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Symmetric space property and an inverse scattering formulation of the SAS Einstein–Maxwell field equations

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We formulate stationary axially symmetric (SAS) Einstein–Maxwell fields in the framework of harmonic mappings of Riemannian manifolds and show that the configuration space of the fields is a symmetric space. This result enables us to embed the configuration space into an eight-dimensional flat manifold and formulate SAS Einstein–Maxwell fields as a σ-model. We then give, in a coordinate free way, a Belinskii–Zakharov type of an inverse scattering transform technique for the field equations supplemented by a reduction scheme similar to that of Zakharov–Mikhailov and Mikhailov–Yarimchuk.

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1. INTRODUCTION

Completely integrable systems and, in connection with these, Bäcklund transformations have attracted much attention in recent years. As a result we now have a better understanding of the nature of certain nonlinear partial differential equations of mathematical physics, and this enables us to devise methods for systematic generation of exact solutions at least in two dimensions. One of these methods is the inverse scattering transform technique of Belinskii and Zakharov.[1,2] It consists of (1) representation of the nonlinear system in the form of compatibility conditions of a more general overdetermined system of linear matrix equations depending on a complex spectral parameter; (2) explicit integration of a Bäcklund transformation for these equations, thus generating new solutions from the known ones.

The Belinskii–Zakharov integration technique was first applied, by the authors, to the Einstein vacuum field equations where the space-time admits two commuting Killing vectors. These authors obtained all multisoliton solutions of Einstein's equations for stationary axially symmetric (SAS) vacuum and colliding plane gravitational wave space-times. The method was later extended and applied by Aleksejev[3] to SAS Einstein–Maxwell equations. In both formulations the parametrization of the problem was such that the relevant linear eigenvalue equation contained the space-time metric functions directly.

In this work we present, in a coordinate-free way, a different formulation of the inverse scattering transform technique of Belinskii and Zakharov for the integration of SAS Einstein–Maxwell field equations written in terms of complex Ernst[4] potentials. In Sec. 2, we first show that the use of Ernst potentials enables one to formulate SAS Einstein–Maxwell field equations as equations determining harmonic mappings[5,6] from a base manifold, which is a three-dimensional flat space, to a four-dimensional Riemannian manifold called the configuration space of the fields. It turns out that the configuration space is a Riemannian symmetric space with the isometry group SU(2,1), in agreement with previous results[7–9] that Ernst equations are invariant under the action of this group. This property enables us to write the kinematical content of the theory in terms of the Maurer–Cartan equations for SU(2,1) while the dynamical content (i.e., the field equations) appear in the form of conservation of Noether currents, both expressed in terms of the set of eight Killing vectors that the configuration space admits. Using 3×3 matrix representation of the generators of SU(2,1) we define a Lie algebra valued flat connection 1-form which can be integrated to give the 3×3 Hermitian matrix, found recently by Gürses and Xanthopoulos,[10] characterizing SAS Einstein–Maxwell fields as a σ-model. The symmetric space property of the configuration space is reflected by the fact that this matrix leaves invariant the metric of the three-dimensional complex vector space on which the group SU(2,1) acts. As a final remark of Sec. 2, we note that the configuration space can be embedded into an eight-dimensional flat space, generalizing the result of Matzner and Misner[11] for SAS Einstein vacuum to the electrovacuum case.

In Sec. 3, we present a Belinskii–Zakharov type of inverse scattering formulation for the integration of SAS Einstein–Maxwell field equations. Using the flat connection 1-form of Sec. 2 and its Hodge dual with respect to a two-dimensional Euclidean space E^2 and introducing a complex spectral parameter, we construct a new connection 1-form defined on $E^2\times C$ whose curvature vanishes modulo the field equations. The associated linear eigenvalue equation follows immediately while gauge transformations of the connection are nothing but Bäcklund transformation for the field equations. Using the technique invented by Belinskii and Zakharov, a particular form of Bäcklund transformation can be integrated explicitly for the 3×3 matrix, characterizing the solution in terms of a known solution. The procedure, however, does not guarantee the symmetric space

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property of the 3×3 matrix, which is crucial for the parametrization of SAS Einstein–Maxwell fields. To solve this problem, one has to modify the integration scheme by imposing additional conditions an the Bäcklund transformation. This, the so-called reduction problem, was solved by Zakharov and Mikhailov[12] and Mikhailov and Yarimchuk,[13] and is the subject of Sec. 4. We also include an appendix for the details of Sec. 2.

2. THE CONFIGURATION SPACE FOR SAS EINSTEIN–MAXWELL FIELDS

The equations governing SAS Einstein–Maxwell fields in terms of complex Ernst potentials $\epsilon(\rho,z)$ and $\Phi(\rho,z)$ are

$$(\epsilon + \bar{\epsilon} + 2\Phi\bar{\Phi})\nabla^2\epsilon = 2(\nabla\epsilon + 2\bar{\Phi}\nabla\Phi)\cdot\nabla\epsilon,$$
$$(\epsilon + \bar{\epsilon} + 2\Phi\bar{\Phi})\nabla^2\Phi = 2(\nabla\epsilon + 2\bar{\Phi}\nabla\Phi)\cdot\nabla\Phi. \quad (2.1)$$

