

Second Homework Problems: Chapter 4 of the Text Book

Nov. 4

Exercises

4.1 If t is a rank-2 tensor, show that

$$t(u+v, w+z) = t_{ab}(u^a + v^a)(w^b + z^b).$$

4.2 If $s_{ab} = s_{ba}$ and $t_{ab} = -t_{ba}$ are the components of a symmetric and an antisymmetric tensor respectively, show that $s_{ab}t^{ab} = 0$.

4.3 If t_{ab} are the components of an antisymmetric tensor and v_a the components of a vector, show that

$$v_{[a}t_{bc]} = \frac{1}{3}(v_a t_{bc} + v_c t_{ab} + v_b t_{ca}).$$

4.4 If t_{ab} are the components of a symmetric tensor and v_a the components of a vector, show that if

$$v_a t_{bc} + v_c t_{ab} + v_b t_{ca} = 0$$

then either $t_{ab} = 0$ or $v_a = 0$.

- 4.5 If the tensor t_{abcd} satisfies $t_{abcd}v^a w^b v^c w^d = 0$ for arbitrary vectors v^a and w^a , show that

$$t_{abcd} + t_{cdab} + t_{cbad} + t_{adcb} = 0.$$

- 4.6 Consider the infinitesimal coordinate transformation

$$x'^a = x^a + \epsilon v^a(x),$$

where $v^a(x)$ is a vector field and ϵ is a small scalar quantity. Show that, to first order in ϵ ,

$$g'_{ab}(x') = g_{ab}(x) - \epsilon(g_{ac}\partial_b v^c + g_{cb}\partial_a v^c).$$

- 4.7 By investigating their transformation properties, show that $\nabla_b v^a$ are the mixed components of a rank-2 tensor.
- 4.8 If v_a are the covariant components of a vector and A_{ab} are the components of an antisymmetric rank-2 tensor, show that

$$\nabla_a v_b - \nabla_b v_a = \partial_a v_b - \partial_b v_a,$$

$$\nabla_a A_{bc} + \nabla_c A_{ab} + \nabla_b A_{ca} = \partial_a A_{bc} + \partial_c A_{ab} + \partial_b A_{ca}.$$

Determine the symmetry properties of the rank-3 tensor

$$B_{abc} = \partial_a A_{bc} + \partial_c A_{ab} + \partial_b A_{ca}.$$

- 4.9 Show that covariant differentiation obeys the usual product rule, e.g.

$$\nabla_a (A_{bc} B^{cd}) = (\nabla_a A_{bc}) B^{cd} + A_{bc} (\nabla_a B^{cd}).$$

Hint: Use local Cartesian coordinates.

- 4.10 For a general rank-2 tensor T^{ab} , show that the covariant divergence is given by

$$\nabla_a T^{ab} = \frac{1}{\sqrt{|g|}} \partial_a (\sqrt{|g|} T^{ab}) + \Gamma^b_{ca} T^{ac}.$$

Show further that if $A^{ab} = -A^{ba}$ are the components of an antisymmetric rank-2 tensor then

$$\nabla_a A^{ab} = \frac{1}{\sqrt{|g|}} \partial_a (\sqrt{|g|} A^{ab}).$$

Hence show that if the antisymmetric tensor field A^{ab} vanishes on a hypersurface S that bounds a region V of an N -dimensional manifold then

$$\int_V (\nabla_a A^{ab}) \sqrt{-g} d^N x = 0.$$

- 4.11 Any coordinate transformation $x^a \rightarrow x'^a$ under which the metric is *form invariant*, i.e. such that

$$g'_{ab}(x) = g_{ab}(x)$$

is called an *isometry* (note that the argument is the same on both sides of the above equation). Show that the infinitesimal coordinate transformation in Exercise 4.6 is an isometry, to first order in ϵ , provided that v^a