On Spacetimes with Given Kinematical Invariants: Construction and Examples

M. Gürses, M. Plaue, M. Scherfner, T. Schönfeld, and L. A. M. de Sousa

ABSTRACT. As one of two objectives we present a useful method for the construction of cosmological models by solving the differential equations arising from calculating the kinematical invariants of an observer field in proper time description. The main purpose of this toolkit here is to create Gödel-type metrics with a particular causal and kinematical behavior. Additionally we investigate the geodesic structure of these and other models – like A. Ori's time machine – by visualizing the behavior of geodesics and that of light cones. We concentrate on showing the formation of caustics and the tipping of light cones since these effects are related to closed timelike or null curves. The described method and the underlying routines can easily be applied for investigating other spacetimes.

1. Introduction

The construction of viable cosmological and astrophysical models often requires particular restrictions on the kinematic properties of those models. For example, parallax-free models must necessarily be shear-free [**7**].

Here the cosmological observer field is analyzed in proper time description to obtain expressions for the metric that depend explicitly on the kinematical invariants. This approach leads to a useful toolkit for constructing cosmological models with given kinematical properties. In addition, the analysis given here leads to a deeper understanding of the kinematical invariants, the relations between them and their influence on causality. Of course, restrictions regarding the kinematical quantities also give rise to a limited range of possible matter models – it is often not possible to give energy–momentum tensors representing simple matter like a perfect fluid with a particular combination of rotation, shear, expansion and acceleration. Recall for example the well-known shear-free fluid conjecture which has been proven for a number of special cases (cf. [**6**]).

As an application of our method, we will construct a spacetime that generalizes the well-known Gödel metric [**1**] which represents a dust model with negative cosmological constant that has non-vanishing rotation, but vanishing shear, acceleration and expansion. In this paper, we will construct one generalization of the Gödel spacetime for which the acceleration vanishes as well as another model that

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is parallax-free. In order to give a deeper understanding of such models, we investigate the geodesic structure by visualizing the behavior of geodesics and light cones. Here we concentrate on showing the formation of caustics and the tipping of light cones since these effects are directly related to the study of closed timelike or null curves.

2. Preliminaries

We consider an $(N+1)$-dimensional smooth Lorentzian manifold (M, g) of signature $(+, -, \ldots, -)$. Given coordinates $(x^0, ..., x^N)$, by ∂_l we denote the partial derivative with respect to the l^{th} coordinate and ∇_j stands for the covariant derivative in direction of the j^{th} coordinate vector. Latin indices will take the values $0, \ldots, N$, whereas greek indices will range from 1 to N. In the following we fix a timelike unit vector field X^i, the cosmological observer field. The usual decomposition of the covariant derivative of X_i into irreducible parts (see for example [**3**]) reads

$$\nabla_k X_i = \omega_{ik} + \sigma_{ik} + \frac{1}{N}\Theta P_{ik} - \dot{X}_i X_k \tag{1}$$

with the antisymmetric part ω_{ik} (rotation), the symmetric traceless part σ_{ik} (shear) and the trace Θ (expansion) itself. In detail, the parts read

$$\omega_{ik} = \nabla_{[i} X_{k]} - \dot{X}_{[i} X_{k]}, \tag{2}$$

$$\sigma_{ik} = \nabla_{(i} X_{k)} - \dot{X}_{(i} X_{k)} - \frac{1}{N}\Theta P_{ik}, \tag{3}$$

and

$$\Theta = \nabla_a X^a. \tag{4}$$

Here the brackets (parentheses) denote the antisymmetric (symmetric) parts and

$$P_{ik} = g_{ik} - X_i X_k \tag{5}$$

is the projection tensor on the N-dimensional subspace perpendicular to X^i. The acceleration is given by

$$\dot{X}_i = X^a \nabla_a X_i. \tag{6}$$

In addition we have

$$\dot{X}_a X^a = 0, \quad \sigma_{ai} X^a = \omega_{ai} X^a = P_{ai} X^a = 0. \tag{7}$$

3. Proper time description of the observer field

We adopt comoving coordinates with respect to the observer field such that $X^i = \delta^i_0$; this is also known as the proper time description. From X^i being a timelike unit vector field we infer $1 = X_a X^a = g_{ab} X^a X^b = g_{00}$. The first coordinate denotes the proper time of the cosmological observer; a fact which we will from now on emphasize by denoting x^0 with t.

In proper time description the components of the metric can be expressed by components of the observer field – this is crucial to our analysis and we have:

$$X_i = g_{ai} X^a = g_{0i} \tag{8}$$

and

$$P_{ik} = g_{ik} - g_{0i} g_{0k}. \tag{9}$$

For the covariant derivative one has

$$
\begin{aligned}
\nabla_k X_i &= \partial_k X_i - \Gamma^a_{ik} X_a \\
&= \partial_k X_i - \Gamma^a_{ik} g_{0a} \\
&= \partial_k g_{0i} - \frac{1}{2} g^{ab} g_{0a} (\partial_k g_{ib} + \partial_i g_{kb} - \partial_b g_{ik}) \\
&= \partial_k g_{0i} - \frac{1}{2} \delta^b_0 (\partial_k g_{ib} + \partial_i g_{kb} - \partial_b g_{ik}) \\
&= \frac{1}{2} (\partial_k g_{0i} - \partial_i g_{0k} + \partial_0 g_{ik}).
\end{aligned} \tag{10}
$$

For the acceleration, one has in particular:

$$
\begin{aligned}
\dot{X}_i &= X^a \nabla_a X_i \\
&= \nabla_0 X_i \\
&= \partial_0 g_{0i} - \frac{1}{2} \partial_i g_{00} \\
&= \partial_0 g_{0i}.
\end{aligned} \tag{11}
$$

In the same manner, we obtain similar expressions for the other kinematical quantities in proper time description:

$$
\begin{aligned}
\Theta &= \frac{1}{2} g^{ia} \partial_0 g_{ia} && (12)\\
&= \frac{1}{2} \partial_0 \left(\log \det \left(g_{ik} \right) \right), \\
\omega_{ik} &= \frac{1}{2} (g_{0k} \partial_0 g_{0i} - g_{0i} \partial_0 g_{0k} - \partial_k g_{0i} + \partial_i g_{0k}), && (13)\\
\sigma_{ik} &= \frac{1}{2} (\partial_0 g_{ik} - g_{0k} \partial_0 g_{0i} - g_{0i} \partial_0 g_{0k}) - \frac{1}{N} \Theta P_{ik} \\
&= \frac{1}{2} \partial_0 P_{ik} - \frac{1}{N} \Theta P_{ik}. && (14)
\end{aligned}
$$

