

Solutions of the 3rd homework problems

Exercises

- 5.1 Show that the transformation matrix for a Lorentz transformation from S to S' in standard configuration is given by (5.4).
- 5.2 Show that, under a Lorentz transformation, the covariant components of a vector transform as $v'_\mu = \Lambda_\mu{}^\nu v_\nu$. Hence show explicitly in component form that, for two 4-vectors v and w , the scalar product $v \cdot w$ is invariant under a Lorentz transformation.
- 5.3 Prove that, for any timelike vector v in Minkowski space, there exists an inertial frame in which the spatial components are zero.
- 5.4 Prove (a) that the sum of any two spacelike vectors is spacelike; and (b) that a timelike vector and a null vector cannot be orthogonal.
- 5.5 For the spaceship discussed in Section 1.14, which maintains a uniform acceleration a in the x -direction of some inertial frame S , the worldline is given by

$$t(\tau) = \frac{c}{a} \sinh \frac{a\tau}{c}, \quad x(\tau) = \frac{c^2}{a} \left(\cosh \frac{a\tau}{c} - 1 \right), \quad y(\tau) = 0, \quad z(\tau) = 0,$$

where τ is the proper time of an astronaut on the spaceship. Show that the 4-velocity of the rocket in the coordinate system (ct, x, y, z) is given by

$$[u^\mu] = \left(c \cosh \frac{a\tau}{c}, c \sinh \frac{a\tau}{c}, 0, 0 \right).$$

Hence show explicitly that $u^\mu u_\mu = c^2$ and that the spaceship's 3-velocity is

$$\vec{u} = \left(c \tanh \frac{a\tau}{c}, 0, 0 \right).$$

- 5.6 Show that the 4-acceleration of the spaceship in Exercise 5.5 is given by

$$[a^\mu] = \left(a \sinh \frac{a\tau}{c}, a \cosh \frac{a\tau}{c}, 0, 0 \right).$$

Hence show that $a^\mu a_\mu = a^2$ and that the magnitude of the spaceship's 3-acceleration in its own instantaneous rest frame is also a .

- 5.7 A spaceship has constant acceleration g in the x -direction in its locally comoving frame, i.e. the IRF. Show that, in an inertial frame, the spaceship's 4-velocity $[u^\mu] =$

$(u^0, u^1, 0, 0)$ and 4-acceleration $[a^\mu] = (a^0, a^1, 0, 0)$ satisfy $a^1 = gu^0/c$ and $a^0 = gu^1/c$. Show also that

$$\frac{d^2 u^\mu}{d\tau^2} = \frac{g^2 u^\mu}{c^2},$$

where τ is the proper time as measured by an occupant of the spaceship. A spaceship accelerates at a constant rate $g = 9.5 \text{ m s}^{-2}$ in its own locally comoving frame. It starts out towards the centre of the Galaxy 10 kpc distant. After going 5 kpc it decelerates at the same rate to come to rest again at the Galactic centre. The outward journey is then repeated in reverse to come back home. Show that, in the spaceship's frame, the elapsed travel time is 41.5 years. What is the elapsed time for the waiting observer (or descendants) on Earth?

- 5.8 Show that in its own instantaneous rest frame (IRF), a particle's 4-acceleration is given by $[a^\mu] = (0, \vec{a})$, where $\vec{a}$ is the 3-acceleration of the particle in the IRF.
- 5.9 Show that, in an inertial frame in which a particle's 3-acceleration $\vec{a}$ is orthogonal to its 3-velocity $\vec{u}$, the particle's 4-acceleration is given by $[a^\mu] = \gamma_u^2(0, \vec{a})$.
- 5.10 Show that when an electron and a positron annihilate, more than one photon must be produced.
- 5.11 Show that if a photon is reflected from a mirror moving parallel to its plane, then the angle of incidence of the photon is equal to the angle of reflection.
- 5.12 An inertial frame S' moves with constant velocity u along the x -axis with respect to frame S . A photon in frame S' is fired at an angle θ' to the forward direction of motion. Show that the angle θ measured in frame S is

$$\tan \theta = \frac{\tan \theta' (1 - \beta^2)^{1/2}}{1 + \beta \sec \theta'},$$

where $\beta = u/c$.

