Date: April 28, 2012 **Name:**......................
Time: 12:30-14:20
Instructor: Dilek Güvenç

MATH 260 MIDTERM EXAM II

1	2	3	4	TOTAL
20	25	30	25	100

- Check that there are 4 questions in your booklet
- Do NOT use your mobile phone as a calculator. Turn it off during the exam.
- Write your name on each question page and on formula page.
- Do NOT forget to return the tables and the formula page, as well.

********* **GOOD LUCK** *********

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1. Let X be proportion of allotted time that a randomly selected student spends working on a certain aptitude test. Suppose density of X is

$$f(x,\theta)=(\theta+1)x^{\theta} \qquad 0\le x\le 1$$

where $\theta>-1$.

a) Find Maximum Likelihood Estimator of θ. (15 Points)

b) If a random sample of 10 students yields the following data on these proportions, what is the Maximum Likelihood Estimate of θ?
0.92, 0.79, 0.90, 0.65, 0.86, 0.46, 0.73, 0.97, 0.94, 0.77. (5 Points)

a) $L_X(\theta)=\prod_{i=1}^{n}(\theta+1)X_i^{\theta}$

$K_X(\theta)=\ln L_X(\theta)=\sum_{i=1}^{n}\ln[(\theta+1)X_i^{\theta}]=\sum_{i=1}^{n}[\ln(\theta+1)+\theta\ln X_i]$

$=n\ln(\theta+1)+\theta\sum\ln X_i$

$\frac{dK_X(\theta)}{d\theta}=0 \Rightarrow \frac{n}{\theta+1}+\sum\ln X_i=0$ since $\theta>-1$

$\Rightarrow n+(\theta+1)\sum\ln X_i=0 \Rightarrow \theta\sum\ln X_i=-n-\sum\ln X_i$

$\Rightarrow$ M.L.E of θ, $\hat{\theta}=-\left(1+\frac{n}{\sum\ln X_i}\right)$

b) M.L. Estimate, $\hat{\theta}=-1-\frac{10}{\ln(0.92)+\cdots+\ln(0.77)}$

$\hat{\theta}=-1-\frac{10}{-0.08338-\cdots-0.261364}$

$=-1-\frac{10}{-2.45}=-1+4.08=3.08$

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2. Let X be the number of drivers who travel between a particular origin and destination during a designated time period. For a given mean μ, X has Poisson distribution. But μ varies according to an exponential distribution with parameter $\beta = 1$. Find Bayes estimator of μ.

$$f(x \mid \mu) = \frac{e^{-\mu} \mu^{x}}{x!} \qquad x = 0, 1, 2 \ldots$$

$$g(\mu) = e^{-\mu} \qquad \mu > 0$$

$$\Rightarrow \quad f(x, \mu) = g(\mu) \cdot f(x \mid \mu)$$
$$= e^{-\mu} \, \frac{e^{-\mu} \mu^{x}}{x!}$$
$$= \frac{e^{-2\mu} \mu^{x}}{x!} \qquad x = 0, 1, 2 \ldots, \quad \mu > 0$$

$$f(x) = \int_0^\infty \frac{e^{-2\mu} \mu^{x}}{x!} \, d\mu \qquad 2\mu = u \Rightarrow 2\,d\mu = du, \quad \mu = \frac{u}{2}$$

$$= \frac{1}{2} \frac{1}{x!} \int_0^\infty \left(\frac{u}{2}\right)^{x} e^{-u} \, du = \frac{1}{2^{x+1} \, x!} \underbrace{\int_0^\infty u^{x} e^{-u} \, du}_{\Gamma(x+1)}$$

$$= \frac{1}{2^{x+1} \, x!} \Gamma(x+1) = \frac{1}{2^{x+1} \, x!} \, x! = \frac{1}{2^{x+1}} \qquad x = 0, 1, 2 \ldots$$

$\Rightarrow$ Posterior density of μ

$$f(\mu \mid x) = \frac{f(x, \mu)}{f(x)} = \frac{e^{-2\mu} \mu^{x}}{x!} \Big/ \frac{1}{2^{x+1}} = \frac{1}{\Gamma(x+1) \left(\frac{1}{2}\right)^{x+1}} \mu^{x} e^{-2\mu}$$

is Gamma $\left(\alpha = x+1, \ \beta = \frac{1}{2}\right)$

$\Rightarrow$ Bayes estimator of μ, $\tilde{\mu} = \dfrac{x+1}{2}$

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3. The drying time of a certain type of paint under specified test conditions is known to be normally distributed with mean value 75 minutes and standard deviation 9 minutes. Chemists have proposed a new additive designed to decrease average drying time. It is believed that drying times with this additive will remain normally distributed with the same standard deviation. Experimental data consist of drying times of 25 test specimens. Sample mean of 25 drying times is found to be 71 minutes.

