal{U}$. It is that subgroup of $\mathcal{M}$ which fixes the origin and is generated by $\sigma \in \mathcal{S}_n$ and the rotations R_ϑ. $\mathcal{M}^*$ is the component of the identity in $\mathcal{M}$; in other words, $\mathcal{M}^*$ is $\mathcal{M}$ without the action of $\mathcal{S}_n$.

The partial boundaries T_k are *invariant* under the action of $\mathcal{M}$, i.e., $\Psi(T_k) = T_k$ for all $\Psi \in \mathcal{M}$ and $k = 0, \ldots, n$. Further, $T_k \backslash T_{k+1}$ $(k = 0, \ldots, n-1)$ and $\mathbb{T}^n$ are the only $\mathcal{M}$-*orbits* in $\overline{\mathbb{U}}^n$, and T_k for $k = 0, \ldots, n$ are their closures. A function space $\mathcal{G}$ defined on $\overline{\mathbb{U}}^n$ is $\mathcal{M}$-*invariant* if $f \circ \Psi \in \mathcal{G}$ whenever $f \in \mathcal{G}$ and $\Psi \in \mathcal{M}$.

A closed $\mathcal{M}$-invariant subspace (subalgebra) of $\mathcal{C}(\overline{\mathbb{U}}^n)$ will be called an *$\mathcal{M}$-space* (*$\mathcal{M}$-algebra*) for brevity. *$\mathcal{U}$-invariance* and *$\mathcal{U}$-spaces* have similar definitions. Some examples of $\mathcal{M}$-spaces are described in the next two paragraphs.

$\mathcal{C}(\overline{\mathbb{U}}^n)$ is the set of all *continuous* complex functions on the closed unit polydisc and $\mathcal{C}_0(\mathbb{U}^n)$ is its subspace consisting of functions which restrict to 0 on $\partial\mathbb{U}^n$. More generally, for each integer k with $0 \leq k \leq n$, let $\mathcal{E}_k(\mathbb{U}^n)$ denote the collection of functions in $\mathcal{C}(\overline{\mathbb{U}}^n)$ which restrict to 0 on T_k. Then $\mathcal{E}_0(\mathbb{U}^n) = \{0\}$, $\mathcal{E}_1(\mathbb{U}^n) = \mathcal{C}_0(\mathbb{U}^n)$, and by (1.1), we have the inclusions

$$\{0\} \subset \mathcal{C}_0(\mathbb{U}^n) \subset \mathcal{E}_2(\mathbb{U}^n) \subset \cdots \subset \mathcal{E}_{n-1}(\mathbb{U}^n) \subset \mathcal{E}_n(\mathbb{U}^n) \subset \mathcal{C}(\overline{\mathbb{U}}^n), \tag{1.2}$$

all of which are proper. The *polydisc algebra* $\mathcal{A}(\mathbb{U}^n)$ is made up of those members of $\mathcal{C}(\overline{\mathbb{U}}^n)$ which are holomorphic in $\mathbb{U}^n$, and the restriction of $\mathcal{A}(\mathbb{U}^n)$ to $\mathbb{T}^n$ is denoted by $\mathcal{A}(\mathbb{T}^n)$.

Now fix an integer k so that $1 \leq k \leq n-1$, and let W denote either $\mathbb{U}^n$ or T_k. Define $\mathcal{A}_k(W)$ to consist of those $F \in \mathcal{C}(\bar{W})$ that restrict to 0 on T_{k+1} and that coincide with the restriction of some $f \in \mathcal{A}(\mathbb{U}^k)$ to $\mathbb{T}^k$ when considered as a function of $\left(\zeta_{\sigma(1)}, \dots, \zeta_{\sigma(k)}\right) \in \mathbb{T}^k$, for any $\sigma \in \mathcal{S}_n$ and for fixed $\left(z_{\sigma(k+1)}, \dots, z_{\sigma(n)}\right) \in \mathbb{U}^{n-k}$. Similarly define $\mathcal{K}_k(W)$ to consist of functions in $\mathcal{C}(\bar{W})$ which are constants when any k variables are of unit length and the remaining $n-k$ variables are held fixed in addition to being 0 when any $k+1$ variables are of unit length. We will call the functions in $\mathcal{A}_k$ and $\mathcal{K}_k$ *partially holomorphic* and *partially constant*, respectively.

All the function spaces mentioned are closed in the topology of *uniform convergence*, that is, convergence in the *norm*

$$\|f\|_\infty = \|f\|_W = \sup_{z \in W} |f(z)|,$$

where W is $\overline{\mathbb{U}}^n$ or T_k or some other set contained in the domain of definition of the space. The term closed for function spaces will always refer to this topology. All these spaces are also *algebras* with respect to pointwise multiplication.

1.2. Main theorems. In this work, we categorize some $\mathcal{M}$-algebras and $\mathcal{M}$-spaces of $\mathcal{C}(\overline{\mathbb{U}}^n)$. The basic algebras are $\{0\}$, $\mathcal{C}(\overline{\mathbb{U}}^n)$, and the $\mathcal{E}_k(\mathbb{U}^n)$ for $k = 1, \dots, n$; the other cases will be built upon them. We first look at the two extreme ends of (1.2). Our first result is:

Theorem A. *$\mathcal{C}_0(\mathbb{U}^n)$ has no nontrivial proper closed $\mathcal{M}^*$-invariant subalgebra.*

The proof of this case is rather different from those of the other cases and will be given in Section 3. It uses real-analytic approximations and the Stone-Weierstrass Theorem. Next we obtain the $\mathcal{M}$-spaces at the other end, those that contain $\mathcal{E}_n(\mathbb{U}^n)$:

Theorem B. *Let $\mathcal{X}$ be a closed $\mathcal{M}$-invariant subspace of $\mathcal{C}(\overline{\mathbb{U}}^n)$ containing $\mathcal{E}_n(\mathbb{U}^n)$. Then $\mathcal{X}$ is in the form $\mathcal{E}_n(\mathbb{U}^n) + P[\mathcal{Y}]$, where $\mathcal{Y}$ is a closed $\mathcal{M}$-invariant subspace of $\mathcal{C}(\mathbb{T}^n)$.*

The closed $\mathcal{M}$-invariant subspaces and subalgebras of $\mathcal{C}(\mathbb{T}^n)$ have been classified by Gowda [1]. Then we classify the $\mathcal{M}$-algebras between any two consecutive algebras in (1.2):

Theorem C. *Suppose $\mathcal{X}$ is a closed $\mathcal{M}$-invariant subalgebra of $\mathcal{C}(\overline{\mathbb{U}}^n)$ satisfying $\mathcal{E}_k(\mathbb{U}^n) \subset \mathcal{X} \subset \mathcal{E}_{k+1}(\mathbb{U}^n)$ for some k with $1 \leq k \leq n-1$. Then $\mathcal{X}$ is one of*

(i) $\mathcal{E}_k(\mathbb{U}^n)$,
(ii) $\mathcal{K}_k(\mathbb{U}^n)$,
(iii) $\mathcal{A}_k(\mathbb{U}^n)$,
(iv) $\operatorname{conj} \mathcal{A}_k(\mathbb{U}^n)$
(v) $\mathcal{E}_{k+1}(\mathbb{U}^n)$.

If $\mathcal{G}$ is a function space, $\operatorname{conj} \mathcal{G}$ denotes the collection of functions whose complex *conjugates* lie in $\mathcal{G}$. Finally we consider some $\mathcal{M}$-spaces that have known intersections with $\mathcal{E}_n(\mathbb{U}^n)$:

Theorem D. *Let $\mathcal{X}$ be a closed $\mathcal{M}$-invariant subspace of $\mathcal{C}(\overline{\mathbb{U}}^n)$ that satisfies the condition $\mathcal{X} \cap \mathcal{E}_n(\mathbb{U}^n) = \{0\}$. Then $\mathcal{X} = P[\mathcal{Y}]$ for some $\mathcal{Y}$, where $\mathcal{Y}$ is a closed $\mathcal{M}$-invariant subspace of $\mathcal{C}(\mathbb{T}^n)$.*

Theorem E. *Let $\mathcal{X}$ be a closed $\mathcal{M}$-invariant subspace of $\mathcal{C}(\overline{\mathbb{U}}^n)$ that has the property $\mathcal{X} \cap \mathcal{E}_n(\mathbb{U}^n) = \mathcal{E}_k(\mathbb{U}^n)$ for some k with $1 \leq k \leq n-1$. Then $\mathcal{X}$ is in the form $\mathcal{E}_k(\mathbb{U}^n) + P[\mathcal{Y}]$, where $\mathcal{Y}$ is a closed $\mathcal{M}$-invariant subspace of $\mathcal{C}(\mathbb{T}^n)$.*

Then we pick the $\mathcal{M}$-algebras from among them. Above, $P[\mathcal{Y}]$ denote the space of functions defined on $\overline{\mathbb{U}}^n$ consisting of the Poisson integrals of the members of $\mathcal{Y}$. The proofs of the last four theorems are supplied in Section 4.

1.3. Comparison with the ball. Earlier work on the classification of $\mathcal{M}$-spaces concentrated mostly on functions defined on the *unit ball* of $\mathbb{C}^n$ given by

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 < 1\},$$

or on its boundary the *unit sphere*

$$\mathbb{S}_n = \{z \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 = 1\}.$$

The subscript n on $\mathbb{S}_n$ is somewhat misleading since $\dim_{\mathbb{R}}(\mathbb{S}_n) = 2n-1$. Nagel and Rudin [2] showed that $\mathcal{C}_0(\mathbb{B}_n)$ has no proper $\mathcal{M}$-algebra, obtained all the $\mathcal{M}$-algebras of $\mathcal{C}(\overline{\mathbb{B}}_n)$, all the $\mathcal{M}$-spaces of $\mathcal{C}(\mathbb{S}_n)$ and $\mathcal{L}^p(\mathbb{S}_n)$ for $1 \leq p < \infty$. Not all the $\mathcal{M}$-spaces of $\mathcal{C}(\overline{\mathbb{B}}_n)$ are known largely because not all the $\mathcal{M}$-spaces of $\mathcal{C}_0(\mathbb{B}_n)$ are known. Some of these are classified by Rudin in another paper [6]. Later Rudin [5] found all the $\mathcal{M}$-algebras of $\mathcal{C}(\mathbb{B}_n)$, too, but nothing else is known about its $\mathcal{M}$-spaces. Working on the polydisc, Gowda [1] found all the $\mathcal{M}$-spaces and the $\mathcal{M}$-algebras of $\mathcal{C}(\mathbb{T}^n)$ and the $\mathcal{M}$-spaces of $\mathcal{L}^p(\mathbb{T}^n)$ again for $1 \leq p < \infty$.

