MATH 543: METHODS OF APPLIED MATHEMATICS I

Final Exam:

10 January 2001 Wednesday 09.30-12.30 SAZ:19

QUESTIONS: Solve at least **five** questions.

(1). [25] Find the large eigenvalues and the corresponding eigenfunctions of the problem ($\lambda >> 1$).

$$\frac{d^2\,u(x)}{dx^2} + \lambda\,(\pi + x)^4\,u(x) = 0,$$
$$u(0) = u(\pi) = 0.$$

(**Hint**: Use the WKB approximation method)

(2). [25] Verify the following approximations for large λ

$$\textbf{i.} \quad \int_0^\infty e^{-\lambda\,t}\,\ln(1+t^2)dt = \frac{2}{\lambda^3} - \frac{1}{2}\,\frac{4!}{\lambda^5} + O(\frac{1}{\lambda^7})$$
$$\textbf{ii.} \quad \int_0^\pi e^{\lambda \sin t}\,dt = e^\lambda \sqrt{\frac{2\pi}{\lambda}} + \cdots$$

(3). [25] Find uniform asymptotic approximation to the solutions of the following problem on $0 \leq t \leq 1$ with $0 < \varepsilon \leq 1$

$$\varepsilon\,u'' + (x^2+1)\,u' - x^3\,u = 0,$$
$$u(0) = u(1) = 1.$$

(4). [25] Expand the Dirac delta function (distribution) $\delta(\mathbf{x}-\mathbf{y})$ interms of the Legendre polynomials.

(5). [25] Find the solution initial value problem (w is a real constant and $f(x)$ is an integrable function)

$$\frac{d^2u(x)}{dx^2} + w^2 u(x) = f(x), \quad x \in [0,1]$$
$$u(0) = a, \qquad \frac{du}{dx}|_{x=0} = b.$$

(6). [25] Let $P_n(x)$ denote the Legendre polynomials and

$$\frac{1}{\sqrt{1 - 2tx + t^2}} = \sum_{n=0}^{\infty} t^n \, P_n(x).$$

Prove the following

$$\begin{array}{ll} \textbf{i.} & \displaystyle\int_{-1}^{1} P_n(x)\, P_m(x) dx = \frac{2}{2n+1}\, \delta_{nm}, \\ \textbf{ii.} & P'_{n-1} - x\, P'_n + n\, P_n = 0, \\ \textbf{iii.} & (1 - x^2)\, P'_n + nxP_n - nP_{n-1} = 0, \end{array}$$

iv. Using **ii-iii** find the second order differential equation satisfied by $P_n(x)$.

(7). [25] Find the most general solution of the equation

$$2x\, u'' + (1 - 2x)\, u' - u = 0.$$

(**Hint**: Use the series expansion and the Frobenius method)