- 5.13 A photon with energy E collides with a stationary electron whose rest mass is m_0 . As a result of the collision the direction of the photon's motion is deflected through an angle θ and its energy is reduced to E' . Show that

$$m_0 c^2 \left(\frac{1}{E'} - \frac{1}{E} \right) = 1 - \cos \theta.$$

Deduce that the wavelength of the photon is increased by

$$\Delta \lambda = \frac{2h}{m_0 c} \sin^2 \left(\frac{\theta}{2} \right),$$

where h is Planck's constant. At what angle to the initial photon direction does the electron move? Show that, if the photon is deflected through a right angle, and the photon energy satisfies $E \ll m_0 c^2$, then after the interaction the angle of the electron's motion to the direction of the photon's initial motion is $\alpha = -\pi/4$.

- 5.14 Inverse Compton scattering occurs whenever a photon scatters off a particle moving with a speed very nearly equal to that of light. Suppose that a particle of rest mass

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Solutions of the 3rd homework problems

5.1) Lorentz transformation matrix satisfies the equation $\left(\frac{\partial x'^{\mu}}{\partial x^{\alpha}} = \Lambda^{\mu}_{\alpha} \right)$

$$\eta_{\mu\nu} = \Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu} \eta_{\alpha\beta}$$

Since ~~to~~ move only in the x -direction, Λ takes the form

$$\left[\Lambda^{\mu}_{\nu} \right] = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\begin{matrix} \uparrow & \text{row} \\ \downarrow & \text{column} \end{matrix}$

$$\begin{aligned} \Lambda^0_0 &= a & \Lambda^x_0 &= c \\ \Lambda^0_x &= b & \Lambda^x_x &= d \end{aligned}$$

$$\begin{aligned} 1) \quad \eta_{00} &= \Lambda^{\alpha}_0 \Lambda^{\beta}_0 \eta_{\alpha\beta} = \Lambda^0_0 \Lambda^0_0 \eta_{00} \\ &+ \Lambda^1_0 \Lambda^1_0 \eta_{11} = a^2 - c^2 = 1 \end{aligned}$$

$$\begin{aligned} 2) \quad \eta_{01} &= \Lambda^{\alpha}_0 \Lambda^{\beta}_1 \eta_{\alpha\beta} = \Lambda^0_0 \Lambda^0_1 \eta_{00} \\ &+ \Lambda^1_0 \Lambda^1_1 \eta_{11} \\ &= +abd + c(-d) = 0 \end{aligned}$$

(2)

$$\begin{aligned}
 3) \quad y_{10} &= \Lambda^\alpha, \Lambda^\beta, y_{\alpha\beta} \\
 &= \Lambda^0, \Lambda^\beta, y_{0\beta} + \Lambda^1, \Lambda^\beta, y_{1\beta} \\
 &= ba - dc = 0
 \end{aligned}$$

$$\begin{aligned}
 4) \quad y_{11} &= \Lambda^\alpha, \Lambda^\beta, y_{\alpha\beta} = \Lambda^0, \Lambda^\beta, y_{0\beta} \\
 &\quad + \Lambda^1, \Lambda^\beta, y_{1\beta} \\
 &= b^2 - d^2 = -1
 \end{aligned}$$

$$\Rightarrow \begin{aligned} a^2 - c^2 &= 1 \\ d^2 - b^2 &= 1 \end{aligned} \quad ab = dc$$

if $c=0 \Rightarrow ab=0$ but $a \neq 0 \Rightarrow b=0$

$a^2 = 1, d^2 = 1 \Rightarrow$ This is the unit ~~real~~ matrix. This gives no transformation. Hence $c \neq 0$

$$d = \frac{ab}{c} \Rightarrow \frac{a^2 b^2}{c^2} - b^2 = 1$$

$$\Rightarrow (a^2 - c^2) b^2 / c^2 = 1 \Rightarrow b^2 = c^2$$

choose $b=c \Rightarrow d=a$ we have
only $a^2 - c^2 = 1$

(3)

Choose $a = \gamma$, $c = -\beta\gamma$

$$\Rightarrow \gamma^2(1-\beta^2) = 1 \quad \gamma = (1-\beta^2)^{-1/2}$$

$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

let $\beta = \tanh \psi \Rightarrow 1-\beta^2 = 1 - \tanh^2 \psi$
 $= \frac{\cosh^2 \psi - \sinh^2 \psi}{\cosh^2 \psi} = \frac{1}{\cosh^2 \psi}$

$$\gamma = \left(\frac{1}{\cosh^2 \psi} \right)^{-1/2} = \cosh \psi$$

$$\Rightarrow (\Lambda^\mu{}_\nu) = \begin{pmatrix} \cosh \psi & -\sinh \psi & 0 & 0 \\ -\sinh \psi & \cosh \psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3.2) Lorentz transformations.