The classification of all the $\mathcal{U}$-spaces of $\mathcal{C}(\mathbb{S}_n)$ and $\mathcal{L}^p(\mathbb{S}_n)$, the $\mathcal{U}$-algebras of $\mathcal{C}(\mathbb{S}_n)$, and the $\mathcal{U}$-spaces of $\mathcal{C}(\mathbb{T}^n)$ have also been done, by Nagel and Rudin [2] for $\mathbb{S}_n$ and by Gowda [1] for $\mathbb{T}^n$. The description here depends on homogeneous polynomials, and because of their sheer number, we do not get the nice poset diagram of a few well-known spaces as given in Chapter 13 of Rudin [4] for the $\mathcal{M}$-spaces.

In the polydisc, the complicated shape of the boundary gives rise to the partial boundaries, continuous functions vanishing only on them, and the partially holomorphic and constant functions. In one complex dimension and in the unit ball, both the topological boundary and the Shilov boundary are the unit sphere, and the partial boundaries are meaningless. In Section 4, we will see that this disagreement between the ball and the polydisc leads to different sets of $\mathcal{M}$-algebras in the ball and the polydisc. In the classification of the $\mathcal{M}$-algebras of continuous functions on $\mathbb{B}_n$, the ball algebra ($\mathcal{A}(\mathbb{B}_n)$) and the continuous functions vanishing on the boundary ($\mathcal{C}_0(\mathbb{B}_n)$) play central roles. Despite the difference, analyticity and vanishing on (parts of) the boundary remain essential in this work, too.

2. Preliminaries

In this section we collect some of the basic properties of the objects defined in Section 1 and introduce some notation that will be useful in the forthcoming discussion.

2.1. Generalities. Points $z = (z_1, \dots, z_n)$ of $\mathbb{C}^n$ will often be written as $z = (z', z'')$, where $z' = (z_1, \dots, z_k)$ and $z'' = (z_{k+1}, \dots, z_n)$ for some $k < n$ whose value will be clear from the context. z_j will usually be an element of $\mathbb{U}$ and ζ_j of $\mathbb{T}$.

If both p, $q \in \mathbb{U}$, Möbius transformations have the following symmetry properties:

$$(2.1). \qquad |\varphi_p(q)| = |\varphi_q(p)| \quad \text{and} \quad 1 - |\varphi_p(q)|^2 = \frac{(1-|p|^2)(1-|q|^2)}{|1-\bar{p}q|^2}$$

Each Φ_w is an *involution* (its inverse is itself) exchanging 0 and w. $\mathcal{M}$ (even $\mathcal{M}^*$) acts *transitively* on $\mathbb{U}^n$: if a, $b \in \mathbb{U}^n$, then $\Phi_b \circ \Phi_a \in \mathcal{M}^*$ moves a to b (and b to a). $\varphi_{w'}(z')$ will be the short-hand notation for $\big(\varphi_{w_1}(z_1), \dots, \varphi_{w_k}(z_k)\big)$ and similarly for $\varphi_{w''}(z'')$.

We will use λ_n for the Lebesgue measure on $\mathbb{T}^n$ and μ_n for the Lebesgue measure on $\mathbb{U}^n$, both normalized with mass 1.

$$d\kappa_n(z) = \frac{d\mu_n(z)}{\prod_{j=1}^n (1-|z_j|^2)^2}$$

is the $\mathcal{M}$-*invariant measure* on $\mathbb{U}^n$, i.e.,

$$(2.2) \qquad \int_{\mathbb{U}^n} f\, d\kappa_n = \int_{\mathbb{U}^n} (f \circ \Psi)\, d\kappa_n \qquad (\Psi \in \mathcal{M},\ f \in \mathcal{L}^1(\kappa_n)).$$

2.2. More on $\mathcal{M}$-invariant spaces. A very useful tool in our proofs is the *Poisson integral* of an $f \in \mathcal{L}^1(\mathbb{T}^n)$ defined as

$$P[f](z) = \int_{\mathbb{T}^n} f(\zeta) \prod_{j=1}^n \frac{1-|z_j|^2}{|1-z_j\bar{\zeta}_j|^2}\, d\lambda_n(\zeta) \qquad (z \in \mathbb{U}^n).$$

The product, which is positive, is called the *Poisson kernel* $P(z,\zeta)$ for $\mathbb{U}^n$. The Poisson integral is $\mathcal{M}$-invariant in the sense that

$$(2.3) \qquad P[f \circ \Psi] = P[f] \circ \Psi \qquad (\Psi \in \mathcal{M},\ f \in \mathcal{L}^1(\mathbb{T}^n)).$$

Any $f \in \mathcal{A}(\mathbb{U}^n)$ can be recovered from its values on $\mathbb{T}^n$ using the Poisson transform:

$$f(z) = P[f](z) \qquad (z \in \mathbb{U}^n).$$

The collection of the Poisson integrals of functions in $\mathcal{C}(\mathbb{T}^n)$ will be called $\mathcal{P}(\mathbb{U}^n)$, which is also $\mathcal{M}$-invariant. The $\mathcal{M}$-invariance of $\mathcal{A}$ relies on the fact that Möbius transformations are holomorphic; that of $\mathcal{P}$ on the $\mathcal{M}$-invariance of the Poisson integral; and that of $\mathcal{E}_k$, $\mathcal{K}_k$, and $\mathcal{A}_k$ on the $\mathcal{M}$-invariance of T_k. For a more complete discussion of these topics and for the proofs in the unit-ball setting, see Rudin [4].

Let's give alternate definitions for the partially holomorphic and partially constant functions. If F is defined on $\overline{\mathbb{U}}^n$ and $\mathcal{G}$ is any set containing it, for $\sigma \in \mathcal{S}_n$ define F_σ and $\mathcal{G}_\sigma$ by

$$F_\sigma(z_1,\dots,z_n) = F(z_{\sigma(1)},\dots,z_{\sigma(n)}) \quad \text{and} \quad \mathcal{G}_\sigma = \{F_\sigma : F \in \mathcal{G}\}.$$

$F \in \mathcal{A}_k(W)$ if and only if it satisfies

$$F\big|_{T_{k+1}} = 0 \quad \text{and} \quad \int_{\mathbb{T}^k} F_\sigma(\zeta', z'')\, \bar\zeta_1^{\ell_1} \cdots \bar\zeta_k^{\ell_k}\, d\lambda_k(\zeta') = 0 \tag{2.4}$$

for $(\ell_1,\dots,\ell_k) \in \mathbb{Z}^k \backslash \mathbb{N}^k$, $\sigma \in \mathcal{S}_n$, and $z'' \in \mathbb{U}^{n-k}$. In other words, F has the same distinguished boundary values as some function in the polydisc algebra of a lower dimension (k). $\mathbb{N}$ and $\mathbb{Z}$ denote the nonnegative integers and integers, respectively. The integral in (2.4), by definition, is the $(\ell_1,\dots,\ell_k)^{\text{th}}$ *Fourier coefficient* of F_σ considered as a function of ζ'. $F \in \mathcal{K}_k(W)$ if and only if it satisfies (2.4) for $(\ell_1,\dots,\ell_k) \in \mathbb{Z}^k \backslash \{(0,\dots,0)\}$, $\sigma \in \mathcal{S}_n$, and $z'' \in \mathbb{U}^{n-k}$. In either case, having the conditions on all $\sigma \in \mathcal{S}_n$ is redundant; one single $\sigma \in \mathcal{S}_n$ for each k-tuple carrying it to the first k positions suffices. $\mathcal{K}_k(W) = \mathcal{A}_k(W) \cap \operatorname{conj} \mathcal{A}_k(W)$ just as $\mathbb{C} = \mathcal{A}(\mathbb{U}^n) \cap \operatorname{conj} \mathcal{A}(\mathbb{U}^n)$. When $n = 2$, the only possible value of k is 1. Then $\mathcal{A}_1(\mathbb{U}^2)$ consists of those functions in $\mathcal{E}_2(\mathbb{U}^2)$ all of whose negative Fourier coefficients are 0 on all circles of the form

$$Q_1(z_2) = \{(\zeta_1, z_2) : \zeta_1 \in \mathbb{T}\} \quad \text{and} \quad Q_2(z_1) = \{(z_1, \zeta_2) : \zeta_2 \in \mathbb{T}\}, \tag{2.5}$$

and $\mathcal{K}_1(\mathbb{U}^2)$ consists of those functions in $\mathcal{E}_2(\mathbb{U}^2)$ which have constant values on all $Q_1(z_2)$ and $Q_2(z_1)$. Hybrid spaces similar to $\mathcal{A}_k$ and $\mathcal{K}_k$ can also be defined by letting the boundary values be, say, holomorphic in one direction and constant in the other; these are only $\mathcal{M}^*$-invariant.

2.3. Radialization. Another useful tool is the *polyradialization* of an $f \in \mathcal{C}(\overline{\mathbb{U}}^n)$ given by

$$f^\#(z) = \int_{\mathbb{T}^n} f(|z_1|\zeta_1,\dots,|z_n|\zeta_n)\, d\lambda_n(\zeta).$$

$f^\#$ is radial in each variable seperately, i.e., $f^\#(z_1,\dots,z_n) = f^\#(w_1,\dots,w_n)$ if we have $|z_1| = |w_1|,\dots,|z_n| = |w_n|$. By (2.1), for any polyradial g,

$$g\big(\Phi_w(z)\big) = g\big(\Phi_z(w)\big) \qquad (z,\, w \in \mathbb{U}^n). \tag{2.6}$$

If $\mathcal{G}$ is a $\mathcal{U}$-space and $f \in \mathcal{G}$, then $f^\# \in \mathcal{G}$ as well.

Partial polyradialization and partial Poisson integrals can also be defined by operating on some set of k variables and integrating on $\mathbb{T}^k$ for some k, $1 \le k < n$.

