Name:.............................

Dept:.................

Date: March 13, 2014
Time: 17:45-19:30
Instructor: Dilek Güvenç

MATH 260 MIDTERM EXAM I

IMPORTANT

- Check that there are 4 questions in your booklet.
- Show all your work. Correct results without sufficient explanation and correct notation may not get full credit. Please, be neat.
- Write your name on each page of your booklet.
- Do NOT use your mobile phone as a calculator, turn it off during the exam.
- Do NOT forget to return the tables and the "formula page", as well.

1	2	3	4	TOTAL
25	25	25	25	100

********* **GOOD LUCK!** *********

---

1. Let $X_1, X_2, \cdots X_n$ denote a random sample from a gamma distribution with parameters $\alpha = 2$ and β (unknown).

a) Derive a 100(1-α) % large-sample confidence interval (CI) for β. **(20 Points)**

b) Suppose that with n=40 observations 98% CI was found to be (2.46, 2.62). Is true value of β in this interval? Interpret the interval. **(5 Points)**

a) $\mu = E(X) = 2\beta$, $\sigma^2 = Var(X) = 2\beta^2$ For large n we know that $\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\bar{X}-2\beta}{\frac{\sqrt{2}\beta}{\sqrt{n}}}$ is app. standard normal (C.L.T)

$$\Rightarrow P\left(-z_{\frac{\alpha}{2}} < \frac{\bar{X}-2\beta}{\frac{\sqrt{2}\beta}{\sqrt{n}}} < z_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$$\Rightarrow P\left(-\frac{\sqrt{2}\, z_{\frac{\alpha}{2}}}{\sqrt{n}} < \frac{\bar{X}}{\beta} - 2 < \frac{\sqrt{2}\, z_{\frac{\alpha}{2}}}{\sqrt{n}}\right) = 1-\alpha$$

$$\Rightarrow P\left(2-\frac{\sqrt{2}\, z_{\frac{\alpha}{2}}}{\sqrt{n}} < \frac{\bar{X}}{\beta} < 2+\frac{\sqrt{2}\, z_{\frac{\alpha}{2}}}{\sqrt{n}}\right) = 1-\alpha$$

$$\Rightarrow P\left(\frac{2-\sqrt{\frac{2}{n}}\, z_{\frac{\alpha}{2}}}{\bar{X}} < \frac{1}{\beta} < \frac{2+\sqrt{\frac{2}{n}}\, z_{\frac{\alpha}{2}}}{\bar{X}}\right) = 1-\alpha$$

$\longrightarrow$

1. a) (Cont.)

$$\Rightarrow P\left(\frac{\bar{X}}{2-\sqrt{\frac{2}{n}}\,Z_{\frac{\alpha}{2}}} > \beta > \frac{\bar{X}}{2+\sqrt{\frac{2}{n}}\,Z_{\frac{\alpha}{2}}}\right) = 1-\alpha$$

$\Rightarrow$ $100(1-\alpha)\%$ C.I for β

$$\left(\frac{\bar{X}}{2+\sqrt{\frac{2}{n}}\,Z_{\frac{\alpha}{2}}},\ \frac{\bar{X}}{2-\sqrt{\frac{2}{n}}\,Z_{\frac{\alpha}{2}}}\right)$$

b) Can not tell. We are 98 % confident that true β is in the interval. That is; if we continue sampling and obtain 98 % C.I for β, from each sample of size $n=40$ approximately 98 % of these intervals will include true β.

——— o ———

Name:..................

2. Suppose that the distribution of student's grades in a statistics exam have a mean 62 and standard deviation 8.

a) What is the approximate probability that average of such a statistics class with 49 students is at least 65? **(8 Points)**

b) If an instructor teaches two different sections of this course and one section contains 49 other contains 36 students, approximate the probability that the average of one class is at least 3 points more than the average of the other class. **(17 Points)**

$\mu = 62 \quad \sigma = 8$

a) $n = 49$ (large) $\quad E(\bar{X}) = \mu_{\bar{X}} = 62$, $\quad Var(\bar{X}) = \sigma^2_{\bar{X}} = \frac{8^2}{49} \Rightarrow \sigma_{\bar{X}} = \frac{8}{7} = 1.14$

$P(\bar{X} \geq 65) = ?$ By C.L.T $P(\bar{X} \geq 65) \overset{app}{\sim} P(Z \geq \frac{65-62}{1.14})$

$\sim P(Z \geq 2.63)$

$\sim 0.5 - A(2.63) = 0.5 - 0.4957$

~ 0.0043

b) $n_1 = 49 \quad n_2 = 36$ (both large)

Let $\bar{X} = \frac{\Sigma X}{49}$, $\bar{Y} = \frac{\Sigma Y}{49}$, by C.L.T both $\bar{X}$ and $\bar{Y}$ app. normal

$\Rightarrow \bar{X} - \bar{Y}$ app. normal $\quad E(\bar{X}) = E(\bar{Y}) = 62 \quad Var(\bar{X}) = \frac{8^2}{49}$, $Var(\bar{Y}) = \frac{8^2}{36}$

$\Rightarrow E(\bar{X} - \bar{Y}) = E(\bar{X}) - E(\bar{Y}) = 62 - 62 = 0$

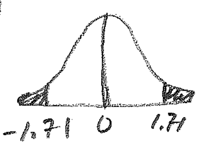
$Var(\bar{X} - \bar{Y}) = Var(\bar{X}) + Var(\bar{Y}) = \frac{8^2}{49} + \frac{8^2}{36} = 1.306 + 1.778 = 3.084$

$\sigma_{\bar{X}-\bar{Y}} = \sqrt{3.084} = 1.756$

$P(|\bar{X} - \bar{Y}| > 3) = P(\bar{X} - \bar{Y} > 3) + P(\bar{X} - \bar{Y} < -3)$

$\sim P(Z > \frac{3-0}{1.756}) + P(Z < \frac{-3-0}{1.756})$

$\sim P(Z > 1.71) + P(Z < -1.71)$



$\sim 1 - 2A(1.71)$

$\sim 1 - 2(0.4564) = 0.0872$

Name:.....................