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$$

Hence
$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \Lambda^{\mu}_{\nu}$$

A vector $\vec{V}^{\alpha}$ in a manifold transform as

$$V'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} V^{\beta} \Rightarrow V'^{\alpha} = \Lambda^{\alpha}_{\beta} V^{\beta}$$

A covariant vector V_{α} in the manifold transform as

$$V'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} V_{\beta} \Rightarrow V'_{\alpha} = \Lambda_{\alpha}^{\beta} V_{\beta}$$

where Λ_{α}^{β} is the inverse of Λ^{α}_{β}

$$\Lambda_{\alpha}^{\beta} = \eta_{\alpha\mu} \Lambda^{\mu}_{\nu} \eta^{\nu\beta}$$

Hence

$$u'^{\alpha} V'_{\alpha} = (\Lambda^{\alpha}_{\beta} u^{\beta}) (\Lambda_{\alpha}^{\gamma} V_{\gamma})$$

$$= (\Lambda^{\alpha}_{\beta} \Lambda_{\alpha}^{\gamma}) u^{\beta} V_{\gamma} = \delta_{\beta}^{\gamma} u^{\beta} V_{\gamma} = u^{\beta} V_{\beta}$$

Invariant under the LT.

Ex. 3) Let $v^\alpha = (a, 0, 0, 0)$ only the zeroth component is different than zero. In a new frame this vector has the components

$$[v'^\alpha] = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma a \\ -\gamma\beta a \\ 0 \\ 0 \end{pmatrix} =$$

Hence any vector $[u^\alpha] = \begin{pmatrix} A \\ B \\ 0 \\ 0 \end{pmatrix}$ can be transformed to vector $[u'^\alpha] = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix}$ if we take

$$\beta = \frac{B}{A} \text{ of the LT.}$$

(6)

Ex 4. (a) Spacelike vectors must be orthogonal

Let u and v be two orthogonal spacelike vectors, i.e. $u^\alpha u_\alpha < 0$, $v^\alpha v_\alpha < 0$ then

$$(u^\alpha + v^\alpha)(u_\alpha + v_\alpha) = u^\alpha u_\alpha + v^\alpha v_\alpha + 2u^\alpha v_\alpha$$

Since u^α and v^α are orthogonal $u^\alpha v_\alpha = 0$

$\Rightarrow$

$$(u^\alpha + v^\alpha)(u_\alpha + v_\alpha) = u^\alpha u_\alpha + v^\alpha v_\alpha$$

Since both $u^\alpha u_\alpha$ and $v^\alpha v_\alpha$ are negative then

$$(u^\alpha + v^\alpha)(u_\alpha + v_\alpha) < 0$$

Hence $u^\alpha + v^\alpha$ is also a spacelike vector

(b) Let u^α be a timelike vector. We know that we can find a frame where u^α takes the form $u^\alpha = (u_0, 0, 0, 0)$. Let v^α be a

null vector. Let $v^\alpha = (v^0, \vec{v})$ so that

$$v^\alpha v_\alpha = (v^0)^2 - \vec{v} \cdot \vec{v} = 0, \text{ It is clear that if}$$

$$v^0 = 0 \Rightarrow \vec{v} = 0. \text{ Therefore } u^\alpha v_\alpha = u^0 v_0$$

Since both u^0 and v_0 can not be zero $\Rightarrow$

$u^\alpha v_\alpha$ cannot be zero. This means that a timelike vector and null vector cannot be orthogonal

⑤ $t(\tau) = \frac{c}{a} \sinh \frac{a\tau}{c}, \quad x(\tau) = \frac{c^2}{a} (\cosh \frac{a\tau}{c} - 1)$

$y(\tau) = 0, \quad z(\tau) = 0$

$\Rightarrow X^\mu(\tau) = (ct, x, y, z)$

$= \left(\frac{c^2}{a} \sinh \frac{a\tau}{c}, \frac{c^2}{a} (\cosh \frac{a\tau}{c} - 1), 0, 0 \right)$