Here ∇ and ∇^2 are, respectively, the flat space gradient and Laplace operators in cylindrical coordinates $\{\rho,z,\phi\}$. Equations in (2.1) can also be regarded as equations determining harmonic mappings $f: M \to M'$, where M and M' are two Riemannian manifolds with metrics

$$M: ds^2 = d\rho^2 + dz^2 + \rho^2\, d\phi^2, \quad (2.2)$$

$$M': ds'^2 = F^{-2}|d\epsilon + 2\bar{\Phi}\, d\Phi|^2 - 4F^{-1}\, d\Phi\, d\bar{\Phi}, \quad (2.3)$$

where $F = \frac{1}{2}(\epsilon + \bar{\epsilon}) + \Phi\bar{\Phi}$. To see this, it is enough to consider the set of basis 1-forms ω^A $(A,B,... = 1,2,3,4)$ of the cotangent space $T^*(M')$ of M', which are orthonormal with respect to the metric g' of M'. These satisfy the integrability conditions $d\omega^A + \omega^A{}_B \wedge \omega^B = 0$, which enables one to determine the connection 1-forms $\omega_{AB} = -\omega_{BA}$. The basis 1-forms ω^A along with the connection 1-forms $\omega^A{}_B$, when pulled back to M using the map f, $\sigma^A = f^*\circ\omega^A$ and $\Omega^A{}_B = f^*\circ\omega^A{}_B$, satisfy

$$d\sigma^A + \Omega^A{}_B \wedge \sigma^B = 0, \quad (2.4)$$

displaying the Riemannian structure of the induced vector bundle $f^{-1}T(M') \to M$ transported from that of the tangent bundle $T(M') \to M'$. Introducing now the Hodge dual operation (*), which is determined by the Riemannian structure of M, the field equations (2.1) can be written as

$$d{*}\sigma^A + \Omega^A{}_B \wedge {*}\sigma^B = 0, \quad (2.5)$$

while the variational principle for the problem is $\delta I = 0$ with

$$I = \frac{1}{2}\int_M \sigma_A \wedge {*}\sigma^A. \quad (2.6)$$

The Riemannian manifold M' with the metric given in Eq. (2.3) is called the configuration space for SAS Einstein–Maxwell fields and is a Riemannian symmetric space. This property implies that there exists eight Killing vectors generating the isometry group SU(2,1) of M' and that the line element (2.3) can be written as (see the Appendix for details)

$$ds'^2 = \tfrac{1}{2}\,\eta_{\mu\nu}\tau^\mu \otimes \tau^\nu, \quad (2.7)$$

where τ^μ $(\mu,\nu,\cdots = 1,2,...,8)$ are the corresponding Killing 1-forms satisfying the Maurer–Cartan equations[14]

$$d\tau^\mu + \tfrac{1}{2}C^\mu{}_{\alpha\beta}\tau^\alpha \wedge \tau^\beta = 0, \quad (2.8)$$

with $C^\mu{}_{\alpha\beta}$ being the structure constants of SU(2,1) and

$$\eta_{\mu\nu} = \tfrac{1}{6}C^\alpha{}_{\mu\beta}C^\beta{}_{\nu\alpha} \quad (2.9)$$

being the constant group metric. (See the Appendix for details.) The invariance of the action integral (2.6) under SU(2,1) implies the existence of eight Noether currents $f^*\circ\tau^\mu$, which are conserved, i.e.,

$$d{*}(f^*\circ\tau^\mu) = 0. \quad (2.10)$$

As it is for the case of a symmetric space, these conservation laws are in one-to-one correspondence with the field equations (2.5) or equivalently (2.1).

Using the 3×3 matrix representation of the generators X_μ of SU(2,1) we can now define Lie algebra valued connection 1-form

$$W = X_\mu \tau^\mu, \quad (2.11)$$

which, because of the Maurer–Cartan equations (2.8), satisfies

$$dW + W \wedge W = 0. \quad (2.12)$$

This implies that the curvature of W vanishes identically; hence

$$W = -(dP)P^{-1}, \quad (2.13)$$

where P is a 3×3 Hermitian matrix with unit determinant. Using Eqs. (2.11) and (2.13) the matrix P can be determined, up to a constant gauge transformation, as

$$P = F^{-1}\begin{pmatrix} 1 & \sqrt{2}\Phi & i(\bar{\epsilon} - \epsilon + 2\Phi\bar{\Phi})/2 \\ \sqrt{2}\bar{\Phi} & -(\epsilon + \bar{\epsilon} - 2\Phi\bar{\Phi})/2 & -i\sqrt{2}\epsilon\bar{\Phi} \\ i(\bar{\epsilon} - \epsilon - 2\Phi\bar{\Phi})/2 & i\sqrt{2}\bar{\epsilon}\Phi & \epsilon\bar{\epsilon} \end{pmatrix}, \quad (2.14)$$

where

$$F = \tfrac{1}{2}(\epsilon + \bar{\epsilon} + 2\Phi\bar{\Phi}),$$

in agreement with the results of Gürses and Xanthopoulos.[10] Because of Eqs. (2.10) and (2.13) the field equations (2.1) can be written in terms of the matrix P as

$$d\,[({*}dP)\,P^{-1}] = 0. \quad (2.15)$$

This form for the SAS Einstein–Maxwell field equations is particularly suitable for the application of inverse scattering transform techniques for generation of soliton solutions.

