Name:......................

Dept:................

Date: March 12, 2015
Time: 17:45-19:30
Instructor: Dilek Güvenç

MATH 260 MIDTERM EXAM I

IMPORTANT

- Check that there are 4 questions in your booklet.
- Show all your work. Correct results without sufficient explanation and correct notation may not get full credit. Please, be neat.
- Write your name on each page of your booklet.
- Do NOT use your mobile phone as a calculator, <u>turn it off</u> during the exam.
- Do NOT forget to return the tables and the "cheat-sheet", as well.

1	2	3	4	TOTAL
25	25	25	25	100

********* **GOOD LUCK!** *********

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1. Let $X_1, X_2, \cdots X_n$ denote a random sample from a normal distribution with mean 0 and variance σ^2.

a) Show that $\hat{\sigma}^2 = \dfrac{\sum_{i=1}^{n} X_i^2}{n}$ is an unbiased estimator of σ^2. **(10 Points)**

b) Use the definition of χ^2 random variable and the fact that $\chi^2(v)$ is *Gamma* $(\alpha = \frac{v}{2}, \beta = 2)$ to show that $\hat{\sigma}$ is a biased estimator of σ. **(15 Points)**

a) $E(\hat\sigma^2) = \frac{1}{n}E(\Sigma X_i^2) = \frac{1}{n}\sum_{i=1}^{n} E(X_i^2) = \frac{1}{n}\sum_{i=1}^{n}(\sigma^2+\mu^2) = \frac{1}{n} n\sigma^2 = \sigma^2 \Rightarrow \hat\sigma^2$ is unbiased

$(X_i \sim N(\mu=0,\sigma^2))$ or: $\frac{\Sigma(X_i-0)^2}{\sigma^2} = \frac{\Sigma X_i^2}{\sigma^2} = \frac{n\hat\sigma^2}{\sigma^2} \sim \chi^2(n) \sim \text{gamma}(\alpha=\frac{n}{2},\beta=2)$

$\Rightarrow E\left(\frac{n\hat\sigma^2}{\sigma^2}\right) = \frac{n}{\sigma^2}E(\hat\sigma^2) = \frac{n}{2}\cdot 2 \Rightarrow E(\hat\sigma^2)=\sigma^2$.

b) $U = \frac{n\hat\sigma^2}{\sigma^2} \sim \text{gamma}(\alpha=\frac{n}{2},\beta=2)$ $\quad \sqrt{U} = \frac{\sqrt n}{\sigma}\cdot\hat\sigma \quad E(\sqrt U) = \frac{\sqrt n}{\sigma}E(\hat\sigma)$

$\Rightarrow E(\hat\sigma) = \frac{\sigma}{\sqrt n}E(\sqrt U)$ where

$E(\sqrt U) = \int_0^\infty u^{\frac12}\frac{1}{\Gamma(\frac n2)2^{\frac n2}}u^{\frac n2-1}e^{-\frac u2}du = \frac{\sqrt2}{\Gamma(\frac n2)}\int_0^\infty \left(\frac u2\right)^{\frac n2 - \frac12}e^{-\frac u2}\frac{du}{2}$

$= \frac{\sqrt2}{\Gamma(\frac n2)}\Gamma\left(\frac n2+\frac12\right) = \frac{\sqrt2}{\Gamma(\frac n2)}\Gamma\left(\frac{n+1}{2}\right)$

$\Rightarrow E(\hat\sigma) = \frac{\sigma}{\sqrt n}\frac{\sqrt2\,\Gamma(\frac{n+1}{2})}{\Gamma(\frac n2)} \neq \sigma$

$\Rightarrow \hat\sigma$ is a biased estimator of σ

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2. Drivers at a rural intersection have been observed to drive through a stop sign 45% of the time. A study is made and 550 drivers travelling through the stop sign are observed.

a) Approximate the probability that more than half of the drivers actually stop at the stop sign. **(10 Points)**

b) Approximate the probability that exactly 240 of the drivers do not stop at the stop sign. **(7 Points)**

c) Suppose that exactly 240 of the 550 drivers did not stop at the stop sign. Based on this information estimate the true proportion of drivers who do not stop at the stop sign, with a 98% confidence interval. Is there any contradiction between the interval and the proportion value given above? Interpret the interval with two sentences, at most. **(8 Points)**

a) X: # of people stop at this stop sign, in 550. $X \sim Bin(n=550, p=0.55)$

$np = 550(0.55) = 302.5$; $\sqrt{npq} = \sqrt{550(0.55)(0.45)} = 11.667$

$P(X>275) = P(X \geq 276) \sim P\left(Z > \frac{276-0.5-302.5}{11.667}\right)$

$\sim P(Z > -2.31) = 0.5 + A(2.31) = 0.5 + 0.4896$

$= 0.9896$

b) Y: # of people do not stop at this stop sign, in 550. $Y \sim Bin(n=550, p=0.45)$