$U^\mu = \frac{dX^\mu}{d\tau} = \left(c \cosh \left(\frac{a\tau}{c} \right), c \sinh \left(\frac{a\tau}{c} \right), 0, 0 \right)$

$U^\mu U_\mu = c^2 \cosh^2 \left(\frac{a\tau}{c} \right) - c^2 \sinh^2 \left(\frac{a\tau}{c} \right) = c^2$

spaceship 3-velocity $\vec{v} = \frac{d\vec{x}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$

$$\frac{dx}{dt} = \frac{dx}{d\tau} \cdot \frac{d\tau}{dt} \quad \text{and} \quad \begin{cases} \frac{dx}{d\tau} = c \sinh \left(\frac{a\tau}{c} \right) \\ \frac{dt}{d\tau} = c \cosh \left(\frac{a\tau}{c} \right) \end{cases}$$

$= c \frac{\sinh(a\tau/c)}{\cosh(a\tau/c)}$

$= c \tanh(a\tau/c) \Rightarrow \vec{v} = (c \tanh(a\tau/c), 0, 0)$

5.6) $a^\mu = \frac{dU^\mu}{d\tau} = \frac{d^2 X^\mu}{d\tau^2}$

$= \left(a \sinh \left(\frac{a\tau}{c} \right), a \cosh \left(\frac{a\tau}{c} \right), 0, 0 \right)$

$$a^\mu a_\mu = c^2 \sinh^2(a\tau/c) - a^2 \cosh^2(a\tau/c) = -a^2$$

Find acceleration vector $\vec{a}$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{v}}{dz} \cdot \frac{dz}{d\tau} = \frac{1}{\cosh(a\tau/c)} \left(a \frac{1}{\cosh^2(a\tau/c)}, 0, 0 \right)$$

$$= \left(\frac{a}{\cosh^2(a\tau/c)}, 0, 0 \right)$$

at the IRF

$$u^\mu = (c, 0, 0, 0)$$

$$\Rightarrow a^\mu = (0, a, 0, 0)$$

5. +) From 5.5

$$(u^\alpha) = (u^0, u^1, 0, 0)$$

since the motion is in
x-direction

$$u^2 - (u^1)^2 = c^2$$

$$a^\alpha = \frac{du^\alpha}{d\tau} = \left(\frac{du^0}{d\tau}, \frac{du^1}{d\tau}, 0, 0 \right) \\ = (a^0, a^1, 0, 0)$$

$$(a^0)^2 - (a^1)^2 = -g^2$$

and $u^\alpha a_\alpha = 0 \Rightarrow (u^0 a^0 - u^1 a^1 = 0)$

$$a^1 = \frac{a^0}{u^1} u^0 \Rightarrow (a^0)^2 - (a^0)^2 \frac{(u^0)^2}{(u^1)^2} = -g^2$$

$$\frac{(a^0)^2}{(u^1)^2} [(u^1)^2 - (u^0)^2] = -g^2 \Rightarrow (a^0)^2 = \frac{g^2}{c^2} (a^1)^2$$

$$\Rightarrow a^0 = \frac{g}{c} u^1 \Rightarrow a^\alpha = \frac{g}{c} u^\alpha$$

or $\frac{du^\alpha}{d\tau} = \frac{g}{c} u^\alpha, \quad \frac{du^1}{d\tau} = \frac{g}{c} u^0$

$$\Rightarrow \frac{d^2 u^\alpha}{d\tau^2} = \frac{g}{c} \frac{du^\alpha}{d\tau} = \frac{g^2}{c^2} u^\alpha$$

$$\Rightarrow \frac{d^2 u^a}{d\tau^2} = \frac{g^2}{c^2} u^a, \quad a=0,1$$

§. 8) Read pages 126 and 127

§. 9) let

$$u^a = \delta_u(c, \bar{u})$$

$$\text{and } a^a = \delta_a(a_0, \bar{a})$$

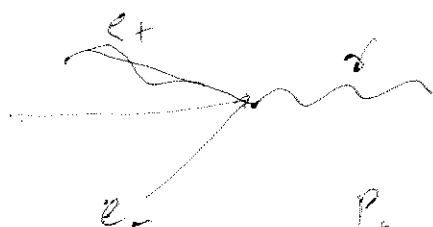
$$\text{since } \bar{u} \cdot \bar{v} = 0$$

