MATH. 346
DIFFERENTIAL GEOMETRY II
ASSIGNED EXERCISES VI

The Calculus of Variations and Geometry (Chapter 8 of Oprea: Differential Geometry and Its Application)

Fixed End Point Problem
Find the curve $y = y(t)$ with $y(t_0) = y_0$ and $y(t_1) = y_1$ such that the following integral is minimized

$$J = \int_{t_0}^{t_1} L(t, y(t), \dot{y}(t))\, dt$$

where $L(t, y(t), \dot{y}(t))$ is a function of t, y and $\dot{y} = \frac{dy}{dt}$. Suppose that $y(t)$ is a curve which minimizes the integral $J = \int_{t_0}^{t_1} L(t, y(t), \dot{y}(t))\, dt$ and let $\bar{y}(t) = y(t) + \epsilon\, \eta(t)$ be a variation of y. That is, we think of ϵ as being small and we require that $\eta(t_0) = 0$ and $\eta(t_1) = 0$. Therefore, the curve $\bar{y}(t)$ still joins the points y_0 and y_1 as well as being "close" to y. Note that $\dot{\bar{y}} = \dot{y} + \epsilon\, \dot{\eta}$. With this notation, we can think of the integral J as a function of the parameter ϵ

$$J(\epsilon) = \int_{t_0}^{t_1} L(t, \bar{y}(t), \dot{\bar{y}}(t))\, dt = \int_{t_0}^{t_1} L(t, y(t) + \epsilon\eta(t), \dot{y}(t) + \epsilon\, \dot{\eta}(t))\, dt$$

In order to recognize the minimum $y(t)$, just as we do in ordinary calculus, we take the derivative of J with respect to ϵ and note that, since $y(t)$ is a minimum by hypothesis, this derivative is zero for $\epsilon = 0$. This is called the variational derivative, δJ, of J,

$$\delta J = (\frac{dJ}{d\epsilon})|_{\epsilon=0} = 0$$

Now,

$$\frac{dJ}{d\epsilon} = \int_{t_0}^{t_1} [\frac{\partial L}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial \epsilon} + \frac{\partial L}{\partial \dot{\bar{y}}} \frac{\partial \dot{\bar{y}}}{\partial \epsilon}]\, dt$$

by the chain rule, hence

$$\frac{dJ}{d\epsilon} = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial \bar{y}} \eta + \frac{\partial L}{\partial \dot{\bar{y}}} \dot{\eta} \right] dt$$

with

$$\left. \frac{dJ}{d\epsilon} \right|_{\epsilon=0} = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial y} \eta + \frac{\partial L}{\partial \dot{y}} \dot{\eta} \right] dt = 0$$

since, at $\epsilon = 0$, $\bar{y}(t) = y(t)$. Now, the second term inside the integral may be integrated by parts

$$0 = \int_{t_0}^{t_1} \eta \left[\frac{\partial L}{\partial y} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) \right] dt + \eta \left. \frac{\partial L}{\partial \dot{y}} \right|_{t0}^{t_1}$$

since $\eta(t_0) = 0$ and $\eta(t_1) = 0$ we get

$$0 = \int_{t_0}^{t_1} \eta \left[\frac{\partial L}{\partial y} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) \right] dt$$

This equation must hold for every function η with $\eta(t_0) = 0$ and $\eta(t_1) = 0$.

Exercise 1. Show that

$$\frac{\partial L}{\partial y} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) = 0$$

This equation is called the *Euler-Lagrange equation* and it gives a necessary condition for $y(t)$ to be a minimum. The function L is called the Lagrange function. So we have the following theorem:

Theorem 1. *If $y(t)$ is a minimum for the fixed endpoint problem then y satisfies the Euler-Lagrange equation*

When there exists more dependent variables we follow a similar approach. Suppose that J depends on two functions $y(t)$ and $z(t)$,

$$J = \int_{t_0}^{t_1} L(t, y(t), z(t), \dot{y}(t), \dot{z}(t))\, dt$$

Show that the variations $\bar{y}(t) = y(t) + \epsilon\,\eta$ and $\bar{z}(t) = z(t) + \epsilon\,\tau$ lead to

$$\begin{aligned} 0 &= \left.\frac{dJ}{d\epsilon}\right|_{\epsilon=0} \\ &= \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial y}\,\eta + \frac{\partial L}{\partial \dot{y}}\,\dot{\eta} + \frac{\partial L}{\partial z}\,\tau + \frac{\partial L}{\partial \dot{z}}\,\dot{\tau}\right] dt, \\ &= \int_{t_0}^{t_1} \eta \left[\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)\right] dt + \int_{t_0}^{t_1} \tau \left[\frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)\right] dt \end{aligned}$$

and that this equation can hold for all η, τ if and only if the following *Euler-Lagrange equations hold*,

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0, \quad \frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0.$$

Example 1. **The Brachistrochrone problem**. Given a point (a, b) in the x, y-plane, find the curve $y(x)$ joining (a, b) to the origin so that a bead, under the influence of gravity, sliding along a frictionless wire in the shape $y(x)$ from (a, b) to $(0, 0)$ minimizes the time of travel

$$J = \frac{1}{\sqrt{2g}} \int_a^0 \frac{\sqrt{1 + (y')^2}}{\sqrt{b - y}} dx$$

with $y(a) = b$ and $y(0) = 0$. Hence the Lagrange function is

$$L = \frac{1}{\sqrt{2g}} \frac{\sqrt{1 + (y')^2}}{\sqrt{b - y}}$$

Letting $u = y - b$ and using the Euler-Lagrange equation we find that

$$\frac{\sqrt{1 + (u')^2}}{\sqrt{u}} - \frac{(u')^2}{\sqrt{u(1 + (u')^2)}} = c \neq 0$$

we find that

$$u(1 + (u')^2) = \frac{1}{c^2}$$

or

$$x = \int \sqrt{\frac{uc^2}{1 - uc^2}} du = -2 \int \sqrt{k^2 - w^2} dw$$

where $w = \sqrt{k^2 - u}$ and $k = \frac{1}{c}$. We find that

$$x = -\frac{c^2}{2}(\phi + \sin\phi) + r$$

where $b - y = u = \frac{1}{c^2}\cos^2\frac{\phi}{2}$, then letting $\zeta = \phi - \pi$, then

$$x(\zeta) = \lambda(\zeta - \sin\zeta) + a, \quad y(\zeta) = \lambda(1 - \cos\zeta) + b$$

This curve is a cycloid.

Example 2.Shortest distance curves in the Plane: Geodesics. Given points (a, b) and (c, d) in the plane, we can ask for the curve $(x(t), y(t))$ which joins them and which minimizes the arc length integral

$$J = \int_{t_0}^{t_1} \sqrt{(\dot{x}(t))^2 + (\dot{x}(t))^2}\, dt$$

hence we have two dependent variable. We have Euler-Lagrange equation for each variable x and y.

$$\frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{(\dot{x}(t))^2 + (\dot{x}(t))^2}} = c, \quad \frac{\partial L}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{(\dot{x}(t))^2 + (\dot{x}(t))^2}} = d,$$

Here c and d are constants. Dividing them we get

$$\frac{\dot{y}}{d} = \frac{\dot{x}}{c}$$

which gives

$$y = \frac{d}{c}\, x + r$$

This is the equation of straight line in the plane.