It can also be easily seen from (7) that the kinematical quantities are characterized by their spatial components:

$$
\dot{X}_0 = 0, \quad \omega_{0i} = \sigma_{0i} = P_{0i} = 0. \tag{15}
$$

4. Models with given kinematical invariants

Equation (14) can be solved for the projection tensor $P_{ik} = g_{ik} - g_{0i} g_{0k}$, which leads to

$$
P_{ik}(t, x^\gamma) = \left(\Sigma_{ik}(t, x^\gamma) + P_{ik}(0, x^\gamma) \right) S^2(t, x^\gamma). \tag{16}
$$

with

$$
S(t, x^\gamma) = \exp \left(\frac{1}{N} \int_0^t \Theta(\tau, x^\gamma) \, d\tau \right) \tag{17}
$$

and

$$
\Sigma_{ik}(t, x^\gamma) = 2 \int_0^t \frac{\sigma_{ik}(\tau, x^\gamma)}{S^2(\tau, x^\gamma)} \, d\tau. \tag{18}
$$

Denoting derivation with respect to t by a dot, we have

$$\Theta = N\frac{\dot{S}}{S} \tag{19}$$

with $S(0,x^\gamma) = 1$; $S(t,x^\gamma)$ is the so-called scale parameter.

We interpret this equation as an evolution equation for the spatial components of the metric for a given "start metric" $g_{\alpha\beta}(0,x^\gamma)$ and a given observer field $g_{0\alpha}(t,x^\gamma)$, expansion $\Theta(t,x^\gamma)$ and shear $\sigma_{\alpha\beta}(t,x^\gamma)$ that drive the evolution:

$$\begin{aligned} g_{\alpha\beta}(t,x^\gamma) &= g_{0\alpha}(t,x^\gamma)g_{0\beta}(t,x^\gamma) + \\ &\quad S^2(t,x^\gamma)\left(g_{\alpha\beta}(0,x^\gamma) - g_{0\alpha}(0,x^\gamma)g_{0\beta}(0,x^\gamma)\right) + \\ &\quad S^2(t)\Sigma_{\alpha\beta}(t,x^\gamma). \end{aligned} \tag{20}$$

Remark: The initial value "$t = 0$" is not a specific coordinate value like e.g. a singularity. In fact, the construction is not valid for singular start metrics. One should regard $g_{\alpha\beta}(0,x^\gamma)$ as the spatial metric at some arbitrary time in the history of the observer.

5. Some special cases

Whereas expansion, shear and the observer field itself enter the evolution equation directly, the rotation and acceleration impose additional constraints on the observer field.

Models with vanishing acceleration. In the case of a vanishing acceleration, we have $\partial_0 g_{0\alpha} = 0$, which implies $g_{0\alpha}(t,x^\gamma) = g_{0\alpha}(0,x^\gamma)$, and (20) reduces to

$$\begin{aligned} g_{\alpha\beta}(t,x^\gamma) &= S^2(t,x^\gamma)g_{\alpha\beta}(0,x^\gamma) + \\ &\quad \left(1 - S^2(t,x^\gamma)\right)g_{0\alpha}(0,x^\gamma)g_{0\beta}(0,x^\gamma) + \\ &\quad S^2(t)\Sigma_{\alpha\beta}(t,x^\gamma). \end{aligned} \tag{21}$$

Irrotational models. The rotation is calculated from the $g_{0\alpha}$ components of the metric alone. The equation for an observer field with vanishing rotation can be solved by setting $g_{0\alpha} = -h(t)\partial_\alpha\phi(x^\gamma)$. If the model also has vanishing acceleration, $h(t)$ is constant and the observer field is a gradient.

Parallax-free models. Spacetimes with an observer field parallel to a conformal vector field are important since such models are precisely the parallax-free models. As was shown in [**7**], this condition holds if and only if X^i is shear-free and the exterior derivative of $\dot{X}_i - \frac{\Theta}{N}X_i$ vanishes. The last condition can be met by assuming

$$\dot{X}_i = \frac{\Theta}{N}X_i - \partial_i f \tag{22}$$

for some function f.

Integrating (22), and substituting into (20) with $\sigma_{\alpha\beta} = 0$, we have

$$\begin{aligned} g_{\alpha\beta}(t,x^\gamma) &= S^2(t,x^\gamma)g_{\alpha\beta}(0,x^\gamma) + \\ &\quad S^2(t,x^\gamma)F_\alpha(t,x^\gamma)F_\beta(t,x^\gamma) - \\ &\quad S^2(t,x^\gamma)\left(F_\alpha(t,x^\gamma)g_{0\beta}(0,x^\gamma) + F_\beta(t,x^\gamma)g_{0\alpha}(0,x^\gamma)\right) \end{aligned} \tag{23}$$

	Spatial metric $g_{\alpha\beta}(t) = g_{\alpha\beta}(t, x^\gamma)$
General	$S^2(t)g_{\alpha\beta}(0) + g_{0\alpha}(t)g_{0\beta}(t) - S^2(t)g_{0\alpha}(0)g_{0\beta}(0)$
$\dot{X}_i = 0$	$S^2(t)g_{\alpha\beta}(0) + \left(1 - S^2(t)\right) g_{0\alpha}(0)g_{0\beta}(0)$
$\Theta = 0$	$g_{\alpha\beta}(0) + g_{0\alpha}(t)g_{0\alpha}(t) - g_{0\alpha}(0)g_{0\alpha}(0)$
$\Theta = 0, \dot{X}_i = 0$	$g_{\alpha\beta}(0)$
$\omega_{ik} = 0$	$S^2(t)g_{\alpha\beta}(0) + \left(h^2(t) - h^2(0)S^2(t)\right) \partial_\alpha\phi\partial_\beta\phi$
Parallax-free	$S^2(t)g_{\alpha\beta}(0) + S^2(t)\left(F_\alpha(t)F_\beta(t) - F_\alpha(t)g_{0\beta}(0) - F_\beta(t)g_{0\alpha}(0)\right)$

Figure 1: Table of shear-free cosmological models

with the functions

$$F_\alpha(t, x^\gamma) = \int_0^t \frac{\partial_\alpha f(\tau, x^\gamma)}{S(\tau, x^\gamma)}\, d\tau. \tag{24}$$

Toolkit. Figure 1 can be used as a toolkit to construct spacetimes with specific kinematic properties. We have only listed shear-free models since a non-vanishing shear can be included via the functions $\Sigma_{\alpha\beta}$. (However, these functions are subject to the trace-free condition $\sigma^i_i = 0$ which generally leads to rather complicated constraint equations.) Also, we have notationally suppressed the dependence on the spatial coordinates x^γ.

