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Metin Gürses

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# Some solutions of stationary, axially-symmetric gravitational field equations[a)]

Metin Gürses

*Physics Department, Middle East Technical University, Ankara, Turkey*
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Stationary, axially-symmetric solutions of the gravitational field equations for vacuum, perfect fluid, and massless scalar field are considered. For the vacuum case, a similiar formulation to the one introduced by Ernst is presented by use of quarternions. Null dust solutions are found, and it is shown that they match with the van Stockum exterior solutions. An extension of the theorem by Eriş and Gürses is given which enables one to construct solutions to the gravitational field equations coupled with a charged dust and a massless scalar field from the solutions of the field equations coupled only with a charged dust.

## 1. INTRODUCTION

Stationary, axially-symmetric solutions of the vacuum and electrovacuum gravitational field equations have been extensively studied, and certain classes of solutions have been found. Some of these classes are the Lewis,[1] van Stockum,[2] Papapetrou,[3] and Tomimatsu—Sato[4] solutions. The first three classes assume a functional relationship between the metric coefficients while the last one has been obtained by using computer logic. It is a three-parameter solution class with parameters describing deformation, mass, and angular momentum where the sum of the squares of the last two parameters is unity. Among all formulations of the vacuum and electrovacuum field equations, the Ernst[5] complex-potential formalism has certain advantages, one of these being generation of new vacuum and electrovacuum solutions from the old ones. Such a generation of solutions follows from the invariance of the Ernst equation under a bilinear transformation. Geroch[6] has shown that the same invariance exists in the vacuum field equations of any space—time having only a nonnull Killing vector.

In the second section of the this work we present a similar formulation of the vacuum field equations to the one introduced by Ernst, by use of quaternionic potentials. In this formalism the Lewis and van Stockum classes of solutions follow immediately. We obtain a Tomimatsu—Sato type class where in this case the difference of the squares of the parameters corresponding to mass and rotation is unity. The field equations are invariant under a quaternionic bilinear transformation.

Construction of stationary, axially-symmetric interior solutions to the gravitational field equations is one of the most difficult problems in general relativity. The difficulty arises from the complexity of the field equations. Some approximate[7] and nonfluid[8] solutions have been found but unfortunately no exact rotating fluid solution exists which matches with the Kerr metric. In order to approach such a solution, one should start with simple systems, such as null and nonnull dust distributions. In the third section we study the interior gravitational field equations for the null-dust case and give a complete solution. We obtain the general relativistic form of the Euler equation for a null rotating perfect fluid and also prove that the energy conservation equation (the Euler equation) is nothing but the integrability condition for one of the metric functions. This equation turns out to be very useful in the integration of the field equations for the case of dust distributions. It is interesting that all null dust solutions can be matched with an appropriate exterior solution of the van Stockum class.

Works on the solutions of the gravitational field equations coupled with a massless scalar field are quite recent and most of them have considered static[9] or conformally flat[10,11] space—times. Recently,[12] it was shown that it is possible to generate the axially-symmetric solutions of the field equations coupled with the electromagnetic and scalar fields from the Einstein—Maxwell solutions. In the fourth section, a generalization of this theorem to gravitational field equations coupled with charged dust and massless scalar field is given.

The components of the Ricci tensor in an orthonormal tetrad are given in the Appendix.

## 2. QUATERNIONIC POTENTIALS FOR VACUUM FIELD EQUATIONS

Gravitational field equations for vacuum are (see Appendix)

$$\nabla^2\psi+\frac{\exp(4\psi)}{2\rho^2}(\nabla\omega)^2=0, \tag{1}$$

$$\nabla\cdot(\rho^{-2}\exp(4\psi)\nabla\omega)=0. \tag{2}$$

Once $\omega$ and $\psi$ are found the remaining metric coefficient $\gamma$ can be found by use of quadratures (A10) and (A11). Defining a new function $f$

$$\psi=\tfrac{1}{2}\ln(\rho/f). \tag{3}$$

Eqs. (1) and (2) become

$$f\nabla^2 f-(\nabla f)^2-(\nabla\omega)^2=0, \tag{4}$$

$$\nabla\cdot(f^{-2}\nabla\omega)=0. \tag{5}$$

We now introduce a quaternionic potential $\varepsilon$ such that

$$\varepsilon=f+e\omega, \tag{6}$$

where $e$ is one of the three quaternionic units with $e^2=1$. Similar to complex conjugation we define the quaternionic conjugation as

$$\varepsilon^*=f-e\omega. \tag{7}$$

Functions $f$ and $\omega$ are, respectively, the scalar and

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vector parts of the quaternionic function $\varepsilon$. Hence it is clear that Eqs. (4) and (5) are the scalar and vector parts of the following differential equation:

$$(\varepsilon + \varepsilon^*)\nabla^2\varepsilon = 2\nabla\varepsilon\cdot\nabla\varepsilon. \tag{8}$$