3. If $\overline{X}$ and $\overline{Y}$ are the means of independent random samples of sizes n_1 and n_2, each from a distribution with mean μ and variance σ^2.

a) Show that $a\overline{X}+(1-a)\overline{Y}$ is an unbiased estimator of μ, where $0 \le a \le 1$. **(7 Points)**

b) Find the value of a, so that variance of the estimator in part (a) is minimum. **(13 points)**

c) If $n_1 = n_2 = n$, is the estimator in part (a) consistent? **(5 Points)**

a) $E[a\bar{X}+(1-a)\bar{Y}] = aE(\bar{X}) + (1-a)E(\bar{Y})$

$= a\mu + (1-a)\mu = \mu$

$\Rightarrow a\bar{X}+(1-a)\bar{Y}$ is an unbiased estimator of μ.

b) $Var[a\bar{X}+(1-a)\bar{Y}] = a^2 Var(\bar{X}) + (1-a)^2 Var(\bar{Y})$

$= a^2\frac{\sigma^2}{n_1} + (1-a)^2\frac{\sigma^2}{n_2}$

$= \sigma^2\left(\frac{a^2}{n_1} + \frac{(1-a)^2}{n_2}\right)$

$$\frac{d\,[Var\; a\bar{X}+(1-a)\bar{Y}]}{da} = 0$$

$\Rightarrow \sigma^2\left(\frac{2a}{n_1} - \frac{2(1-a)}{n_2}\right) = 0$

$\Rightarrow \frac{2a}{n_1} - \frac{2(1-a)}{n_2} = 0 \qquad \frac{a}{n_1} - \frac{1}{n_2} + \frac{a}{n_2} = 0$

$a\left(\frac{1}{n_1} + \frac{1}{n_2}\right) = \frac{1}{n_2} \Rightarrow a = \frac{\frac{1}{n_2}}{\frac{n_1+n_2}{n_1 n_2}} = \frac{1}{\cancel{n_2}} \cdot \frac{n_1\cancel{n_2}}{n_1+n_2} = \frac{n_1}{n_1+n_2}$

c) $n_1 = n_2 = n \Rightarrow Var[a\bar{X}+(1-a)\bar{Y}] = \frac{\sigma^2[a^2+(1-a)^2]}{n}$

$\Rightarrow \lim_{n\to\infty} Var[a\bar{X}+(1-a)\bar{Y}] = \lim_{n\to\infty}\frac{\sigma^2[a^2+(1-a)^2]}{n} = 0$

$\Rightarrow$ It is a consistent estimator.

Name:.....................

4. In testing the strength of chains under heat treatments A and B, the following two sets of strengths were found.

A : 3610 , 3608 , 3672

B : 3416 , 3460 , 3390

a) Find a 90 % confidence interval for $\frac{\sigma_A^2}{\sigma_B^2}$. **(10 Points)**

b) On the basis of the confidence interval in part (a), do you think it likely that $\sigma_A^2 = \sigma_B^2$? Why ? **(3 Points)**

c) Find 95% confidence interval for the difference of true means. State your assumption(s) in parts (a) and (c). **(12 Points).**

$\bar{X}_A = \frac{\Sigma X}{3} = 3630$, $\bar{X}_B = \frac{\Sigma X}{3} = 3422$

$S_A^2 = \frac{\Sigma(X-\bar{X}_A)^2}{2} = 1324 \Rightarrow S_A = 36.3868$

$S_B^2 = \frac{\Sigma(X-\bar{X}_B)^2}{2} = 1252 \Rightarrow S_B = 35.3836$

a) $1-\alpha = 0.90$ $\quad F_{0.05}(2,2) = 19$ $\quad F_{0.95}(2,2) = \frac{1}{F_{0.05}(2,2)} = \frac{1}{19} = 0.053$

$\Rightarrow$ 90 % C.I for $\frac{\sigma_A^2}{\sigma_B^2}$ is $\left(\frac{1324}{1252}(0.053), \frac{1324}{1252}(19)\right)$

or $(0.056 , 20.093)$. We assumed normality of strengths for both heat treatments.

b) Yes, because interval includes 1.

c) $1-\alpha = 0.95$ Assuming normality and $\sigma_A^2 = \sigma_B^2 = \sigma^2$

95 % C.I for $\mu_A - \mu_B$

$\bar{X}_A - \bar{X}_B \pm t_{0.025} \cdot S_p \cdot \sqrt{\frac{1}{3}+\frac{1}{3}}$ where $S_p = \sqrt{\frac{2(1324)+2(1252)}{3+3-2}}$

$S_p = \sqrt{1288} = 35.89$

d.o.f $= 3+3-2 = 4$ $\quad t_{0.025}(4) = 2.776$

$\Rightarrow$ the interval : $3630 - 3422 \pm (2.776)(35.89)\sqrt{\frac{2}{3}}$

208 ± 81.35 or $(126.65, 282.35)$

— o —