Date: March 20, 2009
Time: 18:00-19:50
Instructor: Dilek Güvenç

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MATH 260 MIDTERM EXAM I

1	2	3	4	5	TOTAL
20	25	20	15	20	100

- Check that there are 5 questions in your booklet.
- Show all your work. Correct results without sufficient explanation and correct notation may not get full credit. Please, be neat.
- Write your name on each page of your booklet.
- Do NOT use your mobile phone as a calculator, turn it off during the exam.
- Do NOT forget to return the tables and the "formula page", as well.

********* GOOD LUCK! *********

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1. A random sample of size 54 is selected (with replacement) from a population with distribution

$$f(x) = \begin{cases} \frac{1}{3} & x = 2,4,6 \\ 0 & elsewhere \end{cases}$$

A second random sample of size 81 is selected from a different population having a mean of 3.5 and a variance of 3. Find the probability that sample mean computed from the 54 observations will exceed the sample mean computed from the 81 observations.

$\mu_1 = E(X) = 2\cdot\frac{1}{3} + 4\frac{1}{3} + 6\frac{1}{3} = \frac{12}{3} = 4$

$\sigma_1^2 = Var(X) = E(X^2) - [E(X)]^2 = 4\frac{1}{3} + 16\frac{1}{3} + 36\frac{1}{3} - 4^2 = \frac{56}{3} - 16 = \frac{8}{3}$

$n_1 = 54 \Rightarrow \overline{X}_1$ is app. normal with $\mu_{\overline{X}_1} = 4$, $\sigma^2_{\overline{X}_1} = \frac{8}{(3)(54)}$

$\mu_2 = 3.5 \quad \sigma_2^2 = 3, \; n_2 = 81 \Rightarrow \overline{X}_2$ is app. normal with $\mu_{\overline{X}_2} = 3.5 \quad \sigma^2_{\overline{X}_2} = \frac{3}{81}$

$P(\overline{X}_1 > \overline{X}_2) = P(\overline{X}_1 - \overline{X}_2 > 0) \sim P\left(Z > \frac{-0.5}{\sqrt{\frac{14}{162}}}\right)$

$E(\overline{X}_1 - \overline{X}_2) = 4 - 3{,}5 = 0{,}5$

$Var(\overline{X}_1 - \overline{X}_2) = \frac{8}{(3)(54)} + \frac{3}{\underset{(2)}{81}} = \frac{14}{162}$

$= P(Z > -1.7)$

$= 0{,}5 + A(1.7)$

$= 0{,}5 + 0{,}4554$

$= 0{,}9554$

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2. The weekly downtime X (in hours) for a certain production line has Gamma distribution with parameters α=2 and β.

Let $X_1, X_2, \dots X_n$ be a random sample of weekly downtime of n weeks.

a) Find Maximum Likelihood Estimator, $\hat{\beta}$ of β. **(10 points)**

b) Find Var($\hat{\beta}$).**(5 points)**

c) Show that

$$\frac{\sum_{i=1}^{n} X_i^2}{3n}$$

is an unbiased estimator of Var(X). **(7 points)**

d) During six weeks the following downtimes were observed:

4.1, 3.7, 6.0, 6.9, 4.0, 5.8.

Find the Maximum Likelihood estimate of β.**(3 points)**

$$f(x) = \frac{1}{\Gamma(2)\beta^2} x^{2-1} e^{-\frac{x}{\beta}} = \frac{x}{\beta^2} e^{-\frac{x}{\beta}} \quad x > 0$$

a) $$L_X(\beta) = \prod_{i=1}^{n} \frac{x_i}{\beta^2} e^{-\frac{x_i}{\beta}}$$

$$K_X(\beta) = \ln L_X(\beta) = \sum_{i=1}^{n} \ln\left(\frac{x_i}{\beta^2} e^{-\frac{x_i}{\beta}}\right) = \sum_{i=1}^{n}\left[\ln x_i - 2\ln\beta - \frac{x_i}{\beta}\right]$$