3. MÖBIUS SUBALGEBRAS OF $\mathcal{C}_0(\mathbb{U}^n)$

Our aim in this section is to prove Theorem A. This is one of the two extreme cases mentioned. Note that the full strength of the Möbius group is not required. A similar statement for $\mathcal{M}$-spaces is incorrect; in fact there is an abundance of them, and a full classification still awaits an answer. The proof presented here can be adapted to give a demonstration of the same fact in $\mathbb{B}_n$ different from that of Nagel and Rudin [2], except for Lemma 3.1, which must stay the same. This lemma states that if such an $\mathcal{M}^*$-space existed, it would contain a smooth function F which can be used in constructing an approximate identity.

Lemma 3.1. *If $\mathcal{X}$ is an $\mathcal{M}^*$-invariant closed subspace of $\mathcal{C}_0(\mathbb{U}^n)$ and $\mathcal{X} \neq \{0\}$, then $\mathcal{X}$ contains an F with the following properties:*

(i) *F is polyradial,*
(ii) *F is real-analytic,*
(iii) *$F(0) = 1$,*
(iv) *$|F(z)| < 1$, for every $z \in \overline{\mathbb{U}}^n$ with $z \neq 0$.*

Proof. $\mathcal{X}$ is nontrivial and $\mathcal{M}^*$-invariant, so there is a $g \in \mathcal{X}$ with $g(0) \neq 0$. $g^\# \in \mathcal{X}$ because $\mathcal{X}$ is closed, and $g^\# \neq 0$ since $g^\#(0) = g(0)$. For integer $m \geq 2$, define

$$K_m(w) = c_m \prod_{j=1}^{n} (1 - |w_j|^2)^m \qquad (w \in \overline{\mathbb{U}}^n). \tag{3.1}$$

$K_m \in \mathcal{C}_0(\mathbb{U}^n)$ and c_m are chosen so as to have $\int_{\mathbb{U}^n} K_m \, d\kappa_n = 1$. A quick computation shows that $c_m = (m-1)^n$. Put

$$h_m(z) = \int_{\mathbb{U}^n} (g^\# \circ \Phi_w)(z) K_m(w) \, d\kappa_n(w) \qquad (z \in \overline{\mathbb{U}}^n).$$

By the $\mathcal{M}^*$-invariance of $\mathcal{X}$, the integrand, considered as a function of z, belongs to $\mathcal{X}$ for each $w \in \mathbb{U}^n$. But the integral is a uniform limit of Riemann sums and $\mathcal{X}$ is closed; hence $h_m \in \mathcal{X}$, $m = 2, 3, \ldots$. $m \geq 2$ is required to cancel out the singular behavior of κ_n on $\partial\mathbb{U}^n$. We also have $g^\#(0) = \int_{\mathbb{U}^n} g^\#(0) K_m(w) \, d\kappa_n(w)$ and $h_m(0) = \int_{\mathbb{U}^n} g^\#(w) K_m(w) \, d\kappa_n(w)$.

Given $\varepsilon > 0$, choose $r \in (0,1)$ so that $|g^{\#}(w) - g^{\#}(0)| < \varepsilon$ whenever $w \in r\mathbb{U}^n$, using the continuity of $g^{\#}$. Then for $m \geq 4$,

$$\begin{aligned} |h_m(0) - g^{\#}(0)| &\leq \int_{r\mathbb{U}^n} |g^{\#}(w) - g^{\#}(0)| K_m(w)\, d\kappa_n(w) \\ &\quad + \int_{\mathbb{U}^n \setminus r\mathbb{U}^n} |g^{\#}(w) - g^{\#}(0)| K_m(w)\, d\kappa_n(w) \\ &\leq \varepsilon + 2\|g^{\#}\|_\infty \int_{\mathbb{U}^n \setminus r\mathbb{U}^n} K_m(w)\, d\kappa_n(w) \\ &= \varepsilon + 2\|g^{\#}\|_\infty \left(\frac{c_m}{c_{m-2}}\right) \int_{\mathbb{U}^n \setminus r\mathbb{U}^n} K_{m-2}(w)\, d\mu_n(w). \end{aligned}$$

First note that $\lim_{m\to\infty} c_m/c_{m-2} = 1$. If $w \in \mathbb{U}^n \setminus r\mathbb{U}^n$, at least one of the coordinates of w, say the first, is more than r in absolute value, and then

$$\begin{aligned} K_{m-2}(w) &= c_{m-2} \prod_{j=1}^{n} (1 - |w_j|^2)^{m-2} \\ &\leq c_{m-2}(1 - r^2)^{m-2} \prod_{j=2}^{n} (1 - |w_j|^2)^{m-2} \\ &\leq (m-3)^n (1 - r^2)^{m-2} \qquad (m \geq 4). \end{aligned}$$

Hence $K_{m-2}(w) \to 0$ uniformly on $\mathbb{U}^n \setminus r\mathbb{U}^n$ as $m \to \infty$. Consequently m can be selected to have

$$2\|g^{\#}\|_\infty \left(\frac{c_m}{c_{m-2}}\right) \int_{\mathbb{U}^n \setminus r\mathbb{U}^n} K_{m-2}\, d\mu_n < \varepsilon.$$

Therefore $|h_m(0) - g^{\#}(0)| < 2\varepsilon$ if m is large enough. Since $g^{\#}(0) \neq 0$, for some large (fixed) M, $h_M(0) \neq 0$.

Using first (2.1) and (2.6) together, then (2.2), and then (2.1) and (3.1) together, we obtain

$$\begin{aligned} h_M(z) &= \int_{\mathbb{U}^n} (g^{\#} \circ \Phi_z)(w) K_M(w)\, d\kappa_n(w) \\ &= \int_{\mathbb{U}^n} g^{\#}(w) K_M\big(\Phi_z(w)\big)\, d\kappa_n(w) \\ &= c_M \prod_{j=1}^{n} (1 - |z_j|^2)^M \int_{\mathbb{U}^n} g^{\#}(w) \prod_{j=1}^{n} \frac{(1 - |w_j|^2)^M}{|1 - z_j \bar{w}_j|^{2M}}\, d\kappa_n(w). \end{aligned}$$

Expanding the denominator of the product into a convergent (since h_M is well-defined) power series and then integrating, we see that h_M is the sum of a convergent power series in $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$; this shows, by definition, h_M is real-analytic.

Pick $a \in \mathbb{U}^n$ such that $|h_M(a)| = \|h_M\|_\infty$ and $\max\{1 - |a_j| : 1 \leq j \leq n\}$ is minimal. Let $h = ch_M \circ \Phi_a$, where c is such that $h(0) = ch_M(a) = 1$. Finally let $F = h^{\#}$. Since polyradialization takes the average on a torus, it doesn't allow $|F(z)| = 1$ if $z \neq 0$, because every other point b where $|h_M|$ attains its maximum has at least one coordinate j with $|b_j| < |a_j|$, and in this coordinate, $h_M(b)$ is averaged also with some points at which $|h_M| < 1$. This F is the desired function. □

Moreover, the real-analyticity and the polyradiality of F implies that $|F|$ is a strictly decreasing function of any convex combination of $|z_1|^2, \dots, |z_n|^2$ (in particular, of each $|z_j|^2$) in a small neighborhood of 0.

In the second lemma, we take a function with some of the properties of F of the previous lemma and show that we can suitably normalize it so that the Taylor expansion of the new f has a certain simple form.

Lemma 3.2. *Given a real-analytic $F \in \mathcal{C}_0(\mathbb{U}^n)$ with a unique global maximum modulus of 1 at the origin, there exists a polyradial f having all the properties of F and such that*

$$f(z) = 1 - \sum_{j=1}^{n} \alpha_j |z_j|^2 + \mathcal{O}(|z|^4)$$

with $\Re\alpha_j > 0$, $j = 1, \dots, n$. Further, if F belongs to an $\mathcal{M}^*$-space, so does f.

Proof. Fix $\varepsilon > 0$. Since the maximum of $|F|$ is unique, there is an $r \in (0,1)$ such that $|F(z)| < 1 - \varepsilon$ when $z \in \mathbb{U}^n \backslash r\mathbb{U}^n$. Put $c = \frac{\varepsilon}{n}$. The function

$$|F(z)| + c|z|^2$$

is 1 at $z = 0$ and is strictly less than $1 - \varepsilon + cn = 1$ if $z \in \mathbb{U}^n \backslash r\mathbb{U}^n$, hence attains its maximum at some $p \in r\overline{\mathbb{U}}^n$; thus $|F(z)| + c|z|^2 \leq |F(p)| + c|p|^2$, for all $z \in \overline{\mathbb{U}}^n$. Taking $z \in \mathbb{T}^n$ shows that $|F(p)| \geq c(n - |p|^2) > 0$. So $g = F/F(p)$ is well-defined. Pick $\Psi \in \mathcal{M}^*$ such that $\Psi(0) = p$; e.g., take $\Psi(z) = \Phi_p(z)$. Finally put $f = (g \circ \Psi)^{\#}$. This f satisfies the requirements of the lemma. Indeed,

$$\begin{aligned} 1 - |g(z)| = 1 - \frac{|F(z)|}{|F(p)|} &\geq \frac{c|z|^2 - c|p|^2}{|F(p)|} \\ &\geq c|z|^2 - c|p|^2 = c|z-p|^2 + 2c\Re\langle z-p, p\rangle; \end{aligned}$$

hence

$$|(g \circ \Psi)(z)| \le 1 - c|\Psi(z) - p|^2 - 2c\Re\langle \Psi(z) - p, p\rangle.$$

But

$$\varphi_{p_j}(z_j) - p_j = \frac{(|p_j|^2 - 1)z_j}{1 - \bar{p}_j z_j}$$

and

$$\frac{1 - |p_j|^2}{|1 - \bar{p}_j z_j|} = \frac{(1 - |p_j|)(1 + |p_j|)}{|1 - \bar{p}_j z_j|} \ge \frac{(1 - |p_j|)(1 + |p_j|)}{1 + |\bar{p}_j||z_j|} \ge 1 - |p_j| \ge 1 - r.$$

Thus $|\Psi(z) - p|^2 \ge (1 - r)^2 |z|^2$ and

$$|(g \circ \Psi)(z)| \le 1 - c(1 - r)^2 |z|^2 - 2c\Re\langle \Psi(z) - p, p\rangle.$$

When we polyradialize, since we take the average on a torus, the last term will drop by the mean value property, because it is harmonic in each variable seperately and is 0 at the origin. Therefore

$$|f(z)| = |(g \circ \Psi)^{\#}| \le 1 - c(1 - r)^2 |z|^2.$$

But $f(0) = 1$, f is real-analytic and decreasing in every direction from the origin. Then near 0 we must have

$$f(z) = 1 - \sum_{j=1}^{n} \alpha_j |z_j|^2 + \mathcal{O}(|z|^4)$$

with $\Re\alpha_j > 0$ for $j = 1, \dots, n$. Third-order terms do not appear in the expansion due to polyradialization. Finally, the last assertion is obvious when we consider the way f is obtained from F. □

In the last lemma, we prove that the positive integer powers of f actually form an approximate identity.