6. Generalizations of the Gödel spacetime

We like to generalize the Gödel spacetime $(\mathbb{R}^4, \tilde{g})$ equipped with the usual observer field to a shear-free model with non-vanishing expansion. Written in canonical (cartesian) coordinates (t, x, y, z), the Gödel metric reads:

$$(\tilde{g}_{ik}) = \begin{pmatrix} 1 & 0 & e^{\sqrt{2}\omega_0 x} & 0 \\ 0 & -1 & 0 & 0 \\ e^{\sqrt{2}\omega_0 x} & 0 & \frac{1}{2}e^{2\sqrt{2}\omega_0 x} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{25}$$

In these coordinates, the observer field is already of the desired form, $X^i = \delta^i_0$. We take as a start metric

$$(g_{\alpha\beta}(0, x^\gamma)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2}e^{2\sqrt{2}\omega_0 x} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{26}$$

and as the observer field we choose

$$(g_{0\alpha}(t, x^\gamma)) = \begin{pmatrix} 0 & f(t)e^{\sqrt{2}\omega_0 x} & 0 \end{pmatrix}. \tag{27}$$

The time-dependent function $f(t)$ accounts for a possible non-vanishing acceleration.

In order to evolve this metric in a shear-free manner we use equation (20) with $\sigma_{\alpha\beta} = 0$. To simplify things, we assume that the scale parameter S depends only on the cosmological time parameter t. In this way we arrive at

$$(g_{\alpha\beta}(t, x^\gamma)) = \begin{pmatrix} -S^2(t) & 0 & 0 \\ 0 & \frac{1}{2}\left[S^2(t)\left(1 - 2f^2(0)\right) + 2f^2(t)\right] e^{2\sqrt{2}\omega_0 x} & 0 \\ 0 & 0 & -S^2(t) \end{pmatrix}. \tag{28}$$

We like to model the matter content of this spacetime by a fluid with velocity vector $X^i = \delta^i_0$. Thus, the energy–momentum tensor takes the form

$$T_{ik} = -p g_{ik} + (\epsilon + p) X_i X_k + 2q_{(i} X_{k)} + \pi_{ik}, \tag{29}$$

with energy density ϵ, isotropic pressure p, heat flow q_i and anisotropic pressure π_{ik}.

These quantities may be directly calculated via

$$\epsilon = T_{ab} X^a X^b, \tag{30}$$

$$p = -\frac{1}{3} T_{ab} P^{ab}, \tag{31}$$

$$q_i = T_{ab} X^a P^b{}_i, \tag{32}$$

$$\pi_{ij} = T_{ab} P^a{}_i P^b{}_j + p P_{ij}. \tag{33}$$

Models with vanishing acceleration. First we deal with the case of vanishing acceleration and set $f(t) \equiv 1$. The evolved metric then reads

$$(g_{ik}) = \begin{pmatrix} 1 & 0 & e^{\sqrt{2}\omega_0 x} & 0 \\ 0 & -S^2(t) & 0 & 0 \\ e^{\sqrt{2}\omega_0 x} & 0 & \frac{1}{2}e^{2\sqrt{2}\omega_0 x}\left(2 - S^2(t)\right) & 0 \\ 0 & 0 & 0 & -S^2(t) \end{pmatrix} \tag{34}$$

The Einstein tensor of this metric has the non-vanishing components

$$G_{tt} = \frac{1}{S^4}\left(\omega_0^2(3-2S^2) + (2+3S^2)\dot{S}^2 - 4S\ddot{S}\right), \tag{35}$$

$$G_{tx} = -\sqrt{2}\omega_0\frac{\dot{S}}{S^3}, \tag{36}$$

$$G_{ty} = \frac{e^{\sqrt{2}\omega_0 x}}{S^4}\left(\omega_0^2(3-2S^2) + (2+S^2)\dot{S}^2 - 2S(2-S^2)\ddot{S}\right), \tag{37}$$

$$G_{xx} = \frac{1}{S^2}\left(\omega_0^2 - (2+S^2)\dot{S}^2 + 2S(1-S^2)\ddot{S}\right), \tag{38}$$

$$G_{xy} = -\frac{\sqrt{2}\omega_0 e^{\sqrt{2}\omega_0 x}}{2S^3}\left(2+S^2\right)\dot{S}, \tag{39}$$

$$G_{yy} = \frac{e^{2\sqrt{2}\omega_0 x}}{2S^4}\left(3\omega_0^2(2-S^2) + (4-S^4)\dot{S}^2 - 2S(2-S^2)^2\ddot{S}\right), \tag{40}$$

$$G_{zz} = \frac{1}{S^2}\left(\omega_0^2(-1+2S^2) - (2+S^2)\dot{S}^2 + 2S(1-S^2)\ddot{S}\right). \tag{41}$$