$$= \sum_{i=1}^{n} \ln x_i - 2\sum_{i=1}^{n} \ln\beta - \frac{\sum x_i}{\beta} = \sum_{i=1}^{n} \ln x_i - 2n\ln\beta - \frac{\sum x_i}{\beta}.$$

$$\frac{dK_X(\beta)}{d\beta} = -\frac{2n}{\beta} + \frac{\sum x_i}{\beta^2} = 0 \quad \Rightarrow -2n\beta + \sum x_i = 0$$

$$\Rightarrow \text{M.L.E of } \beta, \quad \hat{\beta} = \frac{\sum x_i}{2n} = \frac{\bar{X}}{2}.$$

b) $$Var(\hat{\beta}) = Var\left(\frac{\bar{X}}{2}\right) = \frac{1}{4} Var(\bar{X}) = \frac{1}{4}\,\frac{2.\beta^2}{n} = \frac{\beta^2}{2n}.$$

c) $$E\left(\frac{\sum X_i^2}{3n}\right) = \frac{1}{3n}\sum_{i=1}^{n} E(X_i^2) = \frac{1}{3n}\sum_{i=1}^{n}\left[Var(X_i) + [E(X_i)]^2\right]$$

$$= \frac{1}{3n}\sum_{i=1}^{n}\left[2\beta^2 + 4\beta^2\right] = \frac{1}{3n}\, 6n\beta^2 = 2\beta^2 = Var(X)$$

d) M.L. Estimate of β, $$\hat{\beta} = \left(\frac{4.1+3.7+6.0+6.9+4+5.8}{6}\right)\Big/2$$

$$= \frac{5.08}{2} = 2.54$$

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3. Waiting times at a service counter in a pharmacy are normally distributed random variable with mean μ and variance σ^2. For 6 customers the following waiting times (rounded to nearest whole minute) are recorded:

12 6 10 14 16 8

a) Find the sample mean, median and variance. **(6 points)**
b) Estimate the true mean waiting time at this service counter with a 90% confidence interval. **(6 points)**
c) Find a 95% confidence interval for the true standard deviation of the waiting times. **(4 points)**
d) Predict the 7-th customer's waiting time with a 95% interval. Interpret the interval. **(4 points)**

a) $\bar{X} = \dfrac{12+6+10+14+16+8}{6} = 11$

Sorted data: 6, 8, 10, 12, 14, 16

$\Rightarrow$ Median $= \dfrac{10+12}{2} = 11$

$$S^2 = \frac{\sum X_i^2 - n\bar{X}^2}{6-1} = \frac{796-726}{5} = 14.$$

b) $1-\alpha = 0{,}90$ $\quad \frac{\alpha}{2} = 0{,}05$ $\quad$ d.o.f $= 5$ $\quad t_{0{,}025}(5) = 2.015$

$S = 3.74$

90% C.I for μ is

$\bar{X} \pm (2.015)\dfrac{S}{\sqrt{6}}$, $\quad 11 \pm (2.015)\dfrac{3.74}{\sqrt{6}}$ $\quad 11 \pm 3.08$

or $(7.92,\ 14.08)$

c) $\chi^2_{0{,}025}(5) = 12.83$ $\quad \chi^2_{0{,}975}(5) = 0{,}83$

95% C.I for σ $\quad \left(3.74\sqrt{\dfrac{5}{12{,}83}},\ 3.74\sqrt{\dfrac{5}{0.83}}\right)$

$(2.33,\ 9.18)$

d) $\bar{X}_6 = 11$ $\quad t_{0{,}025}(5) = 2.571$

95% prediction interval for X_7,

$11 \pm (2.571)(3.74)\sqrt{1+\frac{1}{6}}$

11 ± 10.39 $\quad$ or $\quad (0.61,\ 21.39)$

With 95% probability 7-th customer's waiting time will be between 0,61 and 21.39 minutes.

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4. A certain geneticist is interested in the proportion of males and females in the population that have certain minor blood disorder. In a random sample of 1000 males, 250 are found to be afflicted, whereas 275 of 1000 females tested appear to have the disorder. Estimate the difference between the proportion of males and females that have the blood disorder with a 95% confidence interval. Based on the interval can we conclude that there might be no difference between the male and female blood disorder proportions? Why?

