Name: Grade: /10

Math 102, Calculus II, Spring 2024, Sec. 3 & 13, HTK
Quiz 3, Tue., Mar. 12
Show all your work and name any tests you use.

1. Find the interval and radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{9^n + (-7)^n}{n+1} x^{2n}$.

2. (extra credit) Call the sum of the series $f(x)$ on its interval of convergence and evaluate $f(1/4)$.

1. Find the interval and radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{4^n + (-3)^n}{n+1} x^{2n}$.

2. (extra credit) Call the sum of the series $g(x)$ on its interval of convergence and evaluate $g(1/3)$.

1. Find the interval and radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{9^n + (-5)^n}{n+1} x^{2n+1}$.

2. (extra credit) Call the sum of the series $F(x)$ on its interval of convergence and evaluate $F(1/5)$.

1. Find the interval and radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{4^n + (-1)^n}{n+1} x^{2n+1}$.

2. (extra credit) Call the sum of the series $G(x)$ on its interval of convergence and evaluate $G(1/4)$.

1.) $\lim_{n\to\infty} \left| \frac{9^{n+1} + (-7)^{n+1}}{n+2} \frac{n+1}{9^n + (-7)^n} \frac{x^{2n+2}}{x^{2n}} \right| = |x|^2 \lim_{n\to\infty} \frac{9^{n+1} + (-7)^{n+1}}{9^n + (-7)^n}$

$= x^2 \lim_{n\to\infty} \frac{9(9^n)\left(1 + \left(-\frac{7}{9}\right)^{n+1}\right)}{9^n\left(1 + \left(-\frac{7}{9}\right)^n\right)} = 9x^2 < 1, \quad x^2 < \frac{1}{9}, \ |x| < \frac{1}{3}$ for abs.

convergence by RT. End points: $x = \pm\frac{1}{3}, \ x^2 = \frac{1}{9}, \ x^{2n} = \frac{1}{9^n}$.

The series is $\sum_{n=0}^{\infty} \frac{9^n + (-7)^n}{n+1} \frac{1}{9^n} = \sum_{n=0}^{\infty} \frac{1 + \left(-\frac{7}{9}\right)^n}{n+1} = \sum_{n=0}^{\infty} b_n$. $b_n > 0$.

$\lim_{n\to\infty} \frac{b_n}{1/n} = \lim_{n\to\infty} \left(1 + \left(-\frac{7}{9}\right)^n\right) \frac{n}{n+1} = 1, \quad 0 < 1 < \infty$. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

so $\sum_{n=0}^{\infty} b_n$ diverges by LCT. Interval of convergence $= \left(-\frac{1}{3}, \frac{1}{3}\right)$, $R = \frac{1}{3}$.

2.) $f\left(\frac{1}{4}\right) = \sum_{n=0}^{\infty} \frac{9^n + (-7)^n}{n+1} \frac{1}{16^n} = \sum_{n=0}^{\infty} \frac{\left(\frac{9}{16}\right)^n}{n+1} + \sum_{n=0}^{\infty} \frac{\left(-\frac{7}{16}\right)^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n d^n}{n+1}$

where $c = -\frac{9}{16}$ and $d = \frac{7}{16}$. $\sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n+1} = \frac{1}{c} \sum_{n=0}^{\infty} \frac{(-1)^n c^{n+1}}{n+1} =$

$= \frac{1}{c} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} c^k}{k} = \frac{1}{c} \ln(1+c) = -\frac{16}{9} \ln \frac{7}{16} = A$. Similarly,

$\sum_{n=0}^{\infty} \frac{(-1)^n d^n}{n+1} = \frac{1}{d} \ln(1+d) = \frac{16}{7} \ln\left(\frac{23}{16}\right) = B$. Thus $f\left(\frac{1}{4}\right) = A + B$.