The matter content, assuming that the Einstein equation $G_{ik} = T_{ik}$ for the above energy–momentum tensor holds, computes to

$$\epsilon = \frac{1}{S^4}\left(\omega_0^2(3-2S^2) + (2+3S^2)\dot{S} - 4S\ddot{S}\right), \tag{42}$$

$$p = \frac{1}{3S^4}\left(\omega_0^2(1+2S^2) - (2+3S^2)\dot{S}^2 + 2S(2-3S^2)\ddot{S}\right), \tag{43}$$

$$q_x = -\frac{\sqrt{2}\omega_0\dot{S}}{S^3}, \tag{44}$$

$$q_y = \frac{2e^{\sqrt{2}\omega_0 x}}{S^2}\left(-\dot{S}^2 + S\ddot{S}\right), \tag{45}$$

$$\pi_{xx} = \frac{2}{3S^2}\left(\omega_0^2(1-S^2) - 2\dot{S}^2 + S\ddot{S}\right), \tag{46}$$

$$\pi_{xy} = -\frac{\sqrt{2}}{2}\omega_0 e^{\sqrt{2}\omega_0 x}\frac{\dot{S}}{S}, \tag{47}$$

$$\pi_{yy} = \frac{e^{2\sqrt{2}\omega_0 x}}{3S^2}\left(\omega_0^2(1-S^2) + 4\dot{S}^2 - 2S\ddot{S}\right), \tag{48}$$

$$\pi_{zz} = \frac{2}{3S^2}\left(-2\omega_0^2(1-S^2) - 2\dot{S}^2 + S\ddot{S}\right). \tag{49}$$

Furthermore, the model exhibits an interesting causal structure that we will discuss in the following section.

Parallax-free models. We will now choose $f(t)$ such that the model is parallax-free. Calculating the exterior derivative of $\dot{X}_i - \frac{\Theta}{N}X_i$, we find that the observer field of our already shear-free model (28) is parallel to a conformal vector field if and only if f is a constant multiple of S, $f(t) = \frac{C}{\sqrt{2}}S(t)$ – with $|C| > 1$ for g to be Lorentz.

The resulting metric takes the form

$$(g_{ik}) = \begin{pmatrix} 1 & 0 & \frac{C}{2}\sqrt{2}S(t)e^{\sqrt{2}\omega_0 x} & 0 \\ 0 & -S^2(t) & 0 & 0 \\ \frac{C}{2}S(t)\sqrt{2}e^{\sqrt{2}\omega_0 x} & 0 & \frac{1}{2}S^2(t)e^{2\sqrt{2}\omega_0 x} & 0 \\ 0 & 0 & 0 & -S^2(t) \end{pmatrix}. \tag{50}$$

This model is a conformal perturbation of the classical Gödel model. Since the gradient of the conformal factor cannot be timelike, it is not an observer field and thus the construction of a meaningful energy–momentum tensor as e.g. in [**10**] is difficult.

Rotation scalar and the norm of the acceleration are given by

$$\omega_{ab}\omega^{ab} = \frac{C^2\omega_0^2}{(C^2-1)S^2(t)}, \tag{51}$$

$$-\dot{X}_a\dot{X}^a = \frac{C^2}{C^2-1}\frac{\Theta^2}{9}. \tag{52}$$

The magnitude of the acceleration is always less than $\frac{|\Theta|}{3}$; a nice demonstration of the general fact that conformally stationary models with $\Theta^2 > -9\dot{X}_i\dot{X}^i$ admit a time function [**9**].

7. Closed causal curves

The parallax-free generalization is totally vicious since it is the conformal transform of the classical Gödel model.

As for the model with vanishing acceleration, we note that any function on a smooth closed curve $x^i(s)$ is periodic with respect to the curve parameter s. If the curve is contained in a chart of a Lorentzian manifold, this holds in particular for the coordinate function $x^0(s)$ and there exist maximal and minimal points of $x^0(s)$ for which $\frac{dx^0}{ds} = 0$ holds. Let $x^i(s)$ be such a closed curve that is also causal, i.e.

$$g_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} \geq 0 \tag{53}$$

holds for all s. If s_0 is chosen such that $\frac{dx^0}{ds} = 0$ holds (which is always possible by the arguments above) then

$$g_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds}\bigg|_{s=s_0} = g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\bigg|_{s=s_0} \tag{54}$$

is negative if the matrix consisting of the $g_{\alpha\beta}$ is negative definite – which therefore is a sufficient condition for the absence of closed causal curves.

Following basic linear algebra a real square matrix $A = [a_{\alpha\beta}]$ is negative definite if $-A$ is positive definite. According to the Sylvester criterion, this holds if and only if the determinants of all leading principal minors $-A_\gamma$ (beginning with the element $-a_{11}$) are positive.

Examining the generalized Gödel metric with vanishing acceleration (34), one immediately sees that this condition is met for $S^2(t) > 2$. (It should be noted that this argument only works if the coordinate $x^0 = t$ in question can be globally defined.) On the other hand, since totally viciousness is a stable property [**8**], for some $\epsilon > 0$ the open submanifold $S^{-1}((-\epsilon, \epsilon))$ contains closed timelike curves.

Thus, this spacetime is non-totally vicious but non-chronological if $S^2(t) > 2$ for some t.

8. Visualization Techniques and the Behavior of Geodesics and Light Cones

Geodesic curves γ are solutions of the geodesic equation $\ddot{\gamma} = 0$. When a coordinate frame is chosen and when $\gamma = (x^0, x^1, x^2, x^3)$ is parameterized suitably, the geodesic equation turns into a system of ordinary differential equations of second order:

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} = 0 \qquad \text{for} \qquad \alpha = 0, \ldots, 3. \tag{55}$$

Given an initial position $\gamma(0)$ and an initial direction $\gamma'(0)$, the solution $\gamma(\tau)$ is completely determined.

To visualize this four-dimensional curve, we need to find a three-dimensional representation. This is usually done by simply discarding one of the dimensions, but more elaborate schemes are also conceivable. In some cases, we now also want to transform the remaining three coordinate functions to a more convenient coordinate system. Suppose, for example, that we have a spacetime with one angle and one radial coordinate. To get a meaningful pictorial representation of curves in such a coordinate frame, we will want to convert to Cartesian coordinates. Since this article is concerned with picturing the formation of caustics, we will usually show multiple geodesics originating from the same point in spacetime but with different initial directions, akin to what is described in [**11**].

While the geodesic equation itself is covariant and thus independent of choice of coordinates, we have to specify the initial position and direction of a geodesic in a coordinate frame. Therefore, the resulting pictures do depend on the choice of coordinates and one has to distinguish between effects solely caused by the choice of coordinates and physical effects.