$$\Rightarrow a_0 = 0$$

$$\text{bec } u^a a_a = \delta_u^2(c \cdot a_0) = 0$$

$$\Rightarrow a^a = \delta_a(0, \bar{a})$$

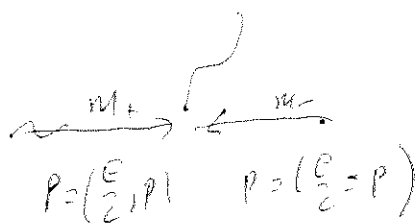
§. 10 |



$$P_+ + P_- = P_{\text{photon}}$$

$$\frac{E_+ + E_-}{c} = \frac{E_{\gamma}}{c} \Rightarrow E_+ + E_- = E_{\gamma}$$

$$2mc^2 = h\nu$$



head on collision

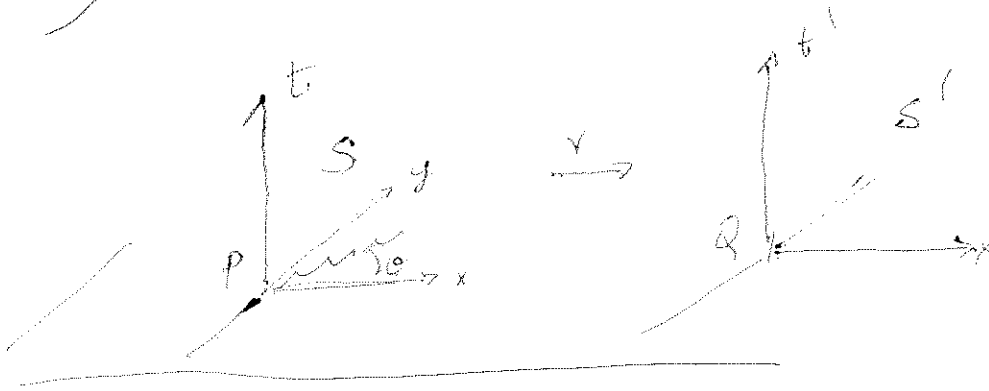
$$2E = h\nu$$

$$0 = \vec{P}_f = 0 \quad \text{contradiction}$$

hence there must be at least two photons after the interaction of e_+ and e_-

5.12)

(11)



$$[k^\mu] = \frac{2\pi}{\lambda} (1, \cos\theta, \sin\theta, 0)$$

$$[k'^\mu] = \frac{2\pi}{\lambda'} (1, \cos\theta', \sin\theta', 0)$$

At P and Q photon 4-vector k^μ has the form
 At Q the components of the 4-wave vector measured by the observer Q at S' is $[k'^\mu]$. k^μ and k'^μ are related by the Lorentz transformation

$$k'^\mu = \Lambda^\mu_\nu k^\nu \Rightarrow \frac{2\pi}{\lambda'} \begin{pmatrix} 1 \\ \cos\theta' \\ \sin\theta' \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_v - \beta\gamma_v & 0 & 0 \\ -\beta\gamma_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{2\pi}{\lambda} \begin{pmatrix} 1 \\ \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

we get

$$(i) \quad \frac{2\pi}{\lambda'} = \frac{2\pi}{\lambda} (\gamma_v - \beta\gamma_v \cos\theta)$$

$$(ii) \quad \frac{2\pi}{\lambda'} \cos\theta' = \frac{2\pi}{\lambda} (-\beta\gamma_v + \gamma_v \cos\theta)$$

$$(iii) \quad \frac{2\pi}{\lambda'} \sin\theta' = \frac{2\pi}{\lambda} \sin\theta$$

$$(iv) \quad 0 = 0$$

From (i)

$$\frac{1}{\lambda'} = \frac{1}{\lambda} \gamma_v (1 - \beta \cos \theta) \quad (1)$$

From (ii)

$$\frac{\cos \theta'}{\lambda'} = \frac{\gamma_v}{\lambda} (-\beta + \cos \theta) \quad (2)$$