In terms of a new quaternionic function $\xi$ defined by

$$\varepsilon = (\xi - 1)/(\xi + 1), \tag{9}$$

Eq. (8) is written as

$$(\xi\xi^* - 1)\nabla^2\xi = 2\xi^*\nabla\xi\cdot\nabla\xi. \tag{10}$$

Equation (10) can be obtained from the Lagrangian density

$$\mathcal{L} = \nabla\xi\cdot\nabla\xi^*/(\xi\xi^* - 1)^2, \tag{11}$$

which is nothing but the Einstein Lagrantian density $\sqrt{-g}R$. This Lagrangian density is invariant under the bilinear transformation

$$\xi' = (a\xi + b)/(c\xi + d) \quad (ad - bc \neq 0) \tag{12}$$

where $a$, $b$, $c$, and $d$ represent eight real constants of which only three are independent. The group of this three-parameter transformation is isomorphic to one of the well-known two-dimensional noncompact groups. Invariance of the Lagrangian density (11) under the transformations (12) leads to the generation of the new solutions of the vacuum field equations from the known solutions. Using the Ernst trick, one may also obtain solutions to the electrovacuum field equations.

The norm of a quaternion $Q = Q_s + eQ_v$ is defined as

$$QQ^* = Q_s^2 - Q_v^2. \tag{13}$$

Hence, vanishing of the norm does not necessarily lead to the vanishing of the scalar and vector parts of $Q$. With the properties of quaternions, we give three simple solutions to Eq. (10). They are

$$\xi = \exp(e\alpha)\coth\Theta, \tag{14}$$

$$\xi = e\exp(e\alpha)\tan\Theta, \tag{15}$$

$$\xi = (1 \pm e)\Theta, \tag{16}$$

where $\Theta$ is a real function satisfying

$$\nabla^2\Theta = 0, \tag{17}$$

and $\alpha$ is an arbitrary real constant. The solutions (14), (15), and (16) are called Lewis, Lewis, and van Stockum classes,[13] respectively. These classes of solutions are not easily seen in the Ernst complex potential formulation. On the other hand, the Papapetrou class can not be directly obtained in our formulation. The other well-known class is the Tomimatsu–Sato solutions. These solutions are the twisting generalization of the Weyl static vacuum metrics. They have three parameters, $\delta$ (distortion), $p$ (mass), and $q$ (twist) with $p^2 + q^2 = 1$. We have a similar class of solutions with $p^2 - q^2 = 1$. For example, in the oblate spheroidal coordinates

$$\xi = px + eqy, \quad (p^2 - q^2 = 1)$$

is one of the solutions of (10), corresponding to $\delta = 1$. One can also generate the NUT parameter using the invariance of (10) under the transformation $\xi' = \exp(e\alpha)\xi$.

## 3. NULL FLUID

The Einstein field equations with a null fluid in a tetrad basis are

$$R_b^a = -\kappa[(p + \epsilon)u^a u_b - p\delta_b^a], \tag{18}$$

where $p$ and $\epsilon$ are the pressure and the energy density of the fluid and $u^a$ are the components of the fluid 4-velocity, with $u^a u_a = 0$. In a stationary, axially symmetric space—time the only nonvanishing components of the fluid velocity vector $u^a$ are $u^0$ and $u^3$. For the case of null fluid

$$u^0 = \eta u^3 = u, \quad \eta = \pm 1,$$

and the field equations are

$$\rho\lambda^{-1}\nabla\cdot(\rho^{-1}\lambda\nabla\psi) + (\exp(4\psi)/2\lambda^2)(\nabla\omega)^2$$
$$= \kappa\exp(2\gamma - 2\psi)(p_+ u^2 + p), \tag{19}$$

$$\lambda^{-1}\rho\nabla\cdot(\rho^{-1}\nabla\lambda) = 2\kappa\exp(2\gamma - 2\psi)p, \tag{20}$$

$$\rho\nabla\cdot(\rho^{-1}\lambda^{-1}\exp(4\psi)\nabla\omega) = -2\kappa\eta\exp(2\gamma)p_+ u^2. \tag{21}$$