Lemma 3.3. *Suppose*

(a) $f\colon \overline{\mathbb{U}}^n \to \overline{\mathbb{U}}$ *is polyradial and continuous,*
(b) $|f(z)| < 1$ *if* $z \ne 0$,
(c) *there are* $0 \ne \alpha_j \in \mathbb{C}$ *such that* $f(z) = 1 - \sum_{j=1}^{n} \alpha_j |z_j|^2 + \mathcal{O}(|z|^4)$ *near* 0.

Fix $p \in \left(\frac{n+1}{2n+4}, \frac{1}{2}\right)$. *For* $m = 2, 3, \dots$, *let* $\varepsilon_m = m^{-p}$, *and take* c_m *so as to satisfy* $\int_{\varepsilon_m \mathbb{U}^n} c_m f^m \, d\mu_n = 1$. *Then as* $m \to \infty$,

(i) $c_m = \mathcal{O}(m^n)$,
(ii) $\int_{\varepsilon_m \mathbb{U}^n} |c_m f^m| \, d\mu_n = \mathcal{O}(1)$,
(iii) $k_m = \sup\{|c_m f^m(z)| : z \in \overline{\mathbb{U}}^n \backslash \varepsilon_m \mathbb{U}^n\} \to 0$.

Proof. Let $a_j = \Re\alpha_j$, $b_j = \Im\alpha_j$, and note that necessarily each $a_j > 0$. The first two terms of f can be factored out to yield

$$f(z) = \prod_{j=1}^{n}(1-\alpha_j|z_j|^2) + \mathcal{O}(|z|^4) = K(z) + \mathcal{O}(|z|^4)$$

near 0. Put $r_j = |z_j|$ and $r = |z|$. We have

$$|1-\alpha_j r_j^2|^2 = 1 - 2a_j r_j^2 + |\alpha_j|^2 r_j^4 = 1 - r_j^2(2a_j - |\alpha_j|^2 r_j^2) \le 1$$

as soon as $r_j^2 \le 2a_j/|\alpha_j|^2$. By restricting our attention to smaller r, we achieve $|K| \le 1$. Now $f^m - K^m = (f-K)(f^{m-1} + f^{m-2}K + \cdots + fK^{m-2} + K^{m-1})$. Each of the m terms in the second factor is not greater than 1 in absolute value in a small enough neighborhood of 0. Hence near 0, $|f^m - K^m| \le m|f-K|$ and

$$f^m(z) = \prod_{j=1}^{n}(1-\alpha_j r_j^2)^m + \mathcal{O}(mr^4). \tag{3.2}$$

When we integrate f^m over the polydisc $\varepsilon_m\mathbb{U}^n$, using Hölder Inequality, the second (error) term of (3.2) becomes

$$\begin{aligned} m\int_{\varepsilon_m\mathbb{U}^n} r^4\,d\mu_n &= m\int_{\varepsilon_m\mathbb{U}^n}(|z_1|^2+\cdots+|z_n|^2)^2\,d\mu_n \\ &\le nm\int_{\varepsilon_m\mathbb{U}^n}(|z_1|^4+\cdots+|z_n|^4)\,d\mu_n \\ &= n^2m\int_{\varepsilon_m\mathbb{U}^n}|z_1|^4\,d\mu_n = 2n^2m\varepsilon_m^{2n-2}\int_0^{\varepsilon_m} t^5\,dt \\ &= \frac{1}{3}n^2m\varepsilon_m^{2n+4}, \end{aligned}$$

and this last form is $\mathcal{O}(m\varepsilon_m^{2n+4}) = \mathcal{O}(m^{1-(2n+4)p}) = \mathrm{o}(m^{-n})$ as $m \to \infty$, because by hypothesis $(2n+4)p > n+1$. And the first term of (3.2) becomes

$$\prod_{j=1}^{n}\int_0^{\varepsilon_m}(1-\alpha_j r_j^2)^m 2r_j\,dr_j.$$

Now looking at the one-dimensional case without the subscripts,

$$\begin{aligned} m\int_0^{\varepsilon_m}(1-\alpha r^2)^m 2r\,dr &= m\int_0^{\varepsilon_m^2}(1-\alpha x)^m\,dx \\ &= \frac{m}{(m+1)\alpha}[1-(1-\alpha\varepsilon_m^2)^{m+1}]. \end{aligned}$$

But $\lim_{m\to\infty}(1-\alpha\varepsilon_m^2)^{m^{2p}} = \lim_{m\to\infty}(1-\alpha m^{-2p})^{m^{2p}} = e^{-\alpha}$. Then since $|e^{-\alpha}| = e^{-a} < 1$ and $2p < 1$, $\lim_{m\to\infty}(1-\alpha\varepsilon_m^2)^{m+1} = 0$. Consequently

$$\lim_{m\to\infty} m\left|\int_0^{\varepsilon_m}(1-\alpha r^2)^m 2r\,dr\right| = \lim_{m\to\infty}\frac{m}{(m+1)|\alpha|} = \frac{1}{|\alpha|}. \tag{3.3}$$

Going back to the multi-dimensional case, using (3.3) we see that

$$\begin{aligned}\lim_{m\to\infty}\left|m^n\int_{\varepsilon_m\mathbb{U}^n} f^m\,d\mu_n\right| &= \lim_{m\to\infty}\prod_{j=1}^n\left|m\int_0^{\varepsilon_m}(1-\alpha_j r_j^2)^m 2r_j\,dr_j\right|\\ &= \prod_{j=1}^n\frac{1}{|\alpha_j|} > 0,\end{aligned}$$

which proves (i).

For small $r > 0$, $|1-\alpha_j r_j^2|^2 = 1 - 2a_j r_j^2 + |\alpha_j|^2 r_j^4 < 1 - \frac{3}{2}a_j r_j^2$. Then

$$|f(z)|^2 < \prod_{j=1}^n(1-a_j r_j^2). \tag{3.4}$$

Applying (3.3) with $m/2$ in place of m and noting that $\varepsilon_m < \varepsilon_{m/2}$, we get

$$\begin{aligned}\limsup_{m\to\infty}\int_{\varepsilon_m\mathbb{U}^n}\left|\left(\frac{m}{2}\right)^n f^m\right|d\mu_n &\le \limsup_{m\to\infty}\int_{\varepsilon_{m/2}\mathbb{U}^n}\left|\left(\frac{m}{2}\right)^n f^m\right|d\mu_n\\ &< \lim_{m\to\infty}\prod_{j=1}^n\frac{m}{2}\int_0^{\varepsilon_{m/2}}(1-a_j r_j^2)^{m/2}2r_j\,dr_j\\ &= \prod_{j=1}^n\frac{1}{a_j} < \infty;\end{aligned}$$

and this proves (ii).

Put $t_m = \sup\{|f(z)| : z \in \overline{\mathbb{U}}^n\setminus\varepsilon_m\mathbb{U}^n\}$. Since the only global maximum of $|f|$ is at 0, for large enough m, this supremum is attained on $\partial(\varepsilon_m\mathbb{U}^n)$, say at z_m. Without loss of generality, $r_{m_1} = |z_{m_1}| = \varepsilon_m$. For such large m, z_m is small enough to use (3.4). Therefore

$$\begin{aligned}t_m^2 = |f(z_m)|^2 &< \prod_{j=1}^k(1-a_j r_{m_j}^2) \le 1 - a_1 r_{m1}^2\\ &= 1 - a_1\varepsilon_m^2 = 1 - a_1 m^{-2p} < \exp(-a_1 m^{-2p})\end{aligned}$$

and for some constant C,

$$k_m \le C m^n t_m^m = C m^n (t_m^2)^{m/2} < C m^n \exp\Big(-\frac{a_1}{2} m^{1-2p}\Big) \to 0$$

as $m \to \infty$, because $2p < 1$. This completes the proof of (iii). □

Now we prove Theorem A by showing that an $\mathcal{M}^*$-algebra of $\mathcal{C}_0(\mathbb{U}^n)$ would actually be dense in $\mathcal{C}_0(\mathbb{U}^n)$. The proof uses the fact that such an algebra would be an ideal in $\mathcal{C}_0(\mathbb{U}^n)$ under the convolution using Möbius transformations in place of translations, which was already used in Lemma 3.1.

Proof of Theorem A. Let $\mathcal{X}$ be an $\mathcal{M}^*$-algebra of $\mathcal{C}_0(\mathbb{U}^n)$ with $\mathcal{X} \ne \{0\}$. By Lemma 3.1, $\mathcal{X}$ has a polyradial real-analytic function f with a unique global maximum modulus of 1 at 0. A normalized f can be expressed as $f(z) = 1 - \sum_{j=1}^n \alpha_j |z_j|^2 + \mathcal{O}(|z|^4)$ near 0 with $\Re \alpha_j > 0$ for $1 \le j \le n$, by Lemma 3.2. Lemma 3.3 applies to f and we obtain ε_m, c_m, and k_m. In addition, κ_n can be freely exchanged with μ_n as needed in Lemma 3.3, because $\mu_n \le \kappa_n \le C\mu_n$ for some real number C, as long as the domain of integration is bounded away from $\partial\mathbb{U}^n$; and $\varepsilon_2\mathbb{U}^n$, which includes all $\varepsilon_m\mathbb{U}^n$ for $m = 2, 3, \ldots$, indeed is.