$np = 247.5$ $\sqrt{npq} = 11.667$

$P(Y=240) \sim P\left(\frac{240-0.5-247.5}{11.667} < Z < \frac{240+0.5-247.5}{11.667}\right)$

$\sim P(-0.69 < Z < -0.60) = A(0.69) - A(0.60) = 0.2549 - 0.2257$

~ 0.0292

c) $\hat{p} = \frac{240}{550} = 0.436$ $1-\alpha = 0.98$ $\alpha = 0.02$ $\frac{\alpha}{2} = 0.01$ $Z_{0.01} = 2.33$

$\Rightarrow$ 98% C.I for p $0.436 \pm (2.33)\sqrt{\frac{(0.436)(0.564)}{550}}$

0.436 ± 0.049 or $(0.387, 0.485)$

No, there is no contradiction; because the interval includes 0.45.

We are 98% confident that, true proportion of people drive through this stop sign, is in this interval. If we continue to observe people and obtain C.I from each sample of size 550, app. 98% of these intervals, will include the true proportion.

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3. To study the effect of alloying on the resistance of electric wires, an engineer measures the resistance of $n_1=10$ standard wires and $n_2=15$ alloyed wires. Sample standard deviations of standard and alloyed wires are found to be 0.0041 ohm and 0.0064 ohm, respectively. Assuming normality of resistance measurements,

a) Obtain 90% confidence interval for the ratio of true population variances of standard and alloyed wire resistance measurements. **(7 Points)**

b) What would be the length of the interval in estimating true mean difference of standard and alloyed wire resistance measurements, with a 98% confidence level? **(8 points)**

c) If it can be assumed that $\sigma_1 = 0.004$ and $\sigma_2 = 0.005$, how should 40 electric wires be allocated to the standard and alloyed wires so that with 98% confidence level maximum error in estimating $\mu_1 - \mu_2$ with $\bar{x}_1 - \bar{x}_2$ will be minimized? **(10 Points)**

a) $S_1 = 0.0041$, $S_2 = 0.0064$ and $n_1 = 10$, $n_2 = 15$; $1-\alpha = 0.90$

$\Rightarrow$ 90% C.I for $\frac{\sigma_2^2}{\sigma_1^2}$ is $\left(\frac{S_2^2}{S_1^2} F_{0.95}(9,14), \frac{S_2^2}{S_1^2} F_{0.05}(9,14)\right)$

where $F_{0.05}(9,14) = 2.65$ $F_{0.95}(9,14) = \frac{1}{F_{0.05}(14,9)} = \frac{1}{3.03} = 0.33$

$\Rightarrow$ the interval $\left(\left(\frac{0.0064}{0.0041}\right)^2(0.33), \left(\frac{0.0064}{0.0041}\right)^2(2.65)\right)$ or $(0.804, 6.457)$

b) Assuming that $\sigma_1^2 = \sigma_2^2 = \sigma^2$ $1-\alpha = 0.98$

Length of the interval for $\mu_1 - \mu_2 = 2 t_{\frac{\alpha}{2}} S_p \sqrt{\frac{1}{10} + \frac{1}{15}}$ d.o.f $= 10+15-2 = 23$

where $S_p = \sqrt{\frac{9S_1^2 + 14 S_2^2}{10+15-2}} = \sqrt{0.00003151} = 0.0056$ $t_{0.01}(23) = 2.5$

The length $= 2(2.5)(0.0056)\sqrt{0.167} = 0.0114$

c) $\sigma_1 = 0.004$, $\sigma_2 = 0.005$ $1-\alpha = 0.98$ since σ_1, σ_2 are known

C.I in estimating $\mu_1 - \mu_2$ is $\bar{X}_1 - \bar{X}_2 \pm Z_{0.01}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$ $Z_{0.01} = 2.33$

where $n_1 + n_2 = 40 \Rightarrow n_2 = 40 - n_1 \Rightarrow B = 2.33\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{40-n_1}}$

$\Rightarrow$ we'll minimize $\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{40-n_1} = f(n_1)$ $\sigma_1 = 0.004$ $\sigma_2 = 0.005$