Another way to gain a better understanding of the geodesic structure of a spacetime is to examine the behavior of light cones. A light cone C at a point p in the spacetime manifold M is a subset of the tangent space T_pM defined by $C = \{v \in T_pM \mid g_p(v, v) = 0\}$ where g is the spacetime metric. That is, a light cone at a point is a hypersurface in T_pM consisting of all tangent vectors at this point with vanishing length.

To get a structure that can be visualized in conjunction with the geodesics obtained by the procedure described above, we can use the exponential map $\exp : T_pM \supset U \to M$ to obtain $\exp(U \cap C)$. We effectively do just this when we choose a representative sample of null vectors at a point and determine the geodesics with these initial values. If, however, we identify a convex neighborhood of p with U, we can also think of the light cone section $U \cap C$ as residing in M. This is what we do in this article.

In a coordinate frame, $g_p(v, v) = 0$ is a quadratic equation for the components of $v = (v_0, v_1, v_2, v_3)$. To get a parametric representation of the light cone, we choose one of the components v_i for $i \in \{0, \ldots, 3\}$ and solve for it. In general, this yields two solutions for v_i corresponding to the past and the future light cone. Since we are unable to visualize three-dimensional hypersurfaces directly, we need a way to discard one dimension. In this article, we do this by setting $v_j = 0$ for $j \in \{0, \ldots, 3\}$, $j \neq i$ right at the beginning. Thus, we get a parametric representation

$\{v_i(v_k, v_l), v_k, v_j\}$ of a two-dimensional hypersurface which can be readily depicted. Just as above, we will also want to apply a coordinate transformation to the vector components for some spacetimes.

9. Implementation

The procedures described above are implemented in a `Mathematica` package called `GeodesicGeometry`[1]. It provides the means to easily plot geodesics and light cones for arbitrary spacetimes.

To use the package, one first specifies the line element of the spacetime via the constructor `GeodesicGeometry`. One then uses the provided plot routines, for example `PlotNullSpray` for plotting null geodesics or `PlotNullCone` for plotting light cones. To solve the geodesic equations, `GeodesicGeometry` uses `Mathematica`'s numeric capabilities.

10. Gödel's Spacetime

In its original form, the line element of Gödel's spacetime in canonical coordinates (x_0, x_1, x_2, x_3) presented in [**12**] reads

$$\mathrm{d}s^2 = \mathrm{d}x_0^2 - \mathrm{d}x_1^2 + \frac{e^{2x_1}}{2}\mathrm{d}x_2^2 - \mathrm{d}x_3^2 + 2e^{x_1}\mathrm{d}x_0\mathrm{d}x_2. \tag{56}$$

To get a better understanding of the symmetries of the line element, we transform to cylindrical coordinates (t, r, ϕ, z) defined by:

$$\begin{aligned} e^{x_1} &= \cosh(2r) + \cos(\phi)\sinh(2r), \\ x_2 e^{x_1} &= \sqrt{2}\sin(\phi)\sinh(2r), \\ \tan\left(\frac{\phi}{2} + \frac{x_0 - 2s}{2\sqrt{2}}\right) &= e^{-2r}\tan\left(\frac{\phi}{2}\right), \\ x_3 &= 2w. \end{aligned} \tag{57}$$

The line element now takes the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - \mathrm{d}z^2 + \Big(\sinh^4(r) - \sinh^2(r)\Big)\,\mathrm{d}\phi^2 + 2\sqrt{2}\sinh^2(r)\mathrm{d}\phi\mathrm{d}t. \tag{58}$$

Where $t, z \in \mathbb{R}$, $r \geq 0$ and $\phi \in [0, 2\pi]$ with $\phi = 0$ and $\phi = 2\pi$ identified. The mostly irrelevant coordinates x_3 and z can be suppressed to obtain a three-dimensional space that can be visualized.

To reproduce the visual representation of the Gödel spacetime from [**13**, pages 168ff] we would have to plot geodesics originating from $r = 0$. This is not possible, since the coordinates are singular there. We can, however, let the geodesics start from a point very close to $r = 0$. This is done in figure 2. It shows a set of geodesics starting from $\{t = 0, r = 10^{-5}, \phi = 0\}$ with different initial directions. The figure also contains a few light cones and a closed timelike curve. The geodesics develop a caustic and then refocus. The light cones tip over as one moves radially outwards, to the point where they contain the direction ∂_ϕ which thus becomes a closed timelike curve. As was pointed out in [**14**] already, the light cones also open up as one moves outwards so that they always include the matter world lines ∂_t. This is not correctly represented in [**13**] where the light cones always have the same apex angle.

[1]The `GeodesicGeometry` package and more information is available on `http://www.math.tu-berlin.de/~schoenf/GeodesicGeometry/`

At the chosen initial position near the coordinate origin, distributing the initial velocities of the geodesics evenly in the tangent space does not yield an even distribution of geodesics in the manifold. To achieve the latter, we need to use initial velocities of the form

$$\cos\alpha\ \partial_r + \frac{\sin\alpha}{r}\ \partial_\phi \quad \text{for } \alpha \in [0, 2\pi]. \tag{59}$$

That is, we have to increase the angular component of the initial velocity in favor of the radial component.

As shown in figure 3, the geodesics tip over just like the light cones as one moves radially outwards, whereas the formation of caustics and the location of the conjugate points are not affected. A similar plot can also be created for the Cartesian coordinate frame, see figure 4. The qualitative behavior is similar to what happens in cylindrical coordinates: geodesics refocus and show caustics, light cones tip over and open up.

For an example of a closed timelike curve in both coordinate frames, see figure 5. Both curves are non-geodesic. Constructing these curves and looking at the light cones at a few specific places on them is of course no proof that they are actually timelike everywhere. This procedure does, however, provide an intuitive way to get an idea where to look for closed timelike curves.

11. Expanding Generalization of Gödel's Spacetime

The line element belonging to the according metric we constructed before reads

$$\mathrm{d}s^2 = \mathrm{d}t^2 - S^2(t)\mathrm{d}x^2 + \frac{1}{2}e^{2\sqrt{2}x}\left(2 - S^2(t)\right)\mathrm{d}y^2 - S^2(t)\mathrm{d}z^2 + 2e^{\sqrt{2}x}\mathrm{d}t\mathrm{d}y. \tag{60}$$

For our purposes, it is sufficient to consider $S(t) = \exp(1/3\ \Theta\ t)$ with constant expansion Θ.