Let $\mathcal{C}_c(\mathbb{U}^n)$ denote the collection of functions in $\mathcal{C}_0(\mathbb{U}^n)$ with compact support in $\mathbb{U}^n$, and pick a polyradial $g \in \mathcal{C}_c(\mathbb{U}^n)$. Define

$$h_m(z) = \int_{\mathbb{U}^n} c_m f^m\big(\Phi_w(z)\big) g(w)\, d\kappa_n(w).$$

Since $\operatorname{supp} g \subset\subset \mathbb{U}^n$, the integral exists and h_m is well-defined for every integer $m \ge 2$.

Claim: $h_m \to g$ uniformly on $\overline{\mathbb{U}}^n$ as $m \to \infty$. As in the proof of Lemma 3.1, $h_m \in \mathcal{X}$; so writing $g(z) = \int_{\mathbb{U}^n} c_m f^m(w) g(z)\, d\mu_n(w)$ and letting $K(w) = \prod_{j=1}^n (1 - |w_j|^2)^2$,

$$\begin{aligned} h_m(z) - g(z) &= \int_{\varepsilon_m\mathbb{U}^n} c_m f^m(w) \big[g\big(\Phi_z(w)\big) - g(z)K(w)\big]\, d\kappa_n(w) \\ &\qquad + \int_{\mathbb{U}^n \setminus \varepsilon_m\mathbb{U}^n} c_m f^m(w) \big[g\big(\Phi_z(w)\big) - g(z)K(w)\big]\, d\kappa_n(w) \\ &= I + II. \end{aligned}$$

Let $w \in \varepsilon_m\mathbb{U}^n$. Then for large m,

$$\big|\varphi_{z_j}(w_j) - z_j\big| = \left|\frac{w_j(1 - |z_j|^2)}{1 - \bar{z}_j w_j}\right| \le \frac{\varepsilon_m}{1 - \varepsilon_m} \le 2\varepsilon_m;$$

so $|\Phi_z(w) - z| \le 2\sqrt{n}\varepsilon_m$, independently of $z \in \overline{\mathbb{U}}^n$. It follows that, as $m \to \infty$ which forces $w \to 0$, $|\Phi_z(w) - z| \to 0$, and by the uniform continuity and the polyradiality of g, $\left|g\big(\Phi_w(z)\big) - g(z)\right| \to 0$; also $\left|g\big(\Phi_w(z)\big) - g(z)K(w)\right| \to 0$, all uniformly in $z \in \overline{\mathbb{U}}^n$. Thus, by (ii) of Lemma 3.3,

$$|I| \le \|g \circ \Phi_w - gK\|_\infty \int_{\varepsilon_m \mathbb{U}^n} \left|c_m f^m\right| d\kappa_n \to 0 \quad \text{as} \quad m \to \infty.$$

By (2.2), (iii) of Lemma 3.3, and since $\operatorname{supp} g \subset\subset \mathbb{U}^n$,

$$\begin{aligned}
|II| &\le k_m \int_{\mathbb{U}^n \setminus \varepsilon_m \mathbb{U}^n} \left|g\big(\Phi_z(w)\big) - g(z)K(w)\right| d\kappa_n(w) \\
&\le k_m \int_{\mathbb{U}^n} \left|g\big(\Phi_z(w)\big) - g(z)K(w)\right| d\kappa_n(w) \\
&\le k_m \left(\int_{\mathbb{U}^n} |(g \circ \Phi_z)(w)|\, d\kappa_n(w) + \int_{\mathbb{U}^n} |g(z)|\, d\mu_n(w) \right) \\
&= k_m \left(\int_{\mathbb{U}^n} |g(w)|\, d\kappa_n(w) + |g(z)| \right) \le k_m \left(\int_{\operatorname{supp} g} |g|\, d\kappa_n + |g(z)| \right) \\
&\le k_m \left(\|g\|_{\mathcal{L}^1(\kappa)} + \|g\|_\infty \right) \to 0 \text{ as } m \to \infty.
\end{aligned}$$

Hence the claim is established and $g \in \mathcal{X}$.

Therefore $\mathcal{X}$ contains the smallest $\mathcal{M}^*$-invariant subalgebra $\mathcal{Y}$ of $\mathcal{C}_c(\mathbb{U}^n)$ which includes all the polyradial functions in $\mathcal{C}_c(\mathbb{U}^n)$. $\mathcal{Y}$ is a self-adjoint subalgebra of $\mathcal{X} \subset \mathcal{C}_0(\mathbb{U}^n)$, so it contains real-valued functions. $\mathcal{Y}$ seperates the points of $\mathbb{U}^n$: If z, $w \in \mathbb{U}^n$ and $z \ne w$, take a $g \in \mathcal{Y}$ with $g(0) = 1$ and $0 \le g(z) < 1$ for $z \ne 0$; then $(g \circ \Phi_w)^\# \in \mathcal{Y}$ seperates z and w. Clearly $\mathcal{Y}$ vanishes nowhere in $\mathbb{U}^n$. Then, since $\mathcal{X}$ is closed, by the Stone-Weierstrass Theorem, $\mathcal{C}_0(\mathbb{U}^n) = \mathrm{cl}(\mathcal{Y}) \subset \mathcal{X} \subset \mathcal{C}_0(\mathbb{U}^n)$, where cl denotes uniform closure. The proof is now complete. □

4. Möbius Subalgebras of $\mathcal{C}(\overline{\mathbb{U}}^n)$

In this section, we present the proofs of Theorems B, C, D, and E. In some of their corollaries, we will state the results in a more detailed form for the case $n = 2$ and $k = 1$, or more specifically for algebras instead of spaces. Often we will use the term function to mean a continuous function on a subset or all of $\overline{\mathbb{U}}^n$.

4.1. $\mathcal{M}$-Algebras Containing $\mathcal{E}_n(\mathbb{U}^n)$. We first find the biggest $\mathcal{M}$-spaces of $\mathcal{C}(\overline{\mathbb{U}}^n)$ in some sense.

Proof of Theorem B. Let $F \in \mathcal{X}$ and $f = F\big|_{\mathbb{T}^n}$. Then $H = F - P[f] \in \mathcal{E}_n(\mathbb{U}^n) \subset \mathcal{X}$ and $P[f] = F - H \in \mathcal{X}$. Now let $\mathcal{Y} = \mathcal{X}\big|_{\mathbb{T}^n}$, take a sequence $\{f_m\}$ in $\mathcal{Y}$, and suppose $f_m \to f$ uniformly on $\mathbb{T}^n$. $f_m = F_m\big|_{\mathbb{T}^n}$ for some $F_m \in \mathcal{X}$ and $F_m = P[f_m] + H_m$ for some $H_m \in \mathcal{E}_n(\mathbb{U}^n)$. Also $\sup_{\mathbb{T}^n} |f_m - f_k| = \sup_{\overline{\mathbb{U}}^n} |F_m - H_m - (F_k - H_k)|$, so there is an $F \in \mathcal{X}$ with $F_m - H_m$ converging uniformly to F on $\overline{\mathbb{U}}^n$. Then $f_m = (F_m - H_m)\big|_{\mathbb{T}^n} \to F\big|_{\mathbb{T}^n}$, i.e., $f = F\big|_{\mathbb{T}^n}$. This shows that $\mathcal{Y}$ is closed; thus $\mathcal{Y}$ is an $\mathcal{M}$-space of $\mathcal{C}(\mathbb{T}^n)$. Poisson integrals of its elements are in $\mathcal{X}$, and $\mathcal{X}$ consists of these Poisson integrals plus the functions in $\mathcal{E}_n(\mathbb{U}^n)$. □

Remark 4.1. The $\mathcal{M}$-spaces of $\mathcal{C}(\mathbb{T}^n)$ are given by Gowda [1] and in dimension n there are $\frac{1}{n+3}\binom{2n+4}{n+2}$ of them. When $n = 2$, this number is a manageable 14, and writing (ζ, η) for $(\zeta_1, \zeta_2) \in \mathbb{T}^2$, the spaces are

(a) $\{0\}$, $\mathbb{C}$, $\mathcal{Z}(\mathbb{T}^2)$, $\operatorname{plh}(\mathbb{T}^2)$, $\mathcal{C}_\zeta(\mathbb{T}) + \mathcal{C}_\eta(\mathbb{T})$, $\mathcal{C}(\mathbb{T}^2)$,
(b) $\mathcal{A}_\zeta(\mathbb{T}) + \mathcal{A}_\eta(\mathbb{T})$, $\mathcal{A}(\mathbb{T}^2)$, $\mathcal{A}(\mathbb{T}^2) + \mathcal{C}_\zeta(\mathbb{T}) + \mathcal{C}_\eta(\mathbb{T})$, $\operatorname{cl}\big(\mathcal{Z}(\mathbb{T}^2) + \mathcal{A}(\mathbb{T}^2)\big)$,
(c) conjugates of the spaces in (b),

where $\mathcal{Z}(\mathbb{T}^2) = \operatorname{cl}(\operatorname{span}\{\bar\zeta^p\eta^q, \zeta^p\bar\eta^q : p,\ q \in \mathbb{N}\})$, $\mathcal{C}_\zeta(\mathbb{T})$ the subspace of $\mathcal{C}(\mathbb{T}^2)$ consisting of functions that depend only on ζ, $\mathcal{A}_\zeta(\mathbb{T})$ the subspace of $\mathcal{C}(\mathbb{T}^2)$ spanned by $\{1, \zeta, \zeta^2, \dots\}$, and $\operatorname{plh}(\mathbb{T}^n)$ those $f \in \mathcal{C}(\mathbb{T}^n)$ whose Poisson integrals are pluriharmonic in $\mathbb{U}^n$. Clearly there are too many spaces to list if $n > 2$. But when we specialize to algebras, there are only 5 in each dimension, and these are also listed in [1]:

(A) $\{0\}$,
(B) $\mathbb{C}$,
(C) $\mathcal{A}(\mathbb{T}^n)$,
(D) $\operatorname{conj}\mathcal{A}(\mathbb{T}^n)$,
(E) $\mathcal{C}(\mathbb{T}^n)$.