   a) Can we conclude that the new additive reduces the average drying time? Write appropriate null and alternative hypothesis and test at 5% significance level? (15 Points)
   b) Calculate P-Value of the test in part (a) (5 Points)
   c) What is probability of making Type II error in part (a) if the true mean drying time is 72 with the new additive? (10 Points)

X: Drying times $\sim N(\mu = 75, \sigma = 9)$

$n = 25 \quad \bar{X} = 71$

a) $H_0: \mu = 75$

$H_A: \mu < 75$

$$Z_{cal} = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{71 - 75}{\frac{9}{\sqrt{25}}} = -2,22$$

$-Z_{0,05} = -1,645$

Since $Z_{cal} = -2,22 < -Z_{0,05} = -1,645$ we reject H_0.

Conclude that new additive reduces average drying time at $\alpha = 0,05$.

b) $P\_Value = P(Z < -2,22) = 0,5 - A(2,22)$

$= 0,5 - 0,4868 = 0.0132$

c) $\beta = P(\text{do not reject } H_0 \mid H_A \text{ is true})$

We reject H_0 if $\dfrac{\bar{X} - 75}{\frac{9}{\sqrt{25}}} \leq -1,645$

or $\bar{X} \leq 75 - (1,645)\frac{9}{5} = 72,039$

$$\beta = P(\bar{X} \geq 72,039 \mid \mu_A = 72) = P\left(\frac{\bar{X} - 72}{\frac{9}{5}} \geq \frac{72,039 - 72}{\frac{9}{5}}\right)$$

$= P(Z > 0,02) = 0,5 - A(0,02) = 0,5 - 0,008 = 0,492$

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4. Biofeedback monitoring devices and techniques in the control of physiologic functions help astronauts control stress. In an experiment related to this topic, 6 subjects were placed in a stressful situation (using video games), followed by a period of biofeedback and adaptation. Another group of 6 subjects was placed under the same stress and then simply told to relax. The first group had an average heart rate of 70.4 and standard deviation of 15.3. The second group had an average heart rate of 74.9 and standard deviation of 16.

   a) Test the equality of population variances at 5% significance level. (10 Points)

   b) Using the conclusion in part (a) , can we say that average heart rate with biofeedback is lower than that without feedback? Test at 10% significance level. (15 Points)

State your assumption(s) for your answers to be valid in parts (a) and (b)

$\bar{X}_1 = 70.4 \quad S_1 = 15.3 \quad n_1 = 6$

$\bar{X}_2 = 74.9 \quad S_2 = 16 \quad n_2 = 6$

Assuming normality of the heartrates for both groups,

a) $H_0: \dfrac{\sigma_2^2}{\sigma_1^2} = 1$

$H_A: \dfrac{\sigma_2^2}{\sigma_1^2} > 1$ $\qquad F_{cal} = \dfrac{S_2^2}{S_1^2} = \dfrac{16^2}{15.3^2} = 1.093$

Since $F_{cal} = 1.093 \not> F_{0.05}(5,5) = 5.05$ we do not reject H_0, conclude that $\sigma_1^2 = \sigma_2^2 = \sigma^2$.

b) $H_0: \mu_1 - \mu_2 = 0$

$H_A: \mu_1 - \mu_2 < 0$ $\quad$ by part (a) $\sigma_1^2 = \sigma_2^2 = \sigma^2$

$$S_p^2 = \frac{5S_1^2 + 5S_2^2}{6+6-2} = \frac{5(16^2 + 15.3^2)}{10} = \frac{490.09}{2} = 245.045$$

$$S_p = 15.6 \qquad T_{cal} = \frac{70.4 - 74.9}{15.6\sqrt{\frac{1}{6}+\frac{1}{6}}} = -\frac{4.5}{9.0067} = -0.50$$

$t_{0.10}(10) = 1.372$

Since $T_{cal} = -0.5 \not< -t_{.10}(10) = -1.372$ we do not reject H_0, and can not conclude that average heart rate with biofeedback is lower than that without feedback