To be able to visually compare this spacetime with Gödel's original spacetime, we used (57) to transform to cylindrical coordinates. This yields a bulky line element which we suppress, but we visualize some aspects related to this representation. Figure 6 shows six sets of geodesics with identical initial conditions but in spacetimes with different values for the expansion Θ. Without expansion, the structure is reminiscent of what happens in the original Gödel spacetime: the geodesics form a caustic and refocus, only the symmetry in ϕ is slightly disturbed. With growing Θ, however, the refocusing is increasingly dissolved.

In the case of (60), we know that the spacetime is causal if $S^2(t) > 2$. In figure 7, the surface $S(t) = 2$ is shown in the cylindrical frame of the spacetime. The area above the surface is thus free of closed timelike or null curves. This is also exemplarily shown by the behavior of light cones, some of which are also shown in figure 7. Below the surface, they show the characteristic tipping at large radii that makes ∂_ϕ a closed timelike curve. When we move upwards towards the surface, some of the light cones tilt back, up to the point of being upright above the surface. Thus, ∂_ϕ is not causal above the surface.

12. Ori's Time Machine Spacetime

In 2005, Ori presented in [**15**] a time machine spacetime which develops closed timelike curves inside a vacuum core part surrounded by a matter field. We focus here on the vacuum core. The line element reads

$$\mathrm{d}s^2 = 2\,\mathrm{d}z\,\mathrm{d}t - \mathrm{d}x^2 - \mathrm{d}y^2 - (e\rho^2 - t)\,\mathrm{d}z^2 - 2\big((2e-a)\,x\,\mathrm{d}x + (2e+a)\,y\,\mathrm{d}y\big)\,\mathrm{d}z, \tag{61}$$

with $\rho^2 = x^2 + y^2$ and $e, a > 0$. The coordinate z is periodic, $z \in [0, L]$ for some $L > 0$, and $z = 0$ and $z = L$ are identified. The other coordinates take all real values. Thus, the spacetime has the somewhat unusual topology of $\mathbb{R}^3 \times S^1$.

We choose e and a such that $e > (2e + a)^2$. Then we have

$$\begin{aligned} g^{tt} &= e\rho^2 - t - (2e-a)^2 x^2 - (2e+a)^2 y^2 \\ &> e\rho^2 - t - (2e+a)^2 x^2 - (2e+a)^2 y^2 \\ &> -t. \end{aligned} \tag{62}$$

So the hypersurfaces $t = \text{const}$ are spacelike at $t < 0$. For $t \geq 0$, the hypersurfaces are mixed: causal for small ρ and spacelike for large ρ. Since $g_{zz} = t - e\rho^2$, the closed curves ∂_z are timelike at $t > e\rho^2$. This region lies completely inside the region where $t = \text{const}$ is causal. The light cones behave accordingly: for growing t, they open up so that $t = \text{const}$ becomes causal, and they also tip over so that ∂_z becomes causal. These observations are depicted in figure 8, where the coordinate y is suppressed.

Interestingly, the criterion for the hypersurfaces $t = \text{const}$ to be spacelike matches the criterion for the absence of closed timelike or null curves developed before. It says that there can be no closed timelike or null curves wherever the spatial part of the metric is negative definite. The determinant of the spatial part of Ori's core metric turns out to be $-g^{tt}$. So this criterion says that there cannot be any closed timelike or null curves wherever $g^{tt} > 0$, which is exactly where the hypersurfaces $t = \text{const}$ are spacelike.

13. Conclusion

The construction of cosmological models with given kinematical invariants is a relevant task. On the one hand, it is an important problem in mathematical cosmology to examine possible connections between the kinematics of a cosmological model and its geometric properties like singularities and causality. On the other hand, astrophysical observations put constraints to the possible kinematical quantities of our universe. With the description of spacetime metrics presented here, these constraints can be directly accounted for.

The usefulness of our method is demonstrated by the given examples. The parallax-free generalization of the Gödel metric presented here seems to be more natural in terms of symmetry since we have just relaxed the property of a Killing observer field to an observer field parallel to a conformal vector field. The model with vanishing acceleration on the other hand has an interesting geometric structure with causality violating regions.

Following the presentation of the visualization method it turns out that this method can be easily applied for other models as we did for Ori's time machine.

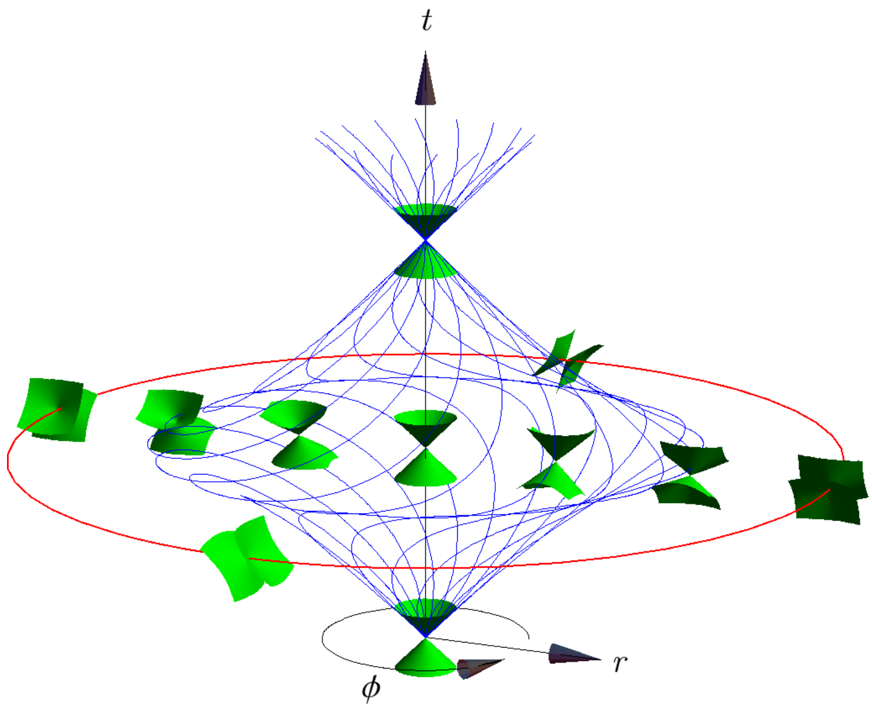


Figure 2: Crucial parts of the figure from [**13**], pages 168ff for Gödel's spacetime in cylindrical coordinates. A set of geodesics starting from very close to the coordinate origin is shown along with a few light cones which exhibit the tipping effect that signals the existence of closed timelike curves, one of which is also depicted.

