Name: Grade: /10

Math 102, Calculus II, Spring 2024, Sec. 3 & 13, HTK
Quiz 1, Tue., Feb. 13
Show all your work and name any tests you use.

1. Show that the sequence defined by $a_1 = \frac{3}{2}$ and $a_{n+1} = 3 - \frac{2}{a_n}$ is increasing and bounded above by 2. Deduce that $\{a_n\}$ converges and find its limit.

Show that the sequence defined by $a_1 = \frac{3}{2}$ and $a_{n+1} = \frac{2}{3 - a_n}$ is decreasing and bounded below by 1. Deduce that $\{a_n\}$ converges and find its limit.

Show that the sequence defined by $a_1 = 2$ and $a_{n+1} = 4 - \frac{3}{a_n}$ is increasing and bounded above by 3. Deduce that $\{a_n\}$ converges and find its limit.

Show that the sequence defined by $a_1 = 2$ and $a_{n+1} = \frac{3}{4 - a_n}$ is decreasing and bounded below by 1. Deduce that $\{a_n\}$ converges and find its limit.

2. Find the exact value of the sum $\displaystyle\sum_{n=1}^{\infty} \frac{(-9)^{n-3}}{10^n}$.

Find the exact value of the sum $\displaystyle\sum_{n=1}^{\infty} \frac{(-7)^{n-5}}{8^n}$.

Find the exact value of the sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$.

Find the exact value of the sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$.

1-) $a_1 = \frac{3}{2}$, $a_2 = 3 - \frac{2}{3/2} = \frac{5}{3} > a_1$, so $a_2 > a_1$.

If $a_n > a_{n-1}$, then $\frac{1}{a_n} < \frac{1}{a_{n-1}}$, $-\frac{1}{a_n} > -\frac{1}{a_{n-1}}$, $3 - \frac{2}{a_n} > 3 - \frac{2}{a_{n-1}}$,

so $a_{n+1} > a_n$. By induction $\{a_n\}$ is increasing.

$a_1 = \frac{3}{2} < 2$. If $a_n < 2$, then $\frac{1}{a_n} > \frac{1}{2}$, $-\frac{1}{a_n} < -\frac{1}{2}$,

$3 - \frac{2}{a_n} < 3 - \frac{2}{2} = 2$, so $a_{n+1} < 2$. By induction, $\{a_n\}$ is bounded above

By MST, $\lim_{n\to\infty} a_n = L$ exists.

$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(3 - \frac{2}{a_n}\right) = 3 - \frac{2}{L}$, $L^2 = 3L - 2$, $L^2 - 3L + 2 = 0$,

$(L-2)(L-1) = 0$, $L = 2$ or $L = 1$. But $a_n \geq \frac{3}{2}$ for all n.

Then $L = 2$.

$a_1 = \frac{3}{2}$, $a_2 = \frac{2}{3 - 3/2} = \frac{4}{3}$, so $a_2 < a_1$.

If $a_n < a_{n-1}$, then $3 - a_n > 3 - a_{n-1}$, $\frac{2}{3-a_n} < \frac{2}{3-a_{n-1}}$, so $a_{n+1} < a_n$.

By induction, $\{a_n\}$ is decreasing.

$a_1 = \frac{3}{2} > 1$. If $a_n > 1$, then $3 - a_n < 2$, $\frac{2}{3-a_n} > 1$, so $a_{n+1} > 1$.

By induction, $\{a_n\}$ is bounded below. By MST, $\lim_{n\to\infty} a_n = L$ exists.

$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{2}{3-a_n} = \frac{2}{3-L}$, $3L - L^2 = 2$, $L^2 - 3L + 2 = 0$

$(L-2)(L-1) = 0$, $L = 2$ or $L = 1$. But $a_n \leq \frac{3}{2}$ for all n. Then $L = 1$.

$a_1 = 2$, $a_2 = 4 - \frac{3}{2} = 3$, so $a_2 > a_1$.

If $a_n > a_{n-1}$, then $\frac{3}{a_n} < \frac{3}{a_{n-1}}$, $4 - \frac{3}{a_n} > 4 - \frac{3}{a_{n-1}}$, so $a_{n+1} > a_n$.