Note. A twice-differentiable function f defined in an open set in $\mathbb{C}^n$ is said to be *pluriharmonic* if

$$\frac{\partial^2 f}{\partial z_j\, \partial \bar z_k} = 0 \qquad (j,\ k = 1, \dots, n).$$

Corollary 4.2. *The $\mathcal{M}$-spaces of $\mathcal{C}(\overline{\mathbb{U}}^2)$ containing $\mathcal{E}_2(\mathbb{U}^2)$ are*

(i) $\mathcal{E}_2(\mathbb{U}^2)$, $\mathcal{E}_2(\mathbb{U}^2) + \mathbb{C}$, $\mathcal{E}_2(\mathbb{U}^2) + \mathcal{Z}(\mathbb{U}^2)$, $\mathcal{E}_2(\mathbb{U}^2) + \operatorname{plh}(\mathbb{U}^2)$,
$\mathcal{E}_2(\mathbb{U}^2) + \mathcal{P}_z(\mathbb{U}) + \mathcal{P}_w(\mathbb{U})$, $\mathcal{C}(\overline{\mathbb{U}}^2)$,
(ii) $\mathcal{E}_2(\mathbb{U}^2) + \mathcal{A}_z(\mathbb{U}) + \mathcal{A}_w(\mathbb{U})$, $\mathcal{E}_2(\mathbb{U}^2) + \mathcal{A}(\mathbb{U}^2)$,
$\mathcal{E}_2(\mathbb{U}^2) + \mathcal{A}(\mathbb{U}^2) + \mathcal{P}_z(\mathbb{U}) + \mathcal{P}_w(\mathbb{U})$, $\mathcal{E}_2(\mathbb{U}^2) + \operatorname{cl}\big(\mathcal{Z}(\mathbb{U}^2) + \mathcal{A}(\mathbb{U}^2)\big)$,

(iii) *conjugates of the spaces in* (ii),

where (z,w) *denotes an element of* $\overline{\mathbb{U}}^n$, $\mathcal{P}_z(\mathbb{U})$ *consists of functions which are Poisson integrals of members of* $\mathcal{C}_\zeta(\mathbb{T})$, *and* $\text{plh}(\mathbb{U}^n)$ *is the set of all* $f \in \mathcal{C}(\overline{\mathbb{U}}^n)$ *that are pluriharmonic in* $\mathbb{U}^n$.

Proof. We take the Poisson integrals of the functions in the spaces in (a), (b), (c) of Remark 4.1, and note that $\mathcal{E}_n(\mathbb{U}^n) + \mathcal{P}(\mathbb{U}^n) = \mathcal{C}(\overline{\mathbb{U}}^n)$. □

Corollary 4.3. *There are precisely* 5 $\mathcal{M}$*-algebras of* $\mathcal{C}(\overline{\mathbb{U}}^n)$ *containing* $\mathcal{E}_n(\mathbb{U}^n)$:

(i) $\mathcal{E}_n(\mathbb{U}^n)$,
(ii) $\mathcal{E}_n(\mathbb{U}^n) + \mathbb{C}$,
(iii) $\mathcal{E}_n(\mathbb{U}^n) + \mathcal{A}(\mathbb{U}^n)$,
(iv) $\mathcal{E}_n(\mathbb{U}^n) + \text{conj}\,\mathcal{A}(\mathbb{U}^n)$,
(v) $\mathcal{C}(\overline{\mathbb{U}}^n)$.

4.2. $\mathcal{M}$-Algebras Between $\mathcal{E}_k(\mathbb{U}^n)$ and $\mathcal{E}_{k+1}(\mathbb{U}^n)$. Next we find the $\mathcal{M}$-algebras between any two consecutive algebras in (1.4). The problem is open for the $\mathcal{M}$-spaces. The following lemma solves essentially the same problem for continuous functions defined on T_k.

Lemma 4.4. *Let* $\mathcal{Y}$ *be a closed* $\mathcal{M}$*-invariant subalgebra of* $\mathcal{C}(T_k)$ *satisfying the condition* $\mathcal{Y}\big|_{T_{k+1}} = \{0\}$ *for some* k, $1 \le k \le n-1$. *Then* $\mathcal{Y}$ *must be one of*

(i) $\{0\}$,
(ii) $\mathcal{K}_k(T_k)$,
(iii) $\mathcal{A}_k(T_k)$,
(iv) $\text{conj}\,\mathcal{A}_k(T_k)$,
(v) $\{f \in \mathcal{C}(T_k) : f\big|_{T_{k+1}} = 0\}$.

Proof. Assume $\mathcal{Y} \neq \{0\}$. Fix $q'' \in \mathbb{U}^{n-k}$ and let

$$Q = Q(q'') = \{(\zeta', q'') : \zeta' \in \mathbb{T}^k\}. \tag{4.1}$$

Topologically Q is just $\mathbb{T}^k$. Put $\mathcal{Z} = \mathcal{Y}|_Q$. $\mathcal{Z}$ is an algebra invariant under the automorphisms of the first k variables, i.e., it is an $\mathcal{M}$-invariant subalgebra of $\mathcal{C}(\mathbb{T}^k)$. To prove $\mathcal{Z}$ is also closed, take a sequence $\{f_m\}$ in $\mathcal{Z}$ with $f_m \to f$ uniformly on Q. Each f_m is the restriction to Q of some $F_m \in \mathcal{Y}$, but we need an extension dominated by multiple of $\|f_m\|_Q$.

$\mathcal{Y}$ has an element G_0 not identically 0. Suppose, with no loss of generality, $\Re G_0$ attains its maximum M_0 at $a = (1', a'') \in T_k$, where $1'$ denotes a

k-tuple of 1's. Then, given (a small) $\varepsilon > 0$, there is a (small) $\delta > 0$ such that $\Re G_0(e^{i\vartheta_1}, \dots, e^{i\vartheta_k}, a'') > M_0 - \varepsilon$ on the set given by $|\vartheta_j| < \delta$, $j = 1, \dots, k$. This set is a cartesian product of (small) arcs. For $j = 1, \dots, k$, choose b_j such that $0 < b_j < 1$ and each so close to 1 that for $\Psi_0(z) = \big(-\varphi_{b'}(z'), z''\big)$ and $G_1 = G_0 \circ \Psi_0$, we have $\Re G_1(e^{i\vartheta_1}, \dots, e^{i\vartheta_k}, a'') > M_0 - \varepsilon$ on the (much larger) set given by $|\vartheta_j| < \pi - \delta$, $j = 1, \dots, k$. (Here, we are actually looking at the restriction of Ψ_0 to the set given by $|z_1| = \cdots = |z_k| = 1$.) Let $\Psi_1(z) = \big(z', \varphi_{a''}(z'')\big)$ and $G_2 = G_1 \circ \Psi_1$. Then $\Re G_2^{\#}(1', 0'') \neq 0$; in fact it is close to M_0, provided that ε and δ are sufficiently small. This procedure assures that $\mathcal{Y}$ possesses a function G_2 whose polyradialization is nontrivial.

There are $c \in T_k$ with the property $|G_2^{\#}(c)| = \|G_2^{\#}\|_{T_k} = M$. Again without loss of generality, $|c_1| = \cdots = |c_k| = 1$ and $|c_j| < 1$ for $j = k+1, \dots, n$. Pick c so that $\max\{1 - |c_j| : k+1 \leq j \leq n\}$ is minimal. Let $\Psi_2(z) = \big(z', \varphi_{c''}(z'')\big)$ and $G_3 = R_{n-k}(G_2^{\#} \circ \Psi_2)$, where R_{n-k} denotes radialization in the last $n-k$ variables. Since $G_2^{\#}$ is radial in the first k variables and Ψ_2 does not change them, we can also write $G_3 = (G_2^{\#} \circ \Psi_2)^{\#}$. We have $|G_3(\zeta', 0'')| = |G_3(c', 0'')| = |G_2^{\#}(c)| = M$ and $|G_3(\zeta', z'')| < M$ if $z'' \neq 0$ due to the action of R_{n-k}.

Next, to obtain $Q(0'')$ as the peak set of some polyradial G_4, choose $d \in T_{n-k}$ such that $1 > |d_1| = \cdots = |d_k| > \max\{|c_j| : k+1 \leq j \leq n\}$. By the definition of c, $|G_2^{\#}(d)| < M$. Let $\Psi_3(z) = \big(\varphi_{d'}(z'), z''\big)$ and $G_4 = (G_3 \circ \Psi_3)^{\#}$. Then $|G_4(\zeta', 0'')| = |G_3(\zeta', 0'')| = M$ and $|G_4(\zeta', z'')| = |G_3(\zeta', z'')| < M$ if $z'' \neq 0$. Also for any $\sigma \in \mathcal{S}_n$, $|G_4(\zeta_{\sigma(1)}, \dots, \zeta_{\sigma(k)}, z_{\sigma(k+1)}, \dots, z_{\sigma(n)})| < M$ if $z'' \neq 0$, by the way d is chosen and by radialization, as explained in the last paragraph of the proof of Lemma 3.1. Finally let $\Psi_4(z) = \big(z', \varphi_{q''}(z'')\big)$ and $G = \frac{1}{M}(G_4 \circ \Psi_4)$. Then $G \in \mathcal{Y}$, $G|_Q = 1$, and $|G| < 1$ elsewhere on T_k.

For any $\ell \in \mathbb{N}$, $G^\ell F_m \in \mathcal{Y}$ is an extension of f_m to T_k. Take a subsequence $\{f_{m_j}\}$ for which $|f_{m_{j+1}} - f_{m_j}| = |F_{m_{j+1}} - F_{m_j}| < 2^{-j-1}$ on Q. Also, for each j, take ℓ_j large enough so that $|F_{m_{j+1}} - F_{m_j}||G^{\ell_j}| \leq 2^{-j}$ on T_k. Such an ℓ_j exists, because $|F_{m_{j+1}} - F_{m_j}| < 2^{-j}$ in a neighborhood of Q in T_k depending on j, and $|G| < \delta_j$ for some $\delta_j < 1$ outside this neighborhood in T_k. Now we define $F = F_{m_1} + \sum_{j=1}^{\infty} G^{\ell_j}(F_{m_{j+1}} - F_{m_j})$. Then $F \in \mathcal{Y}$ since the series is uniformly convergent on T_k; and $F|_Q = \lim_{j\to\infty} F_{m_j}\big|_Q = \lim_{j\to\infty} f_{m_j} = f$. This proves that $\mathcal{Z}$ is closed. Hence $\mathcal{Z}$ is one of the nontrivial $\mathcal{M}$-algebras of $\mathcal{C}(\mathbb{T}^k)$, as listed in (B)–(E) of Remark 4.1.