$\Rightarrow \frac{df(n_1)}{dn_1} = 0 \Rightarrow -\frac{\sigma_1^2}{n_1^2} + \frac{\sigma_2^2}{(40-n_1)^2} = 0 \Rightarrow (40-n_1)^2\sigma_1^2 = n_1^2\sigma_2^2$

$\Rightarrow (40-n_1)\sigma_1 = n_1\sigma_2 \Rightarrow (40-n_1)(0.004) = n_1(0.005) \Rightarrow 0.16 = 0.009 n$

$\Rightarrow n_1 = \frac{0.16}{0.009} \sim 18$ $n_2 = 40 - 18 = 22$

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4. Let $x_1, x_2, \ldots, x_n$ be an observed random sample of random variable X. For $\alpha > 0$ density of X is given by

$$f(x) = \frac{\alpha 3^{\alpha}}{x^{\alpha+1}} \qquad x > 3$$

a) Find M.L.E of α. **(12 Points)**

b) Let $\qquad Y_i = \begin{cases} 2 & if \quad X_i > 5 \\ 0 & if \quad X_i \leq 5 \end{cases}$

For n=20, we observed that $\sum_{i=1}^{20} Y_i = 20$. Find the M.L. Estimate for α. **(13 Points)**

a) $f(X_i, \alpha) = \dfrac{\alpha 3^{\alpha}}{X_i^{\alpha+1}} \qquad X > 3$

$L_X(\alpha) = \prod_{i=1}^{n} \dfrac{\alpha 3^{\alpha}}{X_i^{\alpha+1}} \Rightarrow K_X(\alpha) = \ln L_X(\alpha) = \sum_{i=1}^{n} \ln \dfrac{\alpha 3^{\alpha}}{X_i^{\alpha+1}}$

$\Rightarrow K_X(\alpha) = \sum_{i=1}^{n} (\ln \alpha + \alpha \ln 3 - (\alpha+1)\ln X_i) = n \ln \alpha + n\alpha \ln 3 - (\alpha+1) \sum_{i=1}^{n} \ln X_i$

$\Rightarrow \dfrac{dK_X(\alpha)}{d\alpha} = 0 \Rightarrow \dfrac{n}{\alpha} + n \ln 3 - \sum \ln X_i = 0 \Rightarrow \dfrac{n}{\alpha} = \sum \ln X_i - n \ln 3$

$\Rightarrow$ MLE of α, $\hat{\alpha} = \dfrac{n}{\sum \ln X_i - n \ln 3}$

b) $P(Y_i = 2) = P(X_i > 5) = \int_5^{\infty} \dfrac{\alpha 3^{\alpha}}{x^{\alpha+1}} dx = -\dfrac{3^{\alpha}}{x^{\alpha}} \Big|_5^{\infty} = \left(\dfrac{3}{5}\right)^{\alpha} = (0,6)^{\alpha}$

$P(Y_i = 0) = P(X_i \leq 5) = 1 - (0,6)^{\alpha}$

$n = 20$ and $\sum_{i=1}^{20} Y_i = 20 \Rightarrow$ 10 times $Y_i = 2$ and 10 times $Y_i = 0$

$\Rightarrow L_Y(\alpha) = (0,6)^{10\alpha} (1 - (0,6)^{\alpha})^{10}$

$\Rightarrow K_X(\alpha) = 10\alpha \ln(0,6) + 10 \ln(1 - (0,6)^{\alpha})$

$\dfrac{dK_X(\alpha)}{d\alpha} = 0 \Rightarrow 10 \ln(0,6) - \dfrac{10(0,6)^{\alpha} \ln(0,6)}{1 - (0,6)^{\alpha}} = 0$

$y = 0.6^{\alpha}$
$\ln y = \alpha \ln(0.6)$
$\dfrac{y'}{y} = \ln(0,6)$
$\Rightarrow y' = (0,6)^{\alpha} \ln(0,6)$

$\Rightarrow 10 \ln(0,6) - 10(0,6)^{\alpha} \ln(0,6) - 10(0,6)^{\alpha} \ln(0,6) = 0$

$\Rightarrow 1 - 2(0,6)^{\alpha} = 0 \Rightarrow (0,6)^{\alpha} = \dfrac{1}{2} \Rightarrow \alpha \ln(0,6) = -\ln 2$

$\Rightarrow$ M.L. Estimate of α, $\hat{\alpha} = -\dfrac{\ln 2}{\ln 0,6} = -\dfrac{0.6931}{-0.5108} = 1.35698$