1-) $\sum_{n=0}^{\infty} \frac{4^n+(-3)^n}{n+1}$ converges absolutely for $|x|<\frac{1}{2}$ by RT similarly.
At the endpoints, $x=\pm\frac{1}{2}$, $x^2=\frac{1}{4}$, $x^{2n}=\frac{1}{4^n}$, the series is
$\sum_{n=0}^{\infty} \frac{1+\left(-\frac{3}{4}\right)^n}{n+1}$, and it diverges. Intervall of conv. $=\left(-\frac{1}{2},\frac{1}{2}\right)$, $R=\frac{1}{2}$.

2-) Similarly, $g\left(\frac{1}{}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n c^n}{n+1}+\sum_{n=0}^{\infty}\frac{(-1)^n d^n}{n+1}$ with $c=-\frac{4}{9}$ and
$d=\frac{3}{9}=\frac{1}{3}$. Again $g\left(\frac{1}{4}\right)=\frac{1}{c}\ln(1+c)+\frac{1}{d}\ln(1+d)=$
$=-\frac{9}{4}\ln\left(\frac{5}{9}\right)+3\ln\left(\frac{4}{3}\right)$.

1-) Similarly, $\sum_{n=0}^{\infty}\frac{9^n+(-5)^n}{n+1}x^{2n+1}$ converges absolutely for $|x|<\frac{1}{3}$ by RT.
At the endpoints, $x=\pm\frac{1}{3}$, $x^{2n}=\frac{1}{9^n}$, $x^{2n+1}=\pm\frac{1}{3}\frac{1}{9^n}$, and the series is
$\pm\frac{1}{3}\sum_{n=0}^{\infty}\frac{1+\left(-\frac{5}{9}\right)^n}{n+1}=\pm\frac{1}{3}\sum_{n=0}^{\infty}b_n$. $\sum_{n=0}^{\infty}b_n$ diverges by LCT similarly.
So interval of conv. $=\left(-\frac{1}{3},\frac{1}{3}\right)$ and $R=\frac{1}{3}$.

2-) Similarly, $F\left(\frac{1}{5}\right)=\frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n c^n}{n+1}+\frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n d^n}{n+1}$ with $c=-\frac{9}{25}$ and
$d=\frac{5}{25}=\frac{1}{5}$. So $F\left(\frac{1}{5}\right)=\frac{1}{5}\left(\frac{1}{c}\ln(1+c)+\frac{1}{d}\ln(1+d)\right)=$
$=-\frac{5}{9}\ln\left(\frac{16}{25}\right)+\ln\left(\frac{6}{5}\right)$.

1-) Similarly, $\sum_{n=0}^{\infty}\frac{4^n+(-1)^n}{n+1}x^{2n+1}$ converges absolutely for $|x|<\frac{1}{2}$ by RT.
At the endpoints, $x=\pm\frac{1}{2}$, $x^{2n}=\frac{1}{4^n}$, $x^{2n+1}=\pm\frac{1}{2}\frac{1}{4^n}$, and the series is
$\pm\frac{1}{2}\sum_{n=0}^{\infty}\frac{1+\left(-\frac{1}{4}\right)^n}{n+1}=\pm\frac{1}{2}\sum_{n=0}^{\infty}b_n$, $\sum_{n=0}^{\infty}b_n$ diverges by LCT similarly.
So interval of conv. $=\left(-\frac{1}{2},\frac{1}{2}\right)$ and $R=\frac{1}{2}$.

2-) Similarly, $G\left(\frac{1}{4}\right)=\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n c^n}{n+1}+\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n d^n}{n+1}$ with $c=-\frac{4}{16}=-\frac{1}{4}$
and $d=\frac{1}{16}$. So $G\left(\frac{1}{4}\right)=\frac{1}{4}\left(\frac{1}{c}\ln(1+c)+\frac{1}{d}\ln(1+d)\right)=$
$=-\ln\left(\frac{3}{4}\right)+4\ln\left(\frac{17}{16}\right)$.