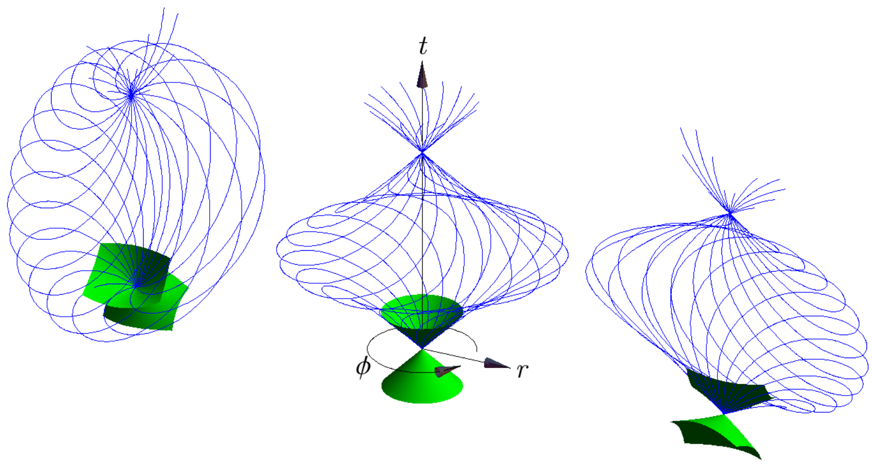


Figure 3: Three sets of geodesics originating from points with different radial coordinates in Gödel's spacetime. Just as the light cones, the geodesics tip over. Their refocusing is not affected.

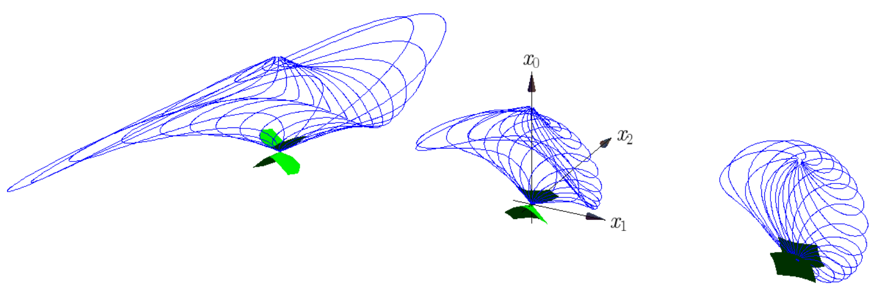
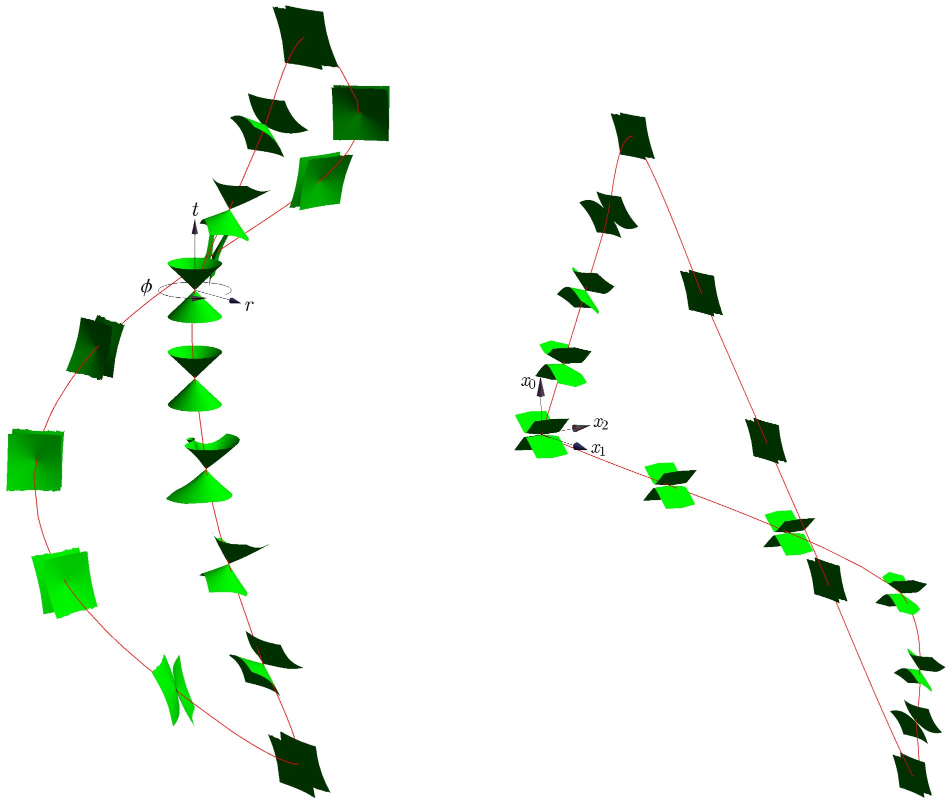