The matrix P given in (2.14) leaves the metric of the complex vector space on which the group SU(2,1) acts invariant, i.e.,

$$P\gamma P = \gamma, \quad (2.16)$$

where

$$\gamma = \begin{pmatrix} 0 & 0 & i \\ 0 & -1 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \gamma^2 = I. \quad (2.17)$$

By defining $Z = \gamma P$, the configuration space can be considered as the four-dimensional hypersurface $Z^2 = I$, embedded in an eight-dimensional flat space with the metric

$$ds'^2 = -\tfrac{1}{2}\,\mathrm{Tr}(dZ \otimes dZ), \tag{2.18}$$

where

$$Z = i\begin{pmatrix} Z_8 - \tfrac{1}{3}iZ_2 & \sqrt{2}(Z_7 + iZ_6) & Z_3 \\ \sqrt{2}(-Z_4 + iZ_5) & i + \tfrac{2}{3}iZ_2 & \sqrt{2}(Z_6 + iZ_7) \\ -Z_1 & \sqrt{2}(-Z_5 + iZ_4) & -Z_8 - \tfrac{1}{3}iZ_2 \end{pmatrix}. \tag{2.19}$$

The embedding equations are

$$\begin{aligned} &Z_1 = F^{-1}, \quad Z_2 = -3\Phi\overline{\Phi}/F, \\ &Z_3 = \epsilon\bar{\epsilon}/F, \quad Z_4 = \tfrac{1}{2}i(\Phi - \overline{\Phi})/F, \\ &Z_5 = \tfrac{1}{2}(\Phi + \overline{\Phi})/F, \quad Z_6 = \tfrac{1}{2}(\bar{\epsilon}\Phi + \epsilon\overline{\Phi})/F, \\ &Z_7 = \tfrac{1}{2}i(\bar{\epsilon}\Phi - \epsilon\overline{\Phi})/F, \quad Z_8 = i(\bar{\epsilon} - \epsilon)/2F, \end{aligned} \tag{2.20}$$

and the line element (2.18) becomes

$$\begin{aligned} -ds'^2 = (&dZ^1 \otimes dZ^3 + \tfrac{1}{3}\,dZ^2 \otimes dZ^2 + 4\,dZ^4 \otimes dZ^7 \\ &+ 4\,dZ^5 \otimes dZ^6 - dZ^8 \otimes dZ^8), \end{aligned} \tag{2.21}$$

which has zero signature. This generalizes the results of Matzner and Misner[11] for the SAS Einstein vacuum to the SAS Einstein–Maxwell case.

3. THE INVERSE SCATTERING TRANSFORM TECHNIQUE

Even though the base manifold M, whose metric is given by Eq. (2.2), is a three-dimensional flat space, the fact that the map f is independent of the azimuthal coordinate ϕ makes M effectively two-dimensional. For a coordinate-free formulation of the inverse scattering transform technique, we shall from now on consider the two-dimensional Euclidean space E^2 as the base manifold M. Therefore, in what follows the Hodge dual operation (*) should be understood as the one defined with respect to the Riemannian structure on E^2. The field equation given by (2.15) will then read

$$d\,[\Delta\,({}^*dP)P^{-1}] = 0, \tag{3.1}$$

where Δ is a scalar function on E^2, satisfying

$$d\,(\Delta^{-1}{}^*\,d\Delta) = 0. \tag{3.2}$$

The explicit functional form of Δ depends on the particular choice of local coordinates by

$$\Delta = |\det g|^{1/2}, \tag{3.3}$$

where g is the metric of the three-dimensional base manifold M of the previous section.

The flat connection 1-form W defined in Eqs. (2.13) and (2.14) is not suitable for the application of an inverse scattering transform technique to generate soliton solutions of the field equations. This is because its curvature vanishes identically without any references to the field equations. What is needed is a connection whose curvature vanishes on the solution submanifold (i.e., modulo the field equations) and, furthermore, contains a complex parameter in such a way that the connection defined in Eq. (2.13) is obtained in the limit that the value of the parameter goes to zero. For this purpose we introduce a generalized exterior derivative operator D, satisfying $D^2 = 0$, by

$$D \equiv d - \left(\frac{\partial\theta}{\partial\lambda}\right)^{-1} d\theta\,\frac{\partial}{\partial\lambda}, \tag{3.4}$$

where λ is a complex parameter independent of the coordinate on E^2 and $O(\lambda,E^2)$ is any scalar function with the property

$$\lim_{\lambda\to 0} D = d. \tag{3.5}$$

Next we consider a linear eigenvalue problem for a 3×3 matrix $\Psi(\lambda,E^2)$ written as

$$D\Psi = -\Omega\Psi, \tag{3.6}$$

where

$$\Omega = aW + b\Delta\,{}^*W, \tag{3.7}$$

and $a(\lambda,E^2)$ and $b(\lambda,E^2)$ are complex functions defined on $E^2\times C$ satisfying

$$\lim_{\lambda\to 0} a(\lambda,E^2) = 1, \quad \lim_{\lambda\to 0} b(\lambda,E^2) = 0. \tag{3.8}$$