A given Q is only one "slice" of T_k, but the same analysis can be done on any Q given by some $(q_{\sigma(k+1)}, \dots, q_{\sigma(n)}) \in \mathbb{T}^k$ and $\sigma \in \mathcal{S}_n$. Since $\mathcal{Y}$ is $\mathcal{M}$-invariant, its restriction to each Q is the same. Therefore $\mathcal{Y}$ must be one of the five algebras referred to in the statement of the lemma. □

Corollary 4.5. *$\mathcal{M}$-algebras of continuous functions on the topological boundary of the unit bidisc which restrict to $\{0\}$ on the distinguished boundary are:*

(i) $\{0\}$,
(ii) *functions that are constant on each circle Q_1 and Q_2,*
(iii) *functions with no negative Fourier coefficients on Q_1 and Q_2,*
(iv) *functions with no positive Fourier coefficients on Q_1 and Q_2,*
(v) $\{f \in \mathcal{C}(\partial\mathbb{U}^2) : f|_{\mathbb{T}^2} = 0\}$.

Q_1 and Q_2 are defined in (2.5). Of course the constant mentioned in (ii) may vary from one circle to another. If we assume $\mathcal{Y}$ to be only $\mathcal{M}^*$-invariant, then $\mathcal{Y}$ can restrict differently on permuted copies of Q. For example, in the special case mentioned in Corollary 4.5, it is possible to have $\mathcal{Y}|_{Q_1} = \mathcal{A}(\mathbb{T})$ and $\mathcal{Y}|_{Q_2} = \operatorname{conj}\mathcal{A}(\mathbb{T})$.

Proof of Theorem C. Let $F \in \mathcal{X}$, $f = F|_{T_k}$, and for $z \in \overline{\mathbb{U}}^n$, put

$$G(z) = F(z) - \sum P_k[f](z), \tag{4.2}$$

where the summation runs over all the $\binom{n}{k}$ distinct k-element subsets of $\{1,\dots,n\}$ and the partial Poisson integral P_k is taken each time over those k variables thus indicated. If $w \in \overline{\mathbb{U}}^n$ has k components of unit length, say $(w_1,\dots,w_k)$, then $P_k[f](w) = f(w) = 0$ if P_k involves any of the last $n-k$ variables, because then in the integral at least $k+1$ arguments of f have length 1 and $\mathcal{X} \subset \mathcal{E}_{k+1}(\mathbb{U}^n)$. So $G(w) = F(w) - P_k[f](w)$ with P_k operating on the first k variables. But then $F(w) = P_k[f](w)$ by the choice of w. Hence $G = 0$ whenever any k of its arguments are of unit length, i.e., $G \in \mathcal{E}_k(\mathbb{U}^n)$.

Put $\mathcal{Y} = \mathcal{X}|_{T_k}$ and consider a sequence $\{f_m\} \subset \mathcal{Y}$ uniformly convergent to f on T_k. Each $f_m = F_m|_{T_k}$ for some $F_m \in \mathcal{X}$. We can find $G_m \in \mathcal{E}_k(\mathbb{U}^n) \subset \mathcal{X}$ so that $F_m = \sum P_k[f_m] + G_m$. Then $\sum P_k[f_m] \in \mathcal{X}$ as well. Define $Q(q'')$ similar to (4.1) (but with varying positions for the k components of unit length) and compute:

$$\begin{aligned}
\binom{n}{k}\sup_{T_k}|f_m - f_\ell| &= \binom{n}{k}\max \sup_{q''\in\overline{\mathbb{U}}^{n-k}}\ \sup_{Q(q'')}\big|P_k[f_m - f_\ell]\big| \\
&\ge \sum \sup_{q''\in\overline{\mathbb{U}}^{n-k}}\ \sup_{Q(q'')}\big|P_k[f_m - f_\ell]\big| \\
&= \sum \sup_{\overline{\mathbb{U}}^n}\big|P_k[f_m - f_\ell]\big| \ge \sup_{\overline{\mathbb{U}}^n}\sum\big|P_k[f_m - f_\ell]\big| \\
&= \sup_{\overline{\mathbb{U}}^n}|F_m - G_m - (F_\ell - G_\ell)|,
\end{aligned}$$

since P_k attains its supremum on $\mathbb{T}^k$. Above, max and $\sum$ run over the same set as in (4.2). Thus $\{F_m - G_m\}$ is uniformly convergent on $\overline{\mathbb{U}}^n$ to a function $F \in \mathcal{X}$. Since $G_m \in \mathcal{E}_k(\mathbb{U}^n)$, $(F_m - G_m)\big|_{T_k} = f_m \to F\big|_{T_k} = f$. This shows that $\mathcal{Y}$ is closed. $\mathcal{Y}$ is also $\mathcal{M}$-invariant since T_k is, and satisfies $\mathcal{Y}\big|_{T_{k+1}} = \{0\}$ since $\mathcal{X}$ does.

The definition of G shows that

$$\mathcal{X} = \mathcal{E}_k(\mathbb{U}^n) + \sum P_k[\mathcal{Y}].$$

Hence $\mathcal{X}$ must be the sum of $\mathcal{E}_k(\mathbb{U}^n)$ and the k-partial Poisson integrals of one of the five algebras obtained in Lemma 4.4. (i) and (v) of that lemma supply $\mathcal{E}_k(\mathbb{U}^n)$ and $\mathcal{E}_{k+1}(\mathbb{U}^n)$. The Poisson integrals of the remaining three provide the other three algebras referred to in the statement of the theorem. Since each of these contain $\mathcal{E}_k(\mathbb{U}^n)$, we get the desired result. □

Corollary 4.6. *$\mathcal{M}$-algebras of $\mathcal{C}(\overline{\mathbb{U}}^2)$ containing $\mathcal{C}_0(\mathbb{U}^2)$ that restrict to $\{0\}$ on $\mathbb{T}^2$ are*

(i) $\mathcal{C}_0(\mathbb{U}^2)$,
(ii) *functions that are constant on each circle Q_1 and Q_2,*
(iii) *functions with no negative Fourier coefficients on Q_1 and Q_2,*
(iv) *functions with no positive Fourier coefficients on Q_1 and Q_2,*
(v) $\mathcal{E}_2(\mathbb{U}^2)$.

Some Other $\mathcal{M}$-Algebras of $\mathcal{C}(\overline{\mathbb{U}}^n)$. We now investigate what happens when we intersect an $\mathcal{M}$-space with $\mathcal{E}_n(\mathbb{U}^n)$. The possible intersections are $\{0\}$, $\mathcal{E}_k(\mathbb{U}^n)$, $\mathcal{K}_k(\mathbb{U}^n)$, $\mathcal{A}_k(\mathbb{U}^n)$, and $\operatorname{conj}\mathcal{A}_k(\mathbb{U}^n)$, for $k = 1, \dots, n-1$. Theorem A can be visualized as the special case when the intersection is $\mathcal{E}_n(\mathbb{U}^n)$.

Proof of Theorem D. We may assume $\mathcal{X} \neq \{0\}$. Define linear functionals L and M on $\mathcal{X}$ by

$$Lf = f(0) \quad \text{and} \quad Mf = \int_{\mathbb{T}^n} f\, d\lambda_n.$$

Suppose there is an $f \in \mathcal{X}$ with $Mf = 0$ but $Lf \neq 0$. Then $f^\#(0) = Lf \neq 0$; so $f^\# \not\equiv 0$, and $f^\#\big|_{\mathbb{T}^n} = Mf^\# = Mf = 0$. But this implies $f^\# \in \mathcal{X} \cap \mathcal{E}_n(\mathbb{U}^n)$, a contradiction. Thus the null space of L contains the null space of M. Since $\mathcal{X} \neq \{0\}$, by the $\mathcal{M}$-invariance of $\mathcal{X}$, L, and hence also M, is clearly onto $\mathbb{C}$. Then by a factorization argument, there exists a linear $c : \mathbb{C} \to \mathbb{C}$ such that $L = c \circ M$. In other words, for some $c \in \mathbb{C}$,

$$f(0) = c \int_{\mathbb{T}^n} f\, d\lambda_n \qquad (f \in \mathcal{X}). \tag{4.3}$$

Take a $z \in \mathbb{U}^n$, pick $\Psi \in \mathcal{M}$ with $\Psi(0) = z$, and apply (4.3) to $f \circ \Psi \in \mathcal{X}$:

$$f(z) = (f \circ \Psi)(0) = c \int_{\mathbb{T}^n} (f \circ \Psi)\, d\lambda_n$$
$$= cP[f \circ \Psi](0) = cP[f]\big(\Psi(0)\big) = cP[f](z),$$

using (1.5). Letting $z \to \zeta \in \mathbb{T}^n$ gives $f(\zeta) = cf(\zeta)$; so $c = 1$. Therefore, every $f \in \mathcal{X}$ is the Poisson integral of its restriction to $\mathbb{T}^n$. Consequently $\sup_{\overline{\mathbb{U}}^n} |f| = \sup_{\mathbb{T}^n} |f|$.