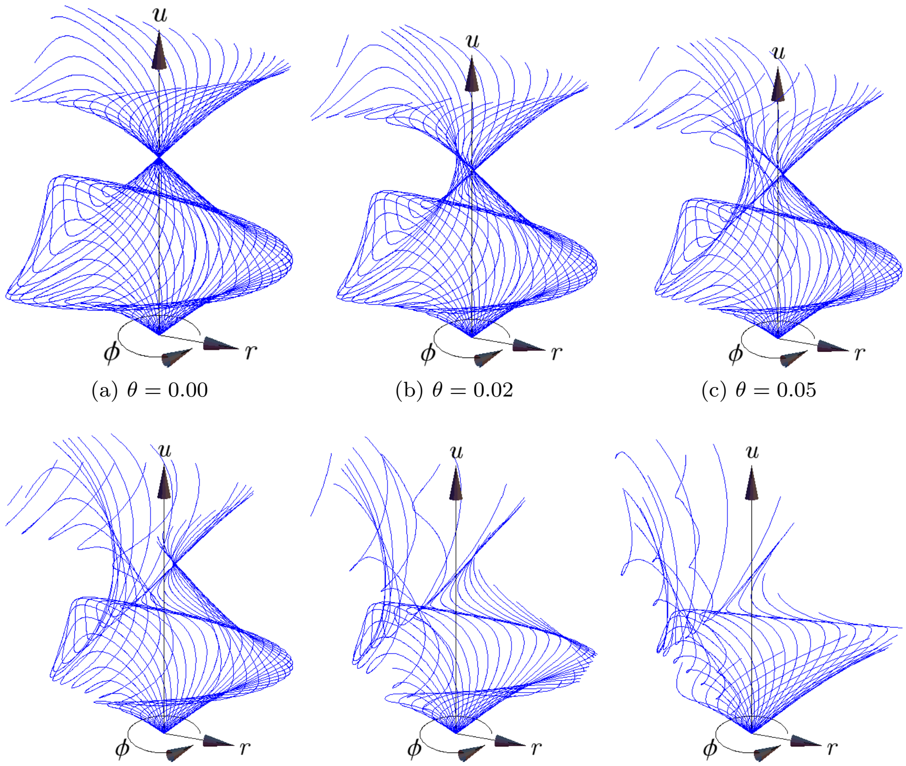
Figure 4: Three sets of geodesics originating from points with different x_1 values in Gödel's spacetime. Just like the light cones, they tip over as one moves to positive x_1 values, and they open up in the x_2 direction as one moves to negative x_1 values.



(a) Cylindrical frame

(b) Cartesian frame

Figure 5: Examples of closed timelike curves in Gödel's spacetime. The shown curves are interpolation curves between the positions of the light cones.



(a) $\theta = 0.00$ (b) $\theta = 0.02$ (c) $\theta = 0.05$

(d) $\theta = 0.10$ (e) $\theta = 0.20$ (f) $\theta = 0.40$

Figure 6: Six sets of geodesics in the expanding spacetime (60) with identical initial conditions but with different values for the expansion Θ.

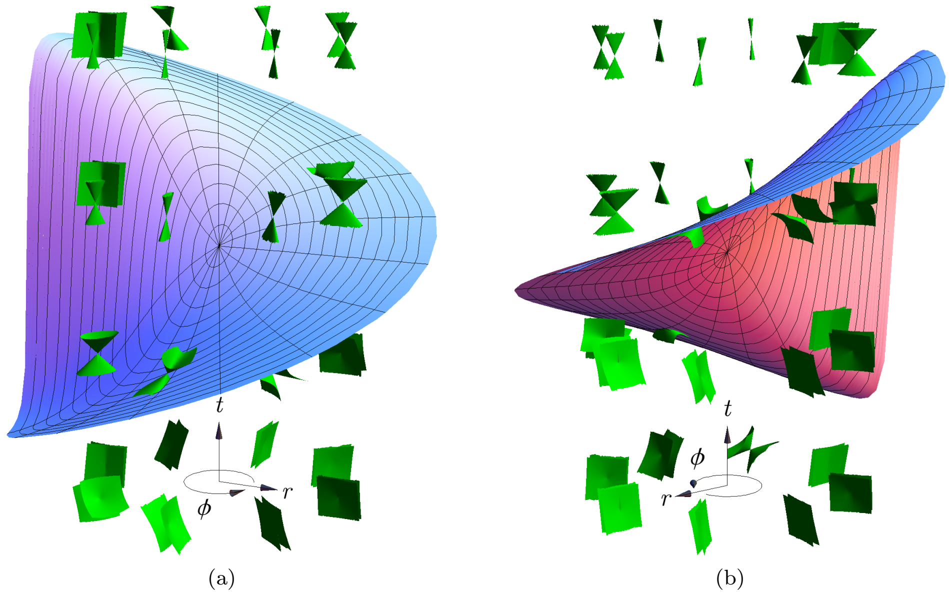


Figure 7: Causality change in the expanding spacetime (60) with $\Theta = 0.4$. The area above the surface is known to be free of closed timelike or null curves.

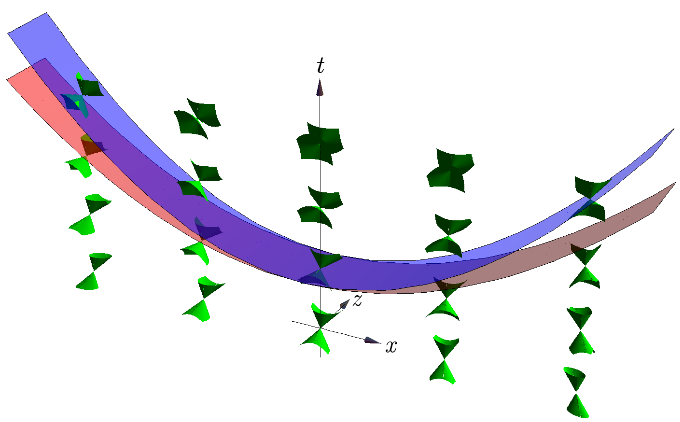


Figure 8: Behavior of light cones in Ori's time machine spacetime. The coordinate y is suppressed, and $e = 1/8$ and $a = 1/16$ are used. The dark contour marks the causality border of the hypersurfaces $t = \text{const}$. Similarly, the lighter contour marks the causality border for the directions ∂_z.

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M. Gürses: Department of Mathematics, Faculty of Sciences Bilkent University, 06800 Ankara - Turkey

M. Plaue, M. Scherfner and T. Schönfeld: Department of Mathematics, Technische Universität Berlin, Str. d. 17. Juni 136, 10623 Berlin, Germany

L. A. M. de Sousa: Department of Mathematics, Universidade Federal do Estado do Rio de Janeiro, Av. Pasteur, 458, Urca, Rio de Janeiro, Brazil