By induction, $\{a_n\}$ is increasing.

$a_1 = 2 < 3$. If $a_n < 3$, then $\frac{3}{a_n} > 1$, $4 - \frac{3}{a_n} < 3$, so $a_{n+1} < 3$.

By induction, $\{a_n\}$ is bounded above. By MST, $\lim_{n\to\infty} a_n = L$ exists.

$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(4 - \frac{3}{a_n}\right) = 4 - \frac{3}{L}$, $L^2 = 4L - 3$, $L^2 - 4L + 3 = 0$,

$(L-3)(L-1) = 0$, $L = 3$ or $L = 1$. But $a_n \geq 2$ for all n. So $L = 3$.

$a_1 = 2$, $a_2 = \frac{3}{4-2} = \frac{3}{2}$, so $a_2 < a_1$.

If $a_n < a_{n-1}$, then $4 - a_n > 4 - a_{n-1}$, $\frac{3}{4-a_n} < \frac{3}{4-a_{n-1}}$, $a_{n+1} < a_n$.

By induction, $\{a_n\}$ is decreasing.

$a_1 = 2 > 1$. If $a_n > 1$, then $4 - a_n < 3$, $\frac{3}{4-a_n} > 1$, so $a_{n+1} > 1$.

By induction, $\{a_n\}$ is bounded below. MST, $\lim_{n\to\infty} a_n = L$ exists.

$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{3}{4-a_n} = \frac{3}{4-L}$, $4L - L^2 = 3$, $L^2 - 4L + 3 = 0$,

$(L-3)(L-1) = 0$, $L = 3$ or $L = 1$. But $a_n \leq 2$ for all n. So $L = 1$.

2-) $\sum_{n=1}^{\infty} \frac{(-9)^{n-3}}{10^n}$ is a geometric series with first term $a = \frac{(-9)^{-2}}{10} =$
$= \frac{1}{810}$ and common ratio $r = -\frac{9}{10}$. Since $|r|<1$, the series converges and has sum $s = \frac{a}{1-r} = \frac{1}{810} \frac{1}{1-(-9/10)} = \frac{1}{(81)(19)}$.

$\sum_{n=1}^{\infty} \frac{(-7)^{n-5}}{8^n}$ is a geometric series with first term $a = \frac{(-7)^4}{8} =$
$= \frac{1}{(49)^2(8)}$ and common ratio $r = -\frac{7}{8}$. Since $|r|<1$, the series converges and has sum $s = \frac{a}{1-r} = \frac{1}{(49)^2(8)} \frac{1}{1-(-7/8)} = \frac{1}{(49)^2(15)}$.

$$\frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1} = \frac{A(2n+1)+B(2n-1)}{(2n-1)(2n+1)}. \qquad \begin{array}{l} A+B=0, \\ A-B=1. \end{array}$$

$A = 1/2$, $B = -1/2$.

$$s_n = \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \sum_{k=1}^{n} \left(\frac{1/2}{2n-1} - \frac{1/2}{2n+1} \right) =$$

$$= \frac{1}{2}\left(1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cdots + \cancel{\frac{1}{2n-1}} \quad \frac{1}{2n+1}\right) = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right).$$

$$s = \lim_{n\to\infty} s_n = \frac{1}{2}.$$

$$\frac{1}{(3n-2)(3n+1)} = \frac{A}{3n-2} + \frac{B}{3n+1} = \frac{A(3n+1)+B(3n-2)}{(3n-2)(3n+1)} \qquad \begin{array}{l} A+B=0, \\ A-2B=1. \end{array}$$

$A = 1/3$, $B = -1/3$.

$$s_n = \sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)} = \sum_{k=1}^{n} \left(\frac{1/3}{3k-2} - \frac{1/3}{3k+1} \right) =$$

$$= \frac{1}{3}\left(1 - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{7}} + \cdots + \cancel{\frac{1}{3k-2}} - \frac{1}{3k+1}\right) = \frac{1}{3}\left(1 - \frac{1}{3k+1}\right.$$

$$s = \lim_{n\to\infty} s_n = \frac{1}{3}.$$