Let $\mathcal{Y} = \mathcal{X}\big|_{\mathbb{T}^n}$, and take a sequence $\{f_m\}$ in $\mathcal{Y}$ such that $f_m \to f$ uniformly on $\mathbb{T}^n$. Each $f_m = F_m\big|_{\mathbb{T}^n}$ for some $F_m \in X$ and $F_m = P[f_m]$. By the equality of the suprema of $f_m - f_k$ and $F_m - F_k$, $F_m \to F$ uniformly on $\overline{\mathbb{U}}^n$ and $F \in \mathcal{X}$. Hence $f = F\big|_{\mathbb{T}^n}$, $f \in \mathcal{Y}$, and $\mathcal{Y}$ is closed. $\mathcal{Y}$ is clearly $\mathcal{M}$-invariant. Thus $\mathcal{Y}$ is one of the $\mathcal{M}$-spaces of $\mathcal{C}(\mathbb{T}^n)$. It follows that $\mathcal{X}$ consists of the Poisson integrals of functions in one of these spaces. □

Corollary 4.7. *$\mathcal{M}$-spaces of $\mathcal{C}(\overline{\mathbb{U}}^2)$ intersecting $\mathcal{E}_2(\mathbb{U}^2)$ in $\{0\}$ are:*

(i) $\{0\}$, $\mathbb{C}$, $\mathcal{Z}(\mathbb{U}^2)$, $\mathrm{plh}(\mathbb{U}^2)$, $\mathcal{P}_z(\mathbb{U}) + \mathcal{P}_w(\mathbb{U})$, $\mathcal{P}(\mathbb{U}^2)$,
(ii) $\mathcal{A}_z(\mathbb{U}) + \mathcal{A}_w(\mathbb{U})$, $\mathcal{A}(\mathbb{U}^2)$, $\mathcal{A}(\mathbb{U}^2) + \mathcal{P}_z(\mathbb{U}) + \mathcal{P}_w(\mathbb{U})$, $\mathrm{cl}\big(\mathcal{Z}(\mathbb{U}^2) + \mathcal{A}(\mathbb{U}^2)\big)$,
(iii) *conjugates of the spaces in* (ii),

where (z,w) *and* $\mathcal{P}_z(\mathbb{U})$ *are as in Corollary 4.2.*

Corollary 4.8. *There are exactly four $\mathcal{M}$-algebras of $\mathcal{C}(\overline{\mathbb{U}}^n)$ which intersect in $\{0\}$ with $\mathcal{E}_n(\mathbb{U}^n)$:*

(i) $\{0\}$,
(ii) $\mathbb{C}$,
(iii) $\mathcal{A}(\mathbb{U}^n)$,
(iv) $\mathrm{conj}\,\mathcal{A}(\mathbb{U}^n)$.

Proof. $\mathcal{P}(\mathbb{U}^n)$ is not an algebra although it is obtained from an algebra, $\mathcal{C}(\mathbb{T}^n)$. □

Proof of Theorem E. Define on $\mathcal{X}$ the linear functionals

$$LF = \int_{\mathbb{T}^k} F(\zeta', 0'')\, d\lambda_k(\zeta') \quad \text{and} \quad MF = \int_{\mathbb{T}^n} F\, d\lambda_n.$$

If there is an $F \in \mathcal{X}$ with $MF = 0$ and $LF \neq 0$, then $F^\#\big|_{\mathbb{T}^n} = MF = 0$, so $F^\# \in \mathcal{E}_n(\mathbb{U}^n)$. Since also $F^\# \in \mathcal{X}$, we conclude $F^\# \in \mathcal{E}_k(\mathbb{U}^n)$ by hypothesis. But

$F^{\#}(\zeta',0'') = LF \neq 0$ contradicts this. Hence the null space of M is contained in the null space of L. Consequently there is a $c \in \mathbb{C}$ such that $LF = cMF$, i.e.,

$$\int_{\mathbb{T}^k} F(\zeta',0'')\,d\lambda_k(\zeta') = c\int_{\mathbb{T}^n} F\,d\lambda_n \qquad (F \in \mathcal{X}). \tag{4.4}$$

If $z'' \in \mathbb{U}^{n-k}$, choose $\Psi \in \mathcal{M}$ so as to have $\Psi(\zeta',0'') = (\zeta',z'')$ and let $G = F \circ \Psi$. Applying (4.4) to G in place of F, we get

$$\begin{aligned}\int_{\mathbb{T}^k} F(\zeta',z'')\,d\lambda_k(\zeta') &= \int_{\mathbb{T}^k} G(\zeta',0'')\,d\lambda_k(\zeta') = c\int_{\mathbb{T}^n} G\,d\lambda_n \\ &= cP[G](0) = cP[F](0',z'').\end{aligned}$$

Letting $z'' \to \zeta'' \in \mathbb{T}^{n-k}$, we obtain

$$\int_{\mathbb{T}^k} F(\zeta',\zeta'')\,d\lambda_k(\zeta') = c\int_{\mathbb{T}^k} F(\zeta',\zeta'')\,d\lambda_k(\zeta'),$$

which shows that $c = 1$. Therefore

$$\begin{aligned}\int_{\mathbb{T}^k} F(\zeta',z'')\,d\lambda_k(\zeta') &= \int_{\mathbb{T}^n} F(\zeta',\zeta'')\prod_{j=k+1}^{n}\frac{1-|z_j|^2}{|1-z_j\bar{\zeta}_j|^2}\,d\lambda_n(\zeta) \\ &= \int_{\mathbb{T}^k} P_{n-k}[F](\zeta',z'')\,d\lambda_k(\zeta')\end{aligned}$$

after integrating in the last $n-k$ variables. Here P_{n-k} is the partial Poisson integral in the last $n-k$ variables. Written in a different form, the above equalities state

$$\int_{\mathbb{T}^k} (F - P_{n-k}[F])(\zeta',z'')\,d\lambda_k(\zeta') = 0 \qquad (F \in \mathcal{X},\ z'' \in \mathbb{U}^{n-k}).$$

Since $\mathcal{X}$ is $\mathcal{M}$-invariant, in particular for any $\Psi \in \mathcal{M}^*$, we also get

$$\int_{\mathbb{T}^k} \big(F\circ\Psi - P_{n-k}[F\circ\Psi]\big)(\zeta',z'')\,d\lambda_k(\zeta') = 0 \qquad (F \in \mathcal{X},\ z'' \in \mathbb{U}^{n-k}).$$

Denoting $F - P_{n-k}[F]$ by G and using the $\mathcal{M}^*$-invariance of the partial Poisson integral, this last equation can also be written as

$$\int_{\mathbb{T}^k} (G\circ\Psi)(\zeta',z'')\,d\lambda_k(\zeta') = 0. \tag{4.5}$$

We want to conclude that G is identically 0 for each $z'' \in \mathbb{U}^{n-k}$. Fix such a z'' and suppose $\Re G(\zeta',z'') \neq 0$. Put $m = \max\{\Re G(\zeta',z'') : \zeta' \in \mathbb{T}^k\}$. Then, as in the proof of Lemma 4.4, given $\varepsilon > 0$, for an appropriate $\Psi \in \mathcal{M}^*$, we can have $\Re(G\circ\Psi) > m - \varepsilon$ on a large part of $\mathbb{T}^k$. Then the integral in (4.5) cannot be equal to 0. This contradiction proves that

$$\begin{aligned}F(\zeta',z'') &= P_{n-k}[F](\zeta',z'') \\ &= P[F](\zeta',z'')\end{aligned} \qquad (F \in \mathcal{X},\ \zeta' \in \mathbb{T}^k,\ z'' \in \mathbb{U}^{n-k}).$$

The second equality holds by extending the Poisson integral to be equal to F on T_k. Fix $p' \in \mathbb{T}^k$ and let

$$P = P(p') = \{(p', z'') : z'' \in \overline{\mathbb{U}}^{n-k}\}.$$

P is a copy of the closed polydisc $\overline{\mathbb{U}}^{n-k}$ in T_k. Then by the $\mathcal{M}$-invariance of $\mathcal{X}$, on every $P(p')$ and on its permuted copies in T_k, every $F \in \mathcal{X}$ can be expressed as the (partial) Poisson integral of its restriction to (a subset of) $\mathbb{T}^n$.

For $F \in \mathcal{X}$, let $f = F\big|_{\mathbb{T}^n}$. Since

$$T_k = \bigcup_{p' \in \mathbb{T}^k} \bigcup_{\sigma \in \mathcal{S}_n} P(p_{\sigma(1)}, \dots, p_{\sigma(k)}),$$

$H = F - P[f] \in \mathcal{E}_k(\mathbb{U}^n) \subset \mathcal{X}$ and $P[f] = F - H \in \mathcal{X}$. If we let $\mathcal{Y} = \mathcal{X}\big|_{\mathbb{T}^n}$, using the equality of the maxima of f and $F - H$, we see that $\mathcal{Y}$ is an $\mathcal{M}$-space of $\mathcal{C}(\mathbb{T}^n)$. Then as in Theorem A, $\mathcal{X}$ consists of the Poisson integrals of members of these plus the functions in $\mathcal{E}_k(\mathbb{U}^n)$. □

Corollary 4.9. *$\mathcal{M}$-spaces of $\mathcal{C}(\overline{\mathbb{U}}^2)$ intersecting $\mathcal{E}_2(\mathbb{U}^2)$ in $\mathcal{C}_0(\mathbb{U}^2)$ are:*

(i) $\mathcal{C}_0(\mathbb{U}^2)$, $\mathcal{C}_0(\mathbb{U}^2) + \mathbb{C}$, $\mathcal{C}_0(\mathbb{U}^2) + \mathcal{Z}(\mathbb{U}^2)$, $\mathcal{C}_0(\mathbb{U}^2) + \mathrm{plh}(\mathbb{U}^2)$, $\mathcal{C}_0(\mathbb{U}^2) + \mathcal{P}_z(\mathbb{U}) + \mathcal{P}_w(\mathbb{U})$, $\mathcal{C}_0(\mathbb{U}^2) + \mathcal{P}(\mathbb{U})$,
(ii) $\mathcal{C}_0(\mathbb{U}^2) + \mathcal{A}_z(\mathbb{U}) + \mathcal{A}_w(\mathbb{U})$, $\mathcal{C}_0(\mathbb{U}^2) + \mathcal{A}(\mathbb{U}^2)$, $\mathcal{C}_0(\mathbb{U}^2) + \mathcal{A}(\mathbb{U}^2) + \mathcal{P}_z(\mathbb{U}) + \mathcal{P}_w(\mathbb{U})$, $\mathcal{C}_0(\mathbb{U}^2) + \mathrm{cl}\big(\mathcal{Z}(\mathbb{U}^2) + \mathcal{A}(\mathbb{U}^2)\big)$,
(iii) *conjugates of the spaces in* (ii),

where the notation is as in Corollary 4.2.

Corollary 4.10. *There are only four $\mathcal{M}$-algebras of $\mathcal{C}(\overline{\mathbb{U}}^n)$ that have an intersection of $\mathcal{E}_k(\mathbb{U}^n)$ with $\mathcal{E}_n(\mathbb{U}^n)$:*

(i) $\mathcal{E}_k(\mathbb{U}^n)$,
(ii) $\mathcal{E}_k(\mathbb{U}^n) + \mathbb{C}$,
(iii) $\mathcal{E}_k(\mathbb{U}^n) + \mathcal{A}(\mathbb{U}^n)$,
(iv) $\mathcal{E}_k(\mathbb{U}^n) + \mathrm{conj}\,\mathcal{A}(\mathbb{U}^n)$.

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