Date: April 27, 2013
Time: 15:30-17:30
Instructor: Dilek Güvenç

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Dept. :..................

MATH 260 MIDTERM EXAM II

1	2	3	4	TOTAL
20	30	25	25	100

- Check that there are 4 questions in your booklet
- Do NOT use your mobile phone as a calculator. Turn it off during the exam.
- Write your name on each question page and on formula page.
- Do NOT forget to return the tables and the formula page, as well.

********* **GOOD LUCK** *********

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1. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Gamma distribution with parameters $\alpha = 3$ and β.

a) Find M.L.E of β. (12 Points)

b) Find M.L.E of the mean and variance of the distribution. (8 points)

a) $f(x,\beta) = \frac{1}{\Gamma(3)\beta^3} x^{3-1} e^{-\frac{x}{\beta}} \quad x>0$

$f(x,\beta) = \frac{1}{2\beta^3} x^2 e^{-\frac{x}{\beta}} \quad x>0$

$L_X(\beta) = \prod_{i=1}^{n} \frac{1}{2\beta^3} x_i^2 e^{-\frac{x_i}{\beta}} \qquad K_X(\beta) = \ln L_X(\beta) = \sum_{i=1}^{n} \ln \left[\frac{x_i^2}{2\beta^3} e^{-\frac{x_i}{\beta}}\right]$

$K_X(\beta) = \sum_{i=1}^{n} \left[\ln \frac{x_i^2}{2} - 3\ln\beta - \frac{x_i}{\beta} \right] = n \ln \frac{x_i^2}{2} - 3n\ln\beta - \frac{\sum x_i}{\beta}$

$\frac{dK_X(\beta)}{d\beta} = 0 \Rightarrow -\frac{3n}{\beta} + \frac{\sum x_i}{\beta^2} = 0 \Rightarrow -3n\beta + \sum x_i = 0$

$\Rightarrow$ M.L.E of β, $\hat{\beta} = \frac{\sum x_i}{3n} = \frac{\bar{X}}{3}$

b) $E(X) = 3\beta \Rightarrow$ M.L.E of $E(X)$, $\widehat{E(X)} = 3\hat{\beta} = 3 \cdot \frac{\bar{X}}{3} = \bar{X}$

$Var(X) = 3\beta^2 \Rightarrow$ M.L.E of $Var(X)$, $\widehat{Var(X)} = 3\hat{\beta}^2$

$= 3 \cdot \frac{\bar{X}^2}{9} = \frac{\bar{X}^2}{3}$

(M.L. Estimators are invariant under transformations of parameter)

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2. An aspirin manufacturer fills bottles by weight rather than by count. Since each bottle should contain 100 tablets, average weight per tablet should be 5 grains. Each of 100 tablets taken from a very large lot is weighed. Sample average weight was found to be 4.87 and sample standard deviation was 0.35 grain.

a) Does this information provide strong evidence for concluding that the company is not filling its bottles as advertised? Write appropriate null and alternative hypothesis and test at 1% significance level? (15 Points)

b) Calculate P-Value of the test in part (a) (5 Points)

c) What is the power of the test in part (a) if the true average mean weight is 4.85 grains? (10 Points)

$n = 100 \quad \bar{X} = 4.87 \quad s = 0.35$

a) $H_0: \mu = 5$

$H_A: \mu < 5$

Under H_0 $\quad Z_{cal} = \dfrac{4.87 - 5}{\frac{0.35}{\sqrt{100}}} = -3.71$

$\alpha = 0.01 \quad Z_{0.01} = 2.33$

Since $Z_{cal} = -3.71 < -Z_{0.01} = -2.33$ we reject H_0, conclude that $\mu < 5$ at 0.01.

b) P_Value $= P(Z < Z_{cal}) = P(Z < -3.71) \sim 0$

c) $\mu_A = 4.85$

Power of the test $= 1 - \beta = P(\text{Reject } H_0 \mid H_A \text{ is true})$

We reject H_0 if $\dfrac{\bar{X} - 5}{\frac{0.35}{\sqrt{100}}} < -Z_{0.01} = -2.33$

$\bar{X} < 5 - (2.33)(0.035) = 4.91$

$\Rightarrow 1 - \beta = P(\bar{X} < 4.91 \mid \mu_A = 4.85)$

$= P\left(Z < \dfrac{4.91 - 4.85}{\frac{0.35}{\sqrt{100}}}\right) = P(Z < 1.96)$

$= 0.5 + A(1.96)$

$= 0.5 + 0.475 = 0.975$

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3. The following data represent the running times of films produced by 2 motion-picture companies:

Company	Time (Minutes)						
1	102	86	98	109	92		
2	81	165	97	134	92	87	114

Assume that distributions of times are normal.