From (iii)

$$\frac{\sin \theta'}{\lambda'} = \frac{\sin \theta}{\lambda} \quad (3)$$

(1) gives the wavelength relation

$$\frac{1}{\lambda'} = \frac{1}{\lambda} \gamma_v (1 - \beta \cos \theta)$$

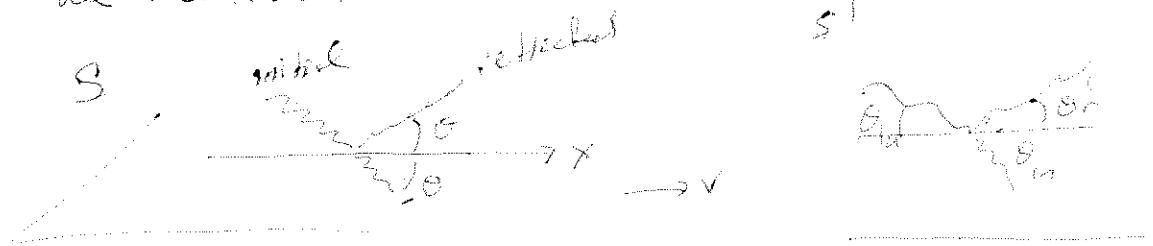
(2) and (3) give

$$\tan \theta' = \frac{\sin \theta}{\gamma_v (-\beta + \cos \theta)} = \frac{\tan \theta}{\gamma_v (1 - \beta \cos \theta)} \quad (4)$$

$$\cos \theta' = \frac{-\beta + \cos \theta}{1 - \beta \cos \theta} \quad (5)$$

$$\sin \theta' = \frac{\sin \theta}{\gamma_v (1 - \beta \cos \theta)} \quad (6)$$

5.11) we use Pr. 5.12



Angle of the γ incident photon $-\theta$
 " " reflected photon θ

under the Lorentz transformation reflected angle

$$\theta \rightarrow \theta' \Rightarrow \tan \theta'_r = \frac{\tan \theta}{\gamma_v (1 - \gamma_v \theta)}$$

$$\text{transformation of the incident angle} \quad \tan \theta'_{in} = \frac{\tan(-\theta)}{\gamma_v (1 - \gamma_v (-\theta))}$$

$$= - \frac{\tan \theta}{\gamma_v (1 - \gamma_v \theta)} = - \tan \theta'_r$$

$$\Rightarrow \theta'_{in} = - \theta'_r$$

5.13) In class we found that (see also the book page 124 eqns (5.24-5.26))

$$\tilde{\gamma} = \gamma \left[1 + \frac{h}{m_0 c^2} \gamma (1 - \cos \theta) \right]^{-1}$$

or

$$\frac{h}{c^2} \gamma \tilde{\gamma} (1 - \cos \theta) = m_0 (\gamma - \tilde{\gamma})$$

since $E = h\gamma \Rightarrow$

$$(1 - \cos \theta) \frac{h}{c^2} \frac{E \tilde{E}}{h^2} = m_0 \left(\frac{E}{h} - \frac{\tilde{E}}{h} \right)$$

$$\Rightarrow 1 - \cos \theta = m_0 c^2 \left(\frac{1}{E} - \frac{1}{\tilde{E}} \right) \quad (*)$$

$$E = h \frac{c}{\lambda} \Rightarrow$$

$$1 - \cos \theta = m_0 c^2 \left(\frac{\tilde{\lambda}}{hc} - \frac{\lambda}{hc} \right)$$

$$\Rightarrow \Delta \lambda = \tilde{\lambda} - \lambda = \frac{h}{m_0 c} (1 - \cos \theta) = \frac{2h \sin^2 \theta/2}{m_0 c} \quad \checkmark$$

$$\cos \theta = 1 - 2 \sin^2 \theta/2$$

• If $\theta = \pi/2$ and $E \ll m_0 c^2$

use the Eqs (5.25) and (5.26) in the book or the lecture notes

$$\left. \begin{aligned} \frac{h\nu}{c} &= \gamma_u m_0 u \cos\phi \\ \frac{h\tilde{\nu}}{c} &= \gamma_u m_0 u \sin\phi \end{aligned} \right\} \Rightarrow \tan\phi = \frac{\tilde{\nu}}{\nu} = \frac{\tilde{E}}{E}$$

but from the (*) Equation in the previous page given

$$\frac{1}{\tilde{E}} - \frac{1}{E} = \frac{1}{m_0 c^2} \Rightarrow \tilde{E} = \frac{1}{\frac{1}{E} + \frac{1}{m_0 c^2}}$$

$$\Rightarrow \tan\phi = \frac{1}{1 + E/m_0 c^2} \approx 1 \Rightarrow \phi = \pi/4.$$

since it is under x-axis $\phi = -\pi/4$