Equations (13) and (14) will be considered as the equations to determine the metric function $\gamma$. Instead of (A11) we take the following equation as one of the field equations which is nothing but the energy conservation equation (contracted Bianchi identity):

$$\nabla p/p_+ = u^2\lambda^{-1}\exp(2\psi)\nabla(\lambda\exp(-2\psi) - \eta\omega), \tag{22}$$

where $p_+ = p + \epsilon$. The $\rho$ component of this equation is obtained through the addition of the derivatives of (A13) and (A14) with respect to $z$ and $\rho$, respectively, while the $z$ component is obtained by subtracting the derivatives of (A13) and (A14) with respect to $\rho$ and $z$, respectively. It is obvious that when $p = 0$ (null-dust or noninteracting null gaseous), the field equations become much simpler. First we can use the coordinate condition (A12) and obtain the following relation between $\omega$ and $\psi$ by use of Eq. (22)

$$\omega = \eta\rho\exp(-2\psi). \tag{23}$$

This relation enables us to integrate completely the quadratures for $\gamma$ given in (A13) and (A14). The result of the integration is

$$\exp(2\gamma - 2\psi) = \rho^{-1/2}. \tag{24}$$

Then the line element becomes

$$ds^2 = -\exp(2\psi)\,dt^2 + 2\eta\rho\,d\phi\,dt + \rho^{-1/2}(d\rho^2 + dz^2), \tag{25}$$

where $\psi$ satisfies

$$\nabla^2 U + VU = 0, \tag{26}$$

with

$$U = \rho^{-1}\exp(2\psi) \tag{27}$$

$$V = 2\kappa\rho^{-1/2}\epsilon u^2. \tag{28}$$

Hence for a given source $\epsilon u^2$, the metric in (25) with (26) defines the gravitational field of a noninteracting null dust (or simply the null electromagnetic field). When $\epsilon u^2$ is a constant, the solution of (26) may be reduced to one type of a Bessel function. These solutions may be matched exactly to the van Stockum exterior metrics which are of the form given in Eq. (25) with

$$\nabla^2 U = 0. \tag{29}$$

In the coordinate basis the only nonvanishing covariant component of the Ricci tensor is $R_{\hat{0}\hat{0}}$; hence

$$R_{\mu\nu} = -\epsilon_0 n_\mu n_\nu, \tag{30}$$

where

$$n_\mu = e^\psi \delta^{\hat{0}}_\mu, \quad n^\mu = \rho^{-1}\delta^\mu_{\hat{3}}, \tag{31}$$

and

$$\epsilon_0 = \kappa\epsilon u^2. \tag{32}$$

## 4. MASSLESS SCALAR FIELD

Recently[13] it was shown that one can generate the solutions to the coupled Einstein—Maxwell massless scalar field equations from the known solutions of the Einstein—Maxwell equations. Here, we give an extension of the above theorem to the case when the source is a charged dust and a massless scalar field.

*Theorem*: Let $\psi, \omega, \gamma, A_0, A_3$, and $\mu$ be a solutions of the Einstein field equations coupled with an electromagnetic field, and a dust distribution, where $A_0$ and $A_3$ are the nonzero components of the electromagnetic vector potential and $\mu$ is the energy density of the dust distribution. Then $\psi, \omega, \gamma+\gamma^\Phi, A_0, A_3, \mu\exp(-2\gamma^\Phi)$, and $\Phi$ form a solution to the Einstein field equations coupled with an electromagnetic field, a dust distribution, and a massless scalar field $\Phi$, where

$$2\gamma^\Phi_{,\rho} = \rho[(\Phi_{,\rho})^2 - (\Phi_{,z})^2],$$

$$2\gamma^\Phi_{,z} = 2\rho\Phi_{,\rho}\Phi_{,z},$$

and

$$\nabla^2\Phi = 0.$$

Using this theorem, one may obtain solutions to the gravitational field equations coupled to a null dust and a massless scalar field once a solution of Eq. (26) is given.

## 5. CONCLUDING REMARKS

We presented a quaternionic potential formulation of the stationary, axially-symmetric vacuum gravitational field equations and obtained a class of "Tomimatsu—Sato"-like solutions which does not contain any algebraically special metrics.

We gave the complete solution of the stationary, axially-symmetric gravitational field equations coupled with a null dust or a null-electromagnetic field. For the case of the null-electromagnetic field, $F_{ab} = 2\epsilon^{1/2}u n_{[a} l_{b]}$, where $l_b$ is a unit spacelike vector which is orthogonal to $n_a$.[14] These solutions match with the van Stockum exterior solutions on a cylindrical boundary. We showed that the energy conservation or the contracted Bianchi identity is the integrability condition for the metric coefficient $\gamma$. This is, in fact, true for any energy—momentum distribution.

We presented a theorem to produce the solutions of the gravitational field equations coupled to a charged dust and a massless scalar field from the solutions of the field equations coupled with a charged dust. In fact, it is possible to extend this theorem further for any covariantly conserved energy—momentum tensor $T^a_b$ plus a massless scalar field when the condition (A12) is satisfied for stationary, axially-symmetric space—times.