a) Test the equality of the variances for running times of the films produced by *Company* 1 and *Company* 2, at significance level of 5%. (10 Points)

b) Can we conclude that the average running time of films produced by *Company* 2 exceeds the average running time of films produced by *Company* 1 by 10 minutes? Use the result of part (a) for your assumption about variances and test at $\alpha = 0.05$. (15 Points)

$n_1 = 5$, $\bar{X}_1 = 97.4$ $\quad S_1^2 = 78.8$ $\quad S_1 = 8.8769$

$n_2 = 7$, $\bar{X}_2 = 110$ $\quad S_2^2 = 913.33$ $\quad S_2 = 30.2214$

a) $H_0: \frac{\sigma_2^2}{\sigma_1^2} = 1$ $\qquad$ Under H_0

$H_A: \frac{\sigma_2^2}{\sigma_1^2} > 1$ $\qquad F_{cal} = \frac{S_2^2}{S_1^2} = \frac{913.33}{78.8} = 11.59$

$F_{0.05}(6,4) = 6.16$

Since $F_{cal} = 11.59 > F_{0.05}(6,4) = 6.16$ we reject H_0, conclude that $\sigma_2^2 > \sigma_1^2$ (true variances are different) at $\alpha = 0.05$.

b) $H_0: \mu_2 - \mu_1 = 10$ $\qquad$ Variances are unequal

$H_A: \mu_2 - \mu_1 > 10$ $\qquad$ Under H_0,

$$T_{cal} = \frac{110 - 97.4 - 10}{\sqrt{\frac{78.8}{5} + \frac{913.33}{7}}} = \frac{2.6}{12.09} = 0.215$$

d.o.f $= 5 - 1 = 4$ (smaller) $\quad t_{0.05}(4) = 2.132$

Since $T_{cal} = 0.215 \not> t_{0.05}(4) = 2.132$ we do not reject H_0

Conclude that average running time (a.r.t) for company 2 may exceed a.r.t for company 1 by 10 minutes at $\alpha = 0.05$

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4. In a shop study, a set of data was collected to determine whether or not proportion of defectives produced by workers was the same for the day, evening, or night shift worked. Data were collected and shown in table below.

	Shift		
	Day	Evening	Night
Defectives	45	55	70
Sample size	950	945	940

a) Use 5 % level of significance to determine if the proportion of defectives is the same for all three shifts. (13 Points)

b) Can we conclude that proportion of defectives for the night shift is larger than the proportion of defectives for the day shift. Test at $\alpha = 0.05$ (12 Points)

a) $H_0: p_1 = p_2 = p_3 = p$ (unknown)

$H_A:$ for at least one i $p_i \neq p$

	Day	Evening	Night	Total
Defectives	45	55	70	170
Non defectives	905	890	870	2665
Sample size	$n_D = 950$	$n_E = 945$	$n_N = 940$	2835

$\hat{p} = \frac{170}{2835}$ $\quad \hat{q} = 1 - \hat{p} = \frac{2665}{2835}$

For defectives:

$E(X_{DD}) = 950 \frac{170}{2835} = 56.97$, $E(X_{DE}) = 945 \frac{170}{2835} = 56.67$

$E(X_{DN}) = 940 \frac{170}{2835} = 56.37$

For Non-defectives:

$E(X_{ND}) = 950 \frac{2665}{2835} = 893.03$, $E(X_{NE}) = 945 \cdot \frac{2665}{2835} = 888.33$

$E(X_{NN}) = 940 \cdot \frac{2665}{2835} = 883.63$

$$\chi^2_{cal} = \frac{(45-56.97)^2}{56.97} + \frac{(55-56.67)^2}{56.67} + \frac{(70-56.37)^2}{56.37} + \frac{(905-893.03)^2}{893.03} + \frac{(890-888.33)^2}{888.33} + \frac{(870-883.63)^2}{883.63} = 6.233$$

$d.o.f = k-1 = 2$ $\quad \chi^2_{0.05}(2) = 5.99$

Since $\chi^2_{cal} = 6.233 > \chi^2_{0.05}(2) = 5.99$ we reject H_0. Conclude that proportion of defectives differs for all 3 shifts at $\alpha = 0.05$.

b) $H_0: P_N = P_D$ or $H_0: P_N - P_D = 0$

$H_A: P_N > P_D$ $H_A: P_N - P_D > 0$

Under H_0 $\hat{p} = \frac{45+70}{950+940} = \frac{115}{1890}$

$\hat{P}_N = \frac{70}{940}$, $\hat{P}_D = \frac{45}{950}$

$$Z_{cal} = \frac{\frac{70}{940} - \frac{45}{950} - 0}{\sqrt{\frac{115}{1890} \cdot \frac{1775}{1890}\left(\frac{1}{950} + \frac{1}{940}\right)}} = \frac{0.0271}{0.010996} = 2.464$$

$Z_{0.05} = 1.645$

Since $Z_{cal} = 2.464 > Z_{0.05} = 1.645$, we reject H_0, conclude that $P_N > P_D$ at $\alpha = 0.05$.

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