Integrability of Eq. (3.6) requires

$$D\Omega + \Omega\wedge\Omega = 0 \tag{3.9}$$

on the solution submanifold which restricts further the functions a and b to satisfy

$$a^2 + \Delta^2 b^2 - a = 0 \tag{3.10}$$

and

$$Da = \Delta\,{}^*Db. \tag{3.11}$$

With this choice we now have a connection 1-form which is integrable because of the field equations (3.1) and which satisfies

$$\lim_{\lambda\to 0} \Omega = W. \tag{3.12}$$

Using Eqs. (3.6) and (3.12), we see that the matrix P can be identified as

$$P(E^2) = \lim_{\lambda\to 0} \Psi(\lambda,E^2). \tag{3.13}$$

Except for the condition given by Eq. (3.5) the function $\theta(\lambda,E^2)$ is arbitrary, with different choices leading to different functions a and b and hence to a different linear eigenvalue problem. But, as will be seen later, its choice is crucial for the explicit integration of a Bäcklund transformation using a solution of the "matrix Riemann problem." In theory, given a proper function $\theta(\lambda,E^2)$ Eqs. (3.10) and (3.11) can be solved for a and b satisfying the conditions given in Eq. (3.8). These functions together with the definition of the connection 1-form Ω, given in Eq. (3.7), determine the linear eigenvalue Eq. (3.6) completely.

We are now at a position in which we can apply the inverse scattering transform technique of Belinskii and Zakharov to the problem under consideration. For this purpose we assume the knowledge of a particular solution P_0 of (3.1) in terms of which we construct the corresponding connection 1-forms $W_0 = -(dP_0)P_0^{-1}$, $\Omega_0 = aW_0 + b\Delta\,{}^*W_0$, and hence the function $\Psi_0(\lambda,E^2)$, by solving Eq. (3.6). A transformation of the form

$$\Psi = \chi\Psi_0 \tag{3.14}$$

defines a new matrix Ψ leading to a new solution P [using Eq. (3.13)] of (3.1), where, by Eq. (3.6), the matrix χ is restricted to solutions of

$$D\chi = \chi\Omega_0 - \Omega\chi. \tag{3.15}$$

Even though the above procedure guarantees that P is a solution to Eq. (3.1), it does not yet provide solutions to SAS Einstein–Maxwell equations. To represent SAS Einstein–Maxwell fields, the new matrix P must also satisfy the symmetric space property (2.16) and must be Hermitian. These additional conditions, which can be demonstrated easily to be consistent with the field equation (3.1), put additional restrictions on the matrices Ψ and χ. This, the so-called reduction problem, will be the subject of the next section.

4. THE REDUCTION AND INTEGRATION

In order for a solution P of (3.1) to represent SAS Einstein–Maxwell fields, it must be consistent with the parametrization given by Eq. (2.14). This means that the matrix P should be Hermitian ($P = P^\dagger$) and, furthermore, must satisfy the symmetric space property

$$(\gamma P)^2 = I, \tag{4.1}$$

where γ is given in Eq. (2.17). It can easily be shown that these conditions restrict the connection 1-form W to satisfy

$$WP - PW^\dagger = 0, \quad \gamma W + W^\dagger\gamma = 0, \tag{4.2}$$

which in turn imply that we should have

$$\Omega(\lambda)P - P\Omega^\dagger(\bar\lambda) = 0, \quad \gamma\Omega(\lambda) + \Omega^\dagger(\bar\lambda)\gamma = 0. \tag{4.3}$$

With these in mind, if we now reconsider the linear eigenvalue equation (3.6) and Eq. (3.15) for χ, we find that

$$P\gamma\Psi(\tau) = \Psi(\lambda)J, \quad \Psi^\dagger(\bar\lambda)\gamma = \gamma\Psi^{-1}(\lambda), \tag{4.4}$$

and that

$$\chi^{-1}(\lambda) = \gamma\chi^\dagger(\bar\lambda)\gamma, \quad P = \chi(\bar\lambda)P_0\chi^\dagger(\tau), \tag{4.5}$$

where J is a 3×3 matrix satisfying $DJ = 0$, $J^2 = I$, and τ: $\lambda \to \tau(\lambda, E^2)$ is a fractional linear transformation on the complex λ plane with $\tau^2 = I$ leaving the function $\theta(\lambda, E^2)$ invariant. The functions $a(\lambda, E^2)$ and $b(\lambda, E^2)$ transform under τ as

$$\begin{aligned} a(\tau, E^2) &= 1 - a(\lambda, E^2),\\ b(\tau, E^2) &= -b(\lambda, E^2), \end{aligned} \tag{4.6}$$

leaving conditions (3.10) and (3.11) invariant.

The remaining problem is explicit construction of the matrix χ, satisfying the requirements in (4.5) with a given P_0. This can be carried out as done by Belinskii and Zakharov,[1,2] Zakharov and Mikhailov,[12] Mikhailov and Yarimchuk,[13] and Eriş and Gürses[15] once the local coordinates on E^2 and the set of functions $\{\theta, a, b\}$ are fixed. For the integration of SAS Einstein–Maxwell equations a convenient choice is the one given by Belinskii and Zakharov, namely,

$$\begin{aligned} &\Delta = \rho, \quad \theta(\lambda,\rho,z) = \rho^2/2\lambda - \lambda/2 - z,\\ &a(\lambda,\rho) = \rho^2/(\lambda^2+\rho^2), \quad b(\lambda,\rho) = \lambda/(\lambda^2+\rho^2),\\ &\tau = -\rho^2/\lambda. \end{aligned} \tag{4.7}$$

Whether this set is unique or not, or whether a different choice satisfying all the requirements mentioned above leads to different solutions to the field equations is still under investigation.