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## APPENDIX

The stationary, axially symmetric metric is given by

$$ds^2 = -\exp(2\psi)(dt - \omega\, d\phi)^2 + \exp(-2\psi)[\exp(2\gamma)(d\rho^2 + dz^2) + \lambda^2\, d\phi^2]. \tag{A1}$$

Choosing the basis 1-forms as

$$\omega^a = h^a{}_\mu\, dx^\mu, \tag{A2}$$

the *vier bein* components $h^a{}_\mu$ are

$$\begin{aligned} &h^0{}_{\hat{0}} = \exp(\psi), \quad h^0{}_{\hat{3}} = -\omega\exp(\psi),\\ &h^3{}_{\hat{3}} = \lambda\exp(-\psi), \quad h^1{}_{\hat{1}} = h^2{}_{\hat{2}} = \exp(\gamma-\psi), \end{aligned} \tag{A3}$$

and the inverse components $h_a{}^\mu$ can be found using relations

$$h^a{}_\nu h_a{}^\mu = \delta^\mu{}_\nu \tag{A4}$$

Here the Latin indices denote the orthonormal tetrad components and run from 0 and 3 and the Greek indices denote the coordinate components running from $\hat{0}$ to $\hat{3}$. Nonvanishing components of the Ricci tensor with the convention

$$R^a{}_b = R^{ac}{}_{cb}, \tag{A5}$$

are found as

$$R^0{}_0 = \exp(2\psi - 2\gamma)[\rho\lambda^{-1}\nabla\cdot(\rho^{-1}\lambda\nabla\psi) + (\exp(4\psi)/2\lambda^2)(\nabla\omega)^2], \tag{A6}$$

$$R^0{}_3 = -\tfrac{1}{2}\rho\exp(-2\gamma)\nabla\cdot(\rho^{-1}\lambda^{-1}\exp(4\psi)\nabla\omega), \tag{A7}$$

$$R^0{}_0 + R^3{}_3 = \rho\lambda^{-1}\exp(2\psi-2\gamma)\nabla\cdot(\rho^{-1}\nabla\lambda), \tag{A8}$$

$$R^1{}_2 = \exp(2\psi-2\gamma)[2\psi_{,\rho}\psi_{,z} - (\exp(4\psi)/2\lambda^2)\omega_{,\rho}\omega_{,z} + \lambda^{-1}(\lambda_{,\rho z} - \lambda_{,\rho}\gamma_{,z} - \lambda_{,z}\gamma_{,\rho})], \tag{A9}$$

$$\begin{aligned} R^1{}_1 - R^2{}_2 = \exp(2\psi-2\gamma)[&2(\psi^2_{,\rho} - \psi^2_{,z})\\ &- (\exp(4\psi)/2\lambda^2)(\omega^2_{,\rho} - \omega^2_{,z})\\ &+ \lambda^{-1}(\lambda_{,\rho\rho} - \lambda_{,zz} - 2\lambda_{,\rho}\gamma_{,\rho} + 2\lambda_{,z}\gamma_{,z})], \end{aligned} \tag{A10}$$

$$\begin{aligned} R^1{}_1 + R^2{}_2 = \exp(2\psi-2\gamma)[&-2\rho\lambda^{-1}\nabla\cdot(\rho^{-1}\lambda\nabla\psi)\\ &+ 2(\nabla\psi)^2 - (\exp(4\psi)/2\lambda^2)(\nabla\omega)^2\\ &+ 2(\gamma_{,\rho\rho} + \gamma_{,zz}) + \lambda^{-1}(\lambda_{,\rho\rho} + \lambda_{,zz})], \end{aligned} \tag{A11}$$

where $\nabla$, $\nabla\cdot$, and $\nabla^2$ are the grad, the divergence, and the Laplace operators defined in flat space cylindrical coordinates, respectively. Throughout this work we consider the case

$$R^0{}_0 + R^3{}_3 = 0, \tag{A12}$$

which enables us to use the coordinate condition $\lambda = \rho$. We will discuss the other possible case $\lambda = $ const in a later communication.

For the perfect fluid case $R^1{}_2$ and $R^1{}_1 - R^2{}_2$ vanish; hence we have

$$\lambda_{,\rho}\gamma_{,z} + \lambda_{,z}\gamma_{,\rho} = \lambda(2\psi_{,\rho}\psi_{,z} - (\exp(4\psi)/2\lambda^2)\omega_{,\rho}\omega_{,z}) + \lambda_{,\rho z}, \tag{A13}$$

$$\lambda_{,\rho}\gamma_{,\rho} - \lambda_{,z}\gamma_{,z} = \lambda(\psi^2_{,\rho} - \psi^2_{,z}) - (\exp(4\psi)/4\lambda)(\omega^2_{,\rho} - \omega^2_{,z}) + \tfrac{1}{2}(\lambda_{,\rho\rho} - \lambda_{,zz}). \tag{A14}$$


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