$n_M = 1000 \quad x_M = 250 \quad p_M = \frac{250}{1000} = 0,25$

$n_F = 1000 \quad x_F = 275 \quad p_F = \frac{275}{1000} = 0,275$

$1-\alpha = 0,95 \quad z_{\frac{\alpha}{2}} = z_{0,025} = 1,96$

95 % C.I for $p_F - p_M$,

$$0,275 - 0,25 \pm (1,96)\sqrt{\frac{(0,25)(0,75)}{1000} + \frac{(0,275)(0,725)}{1000}}$$

$$0,025 \pm 1,96\sqrt{0,0003869}$$

$$0,025 \pm (1,96)(0,01968)$$

$$0,025 \pm 0,039$$

or $(-0,014 , 0,064)$

Yes. Since interval includes 0, there might be no difference between the male and female blood disorder proportions.

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5. For given mean λ, X has a Poisson distribution, but λ varies according to a χ^2 distribution with degrees of freedom 4. Find Bayes estimator of λ.

$$g(\lambda) = \frac{1}{\Gamma(2)2^2} \lambda e^{-\frac{\lambda}{2}} = \frac{\lambda}{4} e^{-\frac{\lambda}{2}} \quad \lambda > 0$$

$$f(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \quad x = 0, 1, 2, \ldots$$

$$f(\lambda|x) = \frac{f(x,\lambda)}{f(x)}$$

$$f(x,\lambda) = \frac{\lambda}{4} e^{-\frac{\lambda}{2}} \cdot \frac{e^{-\lambda}\lambda^x}{x!} = \frac{\lambda^{x+1} e^{-\frac{3\lambda}{2}}}{4x!}$$

or $$f(x,\lambda) = \frac{1}{4\Gamma(x+1)} \lambda^{x+1} e^{-\frac{3\lambda}{2}} \quad \lambda > 0 \quad x = 0, 1, 2 \ldots$$

$$f(x) = \int_0^\infty \frac{1}{4} \frac{1}{\Gamma(x+1)} \lambda^{x+1} e^{-\frac{3\lambda}{2}} d\lambda$$

$\frac{3\lambda}{2} = u \quad \frac{3}{2} d\lambda = du$

$$= \frac{1}{4\Gamma(x+1)} \int_0^\infty \lambda^{x+1} e^{-\frac{3\lambda}{2}} d\lambda$$

$\lambda = \frac{2}{3}u \quad d\lambda = \frac{2}{3}du$

$$= \frac{1}{4\Gamma(x+1)} \int_0^\infty \left(\frac{2}{3}\right)^{x+1} u^{x+1} e^{-u} \frac{2}{3} du$$

$$= \frac{1}{4\Gamma(x+1)} \left(\frac{2}{3}\right)^{x+2} \int_0^\infty u^{x+1} e^{-u} du$$

$$= \frac{1}{4\Gamma(x+1)} \left(\frac{2}{3}\right)^{x+2} \Gamma(x+2)$$

$$f(x) = \frac{1}{4\cancel{\Gamma(x+1)}} \left(\frac{2}{3}\right)^{x+2} (x+1)\cancel{\Gamma(x+1)} = \frac{1}{4}(x+1)\left(\frac{2}{3}\right)^{x+2} \quad x = 0, 1, 2 \ldots$$

$$f(\lambda|x) = \frac{\frac{1}{\cancel{4}\Gamma(x+1)} \lambda^{x+1} e^{-\frac{3\lambda}{2}}}{\frac{1}{\cancel{4}} \left(\frac{2}{3}\right)^{x+2}(x+1)} = \frac{\lambda^{x+1} e^{-\frac{3\lambda}{2}}}{\left(\frac{2}{3}\right)^{x+2} \Gamma(x+2)} \quad \text{is}$$

Gamma $(\alpha = x+2, \beta = \frac{2}{3}) \Rightarrow$ Bayes estimator of λ,

$$\hat{\lambda} = \frac{2(x+2)}{3}$$