For the N-soliton configuration it is assumed that the matrix $\chi(\lambda,\rho,z)$ is of the form

$$\chi = \mathrm{I} + \sum_{i=1}^{2N} \frac{R_i}{\lambda - \mu_i}, \tag{4.8}$$

where the scalar functions μ_i are the roots of the equation

$$\theta(\mu_i, \rho, z) = -w_i \quad \text{(consts)} \tag{4.9}$$

and the matrices R_i are independent of the complex spectral parameter λ. The above form for χ together with Eq. (4.5) implies that

$$P = \chi(0)P_0. \tag{4.10}$$

For reasons that will become clear later we shall choose the $2N$ poles of χ to be related pairwise as

$$\mu_{N+k} = \tau(\mu_k), \quad k = 1,2,\ldots,N. \tag{4.11}$$

The unknown matrices R_i will be determined using Eqs. (3.15) and (4.5). Since R_i are independent of the spectral parameter λ, it suffices to consider these equations at the poles $\lambda = \mu_k$ and look at the residues at these points. Considering the relation $\chi\chi^{-1} = I$ at the poles $\lambda = \mu_k$, we get

$$R_k\chi^{-1}(\mu_k) = 0, \quad k = 1,2,\ldots,2N, \tag{4.12}$$

displaying the fact that the matrices R_k are degenerate. Equation (3.15) evaluated at $\lambda = \mu_k$ gives

$$(DR_k - R_k\Omega_0)\chi^{-1}|_{\lambda=\mu_k} = 0, \quad k = 1,2,\ldots,2N, \tag{4.13}$$

which, using the fact that the matrix Ψ_0^{-1} satisfies

$$D\Psi_0^{-1} - \Psi_0^{-1}\Omega_0 = 0, \tag{4.14}$$

shows that we have

$$R_k = M_k\Psi_0^{-1}(\mu_k), \quad k = 1,2,\ldots,2N, \tag{4.15}$$

where, for the moment, the matrices M_k appear to be arbitrary except that they have to be degenerate because of (4.12). To determine M_k, we consider the two reduction conditions given by Eq. (4.5). The first condition, when evaluated at $\lambda = \mu_k$, requires that χ^{-1} should have poles at $\lambda = \bar\mu_k$ and that

$$R_k + \sum_{i=1}^{2N} \frac{R_k\gamma R_i^\dagger\gamma}{\mu_k - \bar\mu_i} = 0, \quad k = 1,2,\ldots,2N. \tag{4.16}$$

On the other hand, the second condition of Eq. (4.5) requires

$$R_kP_0 + \sum_{i=1}^{2N} \frac{R_kP_0R_i^\dagger}{\tau(\mu_k) - \bar\mu_i} = 0, \quad k = 1,2,\ldots,2N. \tag{4.17}$$

Since the 3×3 matrices M_k are degenerate, writing

$$M_k = m_kn_k^\dagger + p_kq_k^\dagger, \quad k = 1,2,\ldots,2N, \tag{4.18}$$

Eqs. (4.16) and (4.17) reduce to two systems of linear algebraic equations which must consistently be solved for the column matrices m_k, n_k, p_k, and q_k. The consistency of the two systems, apart from the particular ordering of the poles μ_i as given in Eq. (4.11), requires that we should have

$$J^\dagger n_k = n_{N+k}, \quad J^\dagger q_k = q_{N+k}, \quad k = 1,2,\ldots,N, \tag{4.19}$$

which can equivalently be written as

$$P_0\gamma\psi_0(\bar\mu_k)\gamma n_k = \psi_0(\bar\mu_{k+N})\gamma n_{k+N},$$
$$P_0\gamma\psi_0(\bar\mu_k)\gamma q_k = \psi_0(\bar\mu_{k+N})\gamma q_{k+N}. \tag{4.20}$$

With these conditions, Eqs. (4.16) and (4.17) become identical. Hence, using any one of these equations, we obtain the following set of linear algebraic equation:

$$\sum_{i=1}^{2N}\left(\frac{1}{\mu_i-\bar\mu_k}\right)[(n_i^\dagger S_{ik}n_k)m_i + (q_i^\dagger S_{ik}n_k)P_i] = \gamma(\psi_0^\dagger(\mu_k))^{-1}n_k,$$
$$\sum_{i=1}^{2N}\frac{1}{\mu_i-\bar\mu_k}[(n_i^\dagger S_{ik}q_k)m_i + (q_i^\dagger S_{ik}q_k)P_i] = \gamma(\psi_0^\dagger(\mu_k))^{-1}q_k, \tag{4.21}$$

where

$$S_{ik} = \gamma\psi_0^\dagger(\bar\mu_i)\gamma\psi_0(\bar\mu_k)\gamma. \tag{4.22}$$

By use of these equations we determine the column vectors m_k and p_k in terms of the column vectors n_k and q_k.

So far the vectors n_k and q_k which have been choosen as constant vectors by Belinskii and Zakharov as a particular solution of Eq. (4.13) appeared to be arbitrary. We shall show that their choice is indeed the unique solution. Using Eqs. (4.15), (4.18), and (4.14) in Eq. (4.13), we obtain

$$(Dn_k^\dagger)\psi_0^{-1}(\mu_k)\chi^{-1}|_{\lambda=\mu_k} = 0. \tag{4.23}$$

Knowing the fact

$$n_k^\dagger\psi_0^{-1}(\mu_k)\chi^{-1}|_{\lambda=\mu_k} = 0, \tag{4.24}$$

then (4.23) can be written as

$$Dn_k^\dagger = H_k n_k^\dagger, \quad \text{at } \lambda=\mu_k, \tag{4.25}$$

where H_k's are 1-forms (not matrix-valued). Applying D operator ($\lambda=\mu_k$) to (4.25), we get

$$DH_k = 0, \quad \lambda=\mu_k. \tag{4.26}$$

Since $D\wedge D=0$, then $H_k = D\gamma_k|_{\lambda=\mu_k}$, where $\gamma_k\ [=\gamma_k(E^2)]$ are arbitrary functions. Hence $n_k^\dagger$ in Eq. (4.25) can be solved exactly as

$$n_k^\dagger = e^{-\gamma_k}n_{0k}^\dagger, \tag{4.27}$$

where n_{0k}'s are constant vectors. On the other hand, it can be shown from the linear set (4.21) that the vectors m_k's are also scaled by the factors e^{γ_k}. Hence the degenerate matrix M_k given in (4.18) and the new solution P does not contain these functions. The same results are also valid for the vectors q_k and p_k. Therefore we do not lose any generality by taking n_k and q_k as constant vectors.

5. CONCLUDING REMARKS

The symmetric space property and hence the σ-model formulation of the SAS Einstein–Maxwell problem may lead to some other new results. One of them has recently been given by Mazur.[16] He, independently, using such a formulation, proved the uniqueness of the Kerr–Newman black-hole solution. Einstein–Maxwell field equations for space-times admitting only one Killing vector formally look like the field equations given in Eq. (2.1). The crucial difference is that the differential operators are functions of the metric variables for the former case. Nevertheless, generalizing the differential operator D in (3.4), there is a hope of linearizing the E–M field equations for space-times admitting a non-null Killing vector. Work along this direction is in progress.

Although in this work we concentrate on soliton solutions for SAS Einstein–Maxwell fields, the procedure can be applied with minor modifications to every field theory which can be formulated in the framework of harmonic mappings of Riemannian manifolds provided that the configuration space is a Riemannian symmetric space and the base manifold is effectively two-dimensional. Recently,[10] it was shown that there is a close relationship between static axially symmetric self-dual SU(3) Yang–Mills and SAS Einstein–Maxwell fields. Thus, the solution generation technique presented here may also be used to obtain the monopole or the instanton solutions for SU(3). Furthermore, since the dimension of the matrices may be arbitrary, static axially symmetric self-dual SU(N) Yang–Mills fields can be treated as well.

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APPENDIX

Choosing the complex tetrad 1-forms as

$$\omega^1 = F^{-1}(d\epsilon + 2\bar\Phi\, d\Phi), \quad \omega^2 = F^{-1}(d\bar\epsilon + 2\Phi\, d\bar\Phi),$$
$$\omega^3 = 2F^{-1/2}\,d\Phi, \quad \omega^4 = 2F^{-1/2}\,d\bar\Phi. \tag{A1}$$

The line element (2.3) for the configuration space can be written as

$$d's^2 = \omega^1\otimes\omega^2 - \omega^3\otimes\omega^4. \tag{A2}$$

The connection 1-forms are

$$\begin{aligned}
&\omega^1{}_1 = -\omega^2{}_2 = -\tfrac12\omega^1 + \tfrac12\omega^2,\\
&\omega^1{}_3 = \omega^4{}_2 = -\tfrac12\omega^4,\\
&\omega^2{}_4 = \omega^3{}_1 = -\tfrac12\omega^3,\\
&\omega^3{}_3 = \omega^4{}_4 = -\tfrac14\omega^1 + \tfrac14\omega^2.
\end{aligned} \tag{A3}$$

Using these, the tetrad components of the curvature tensor are found as

$$-R'^1{}_{112} = R'^2{}_{212} = R'^3{}_{334} = -R'^4{}_{434} = \tfrac12,$$
$$R'^1{}_{134} = R'^1{}_{314} = -R'^2{}_{234} = R'^2{}_{423} = R'^3{}_{123} = -R'^3{}_{312} = R'^4{}_{214} = R'^4{}_{412} = \tfrac14, \tag{A4}$$

while the nonzero components of the Ricci tensor are

$$R'_{12} = -\tfrac34, \quad R'_{34} = \tfrac34. \tag{A5}$$

Hence, $R' = -6$ is the curvature scalar. Hence we have

$$R'_{AB} = -\tfrac32 g'_{AB} \quad (A,B,\ldots = 1,2,3,4), \tag{A6}$$

where

$$g'_{AB} = \tfrac12\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix}; \tag{A7}$$

therefore (M',g'_{AB}), is an Einstein space. In addition to this property we have

$$
\begin{aligned}
&R'_{GBCD}R'^{G}{}_{AEF} + R'_{AGCD}R'^{G}{}_{BEF} \\
&\quad + R'_{ABGD}R'^{G}{}_{CEF} + R'_{ABCG}R'^{G}{}_{DEF} = 0,
\end{aligned}
\tag{A8}
$$

meaning that the Riemann curvature tensor of M' is covariantly constant. This in turn means that M' is a Riemannian symmetric space. For this case the two properties together mean that M' is a harmonic space in the sense of Walker.[17] Thus M' has eight Killing vectors, $\tau_\mu = h_\mu{}^A\, \partial/\partial y^A$ $(\mu,\nu,\ldots, = 1,2,\ldots,8)$. With $\{y^A\}$ denoting the set $\{\epsilon,\bar\epsilon,\Phi,\bar\Phi\}$ these are

$$
\begin{aligned}
\tau_1 &= i\frac{\partial}{\partial\epsilon} - i\frac{\partial}{\partial\bar\epsilon}, \quad \tau_2 = i\Phi\frac{\partial}{\partial\Phi} - i\bar\Phi\frac{\partial}{\partial\bar\Phi}, \\
\tau_3 &= -i\epsilon^2\frac{\partial}{\partial\epsilon} + i\bar\epsilon^2\frac{\partial}{\partial\bar\epsilon} - i\Phi\epsilon\frac{\partial}{\partial\Phi} + i\bar\Phi\bar\epsilon\frac{\partial}{\partial\bar\Phi}, \\
\tau_4 &= -2\Phi\frac{\partial}{\partial\epsilon} - 2\bar\Phi\frac{\partial}{\partial\bar\epsilon} + \frac{\partial}{\partial\Phi} + \frac{\partial}{\partial\bar\Phi}, \\
\tau_5 &= 2i\Phi\frac{\partial}{\partial\epsilon} - 2i\bar\Phi\frac{\partial}{\partial\bar\epsilon} + i\frac{\partial}{\partial\Phi} - i\frac{\partial}{\partial\bar\Phi}, \\
\tau_6 &= -2i\Phi\epsilon\frac{\partial}{\partial\epsilon} + 2i\bar\Phi\bar\epsilon\frac{\partial}{\partial\bar\epsilon} + i(\epsilon - 2\Phi^2)\frac{\partial}{\partial\Phi} + i(2\bar\Phi^2 - \bar\epsilon)\frac{\partial}{\partial\bar\Phi}, \\
\tau_7 &= 2\Phi\epsilon\frac{\partial}{\partial\epsilon} + 2\bar\Phi\bar\epsilon\frac{\partial}{\partial\bar\epsilon} + (\epsilon + 2\Phi^2)\frac{\partial}{\partial\Phi} + (\bar\epsilon + 2\bar\Phi^2)\frac{\partial}{\partial\bar\Phi}, \\
\tau_8 &= 2\epsilon\frac{\partial}{\partial\epsilon} + 2\bar\epsilon\frac{\partial}{\partial\bar\epsilon} + \Phi\frac{\partial}{\partial\Phi} + \bar\Phi\frac{\partial}{\partial\bar\Phi}.
\end{aligned}
\tag{A9}
$$

These Killing vectors satisfy the SU(2,1) algebra

$$
[\tau_\mu,\tau_\nu] = C^\alpha{}_{\mu\nu}\tau_\alpha,
\tag{A10}
$$

where

$$
\begin{aligned}
&-C^4{}_{16} = C^4{}_{25} = C^4{}_{48} = C^5{}_{17} = -C^5{}_{24} = C^5{}_{58} = -C^6{}_{27}, \\
&C^6{}_{34} = -C^6{}_{68} = C^7{}_{26} = -C^7{}_{35} = -C^7{}_{78} = C^8{}_{13} = 1, \\
&C^1{}_{18} = -C^3{}_{38} = C^8{}_{47} = C^8{}_{56} = 2, \\
&C^1{}_{45} = -C^3{}_{67} = 4, \quad -C^2{}_{46} = C^2{}_{57} = 6.
\end{aligned}
\tag{A11}
$$

The corresponding Killing 1-forms $\tau^\mu = h^\mu{}_A\, dy^A$ are given as

$$
\begin{aligned}
\tau^1 &= -iF^{-2}\{\tfrac12\bar\epsilon^2 + \Phi\bar\Phi\bar\epsilon\}\, d\epsilon + iF^{-2}\{\tfrac12\epsilon^2 + \Phi\bar\Phi\epsilon\}\, d\bar\epsilon + iF^{-2}\bar\Phi\epsilon\bar\epsilon\, d\Phi - iF^{-2}\Phi\epsilon\bar\epsilon\, d\bar\Phi, \\
\tau^2 &= \tfrac32 iF^{-2}\Phi\bar\Phi\, d\epsilon - \tfrac32 iF^{-2}\Phi\bar\Phi\, d\bar\epsilon - \tfrac32 iF^{-2}\bar\Phi(\epsilon+\bar\epsilon)\, d\Phi + \tfrac32 iF^{-2}\Phi(\epsilon+\bar\epsilon)\, d\bar\Phi, \\
\tau^3 &= \tfrac12 iF^{-2}\, d\epsilon - \tfrac12 iF^{-2}\, d\bar\epsilon + iF^{-2}\bar\Phi\, d\Phi - iF^{-2}\Phi\, d\bar\Phi, \\
\tau^4 &= -\tfrac14 F^{-2}\{\bar\epsilon(\Phi+\bar\Phi) + 2\Phi\bar\Phi^2\}\, d\epsilon - \tfrac14 F^{-2}\{\epsilon(\Phi+\bar\Phi) + 2\Phi^2\bar\Phi\}\, d\bar\epsilon, \\
&\quad + \tfrac14 F^{-2}\{\bar\epsilon^2 + \epsilon\bar\epsilon + 2\epsilon\bar\Phi^2\}\, d\Phi + \tfrac14 F^{-2}\{\epsilon^2 + \epsilon\bar\epsilon + 2\bar\epsilon\Phi^2\}\, d\bar\Phi, \\
\tau^5 &= -\tfrac14 iF^{-2}\{\bar\epsilon(\bar\Phi-\Phi) + 2\Phi\bar\Phi^2\}\, d\epsilon + \tfrac14 iF^{-2}\{\epsilon(\Phi-\bar\Phi) + 2\Phi^2\bar\Phi\}\, d\bar\epsilon, \\
&\quad - \tfrac14 iF^{-2}\{\bar\epsilon^2 + \epsilon\bar\epsilon - 2\epsilon\bar\Phi^2\}\, d\Phi + \tfrac14 iF^{-2}\{\epsilon^2 + \epsilon\bar\epsilon - 2\bar\epsilon\Phi^2\}\, d\bar\Phi, \\
\tau^6 &= \tfrac14 iF^{-2}(\Phi+\bar\Phi)\, d\epsilon - \tfrac14 iF^{-2}(\Phi+\bar\Phi) d\bar\epsilon + \tfrac12 iF^{-2}\{\bar\Phi^2 - \tfrac12(\epsilon+\bar\epsilon)\}\, d\Phi - \tfrac12 iF^{-2}\{\Phi^2 - \tfrac12(\epsilon+\bar\epsilon)\}\, d\bar\Phi, \\
\tau^7 &= -\tfrac14 F^{-2}(\Phi-\bar\Phi)\, d\epsilon + \tfrac14 F^{-2}(\Phi-\bar\Phi)\, d\bar\epsilon + \tfrac12 F^{-2}\{\bar\Phi^2 + \tfrac12(\epsilon+\bar\epsilon)\}\, d\Phi + \tfrac12 F^{-2}\{\Phi^2 + \tfrac12(\epsilon+\bar\epsilon)\}\, d\bar\Phi, \\
\tau^8 &= \tfrac12 F^{-2}\{\bar\epsilon + \Phi\bar\Phi\}\, d\epsilon + \tfrac12 F^{-2}\{\epsilon + \Phi\bar\Phi\}\, d\bar\epsilon + \tfrac12 F^{-2}\bar\Phi\{\bar\epsilon - \epsilon\}\, d\Phi + \tfrac12 F^{-2}\Phi\{\epsilon - \bar\epsilon\}\, d\bar\Phi.
\end{aligned}
\tag{A12}
$$

They satisfy the Maurer–Cartan equations

$$
d\tau^\mu + \tfrac12 C^\mu{}_{\alpha\beta}\tau^\alpha \wedge \tau^\beta = 0.
\tag{A13}
$$

Using these, the line element of M' can be written as

$$
\begin{aligned}
d's^2 &= \tfrac12 \eta_{\mu\nu}\tau^\mu \otimes \tau^\nu, \\
d's^2 &= -\tau^1\otimes\tau^3 - \tfrac12\tau^2\otimes\tau^2 - 4\tau^4\otimes\tau^7 \\
&\quad - 4\tau^5\otimes\tau^6 + \tau^8\otimes\tau^8.
\end{aligned}
\tag{A14}
$$

We now define a Lie algebra valued connection 1-form W as

$$
W = X_\mu \tau^\mu,
$$

where X_μ is the 3×3 matrix representation of the generators of the group SU(2,1) satisfying

$$
[X_\mu,X_\nu] = C^\alpha{}_{\mu\nu}X_\alpha.
$$

Since the group metric $\eta_{\mu\nu}$ can also be written as

$\eta_{\mu\nu} = \frac{1}{2}\mathrm{tr}(X_\mu X_\nu + X_\nu X_\mu)$. The line element (A2), or, equivalently, (A14), can also be written as

$$ds'^2 = \frac{1}{2}\,\mathrm{tr}(W \otimes W), \qquad \text{(A15)}$$

where, written out explicitly in terms of the Killing 1-forms,

$$W = \begin{pmatrix} -\frac{1}{3}i\tau^2 + \tau^8 & \sqrt{2}(i\tau^6 + \tau^7) & \tau^3 \\ \sqrt{2}(-\tau^4 + i\tau^5) & \frac{2}{3}i\tau^2 & 2(\tau^6 + i\tau^7) \\ -\tau^1 & \sqrt{2}(i\tau^4 - \tau^5) & -\frac{1}{3}i\tau^2 - \tau^8 \end{pmatrix},$$

because of the Maurer–Cartan equations (A13). The curvature 2-form of this connection vanishes identically, i.e., $dW + W \wedge W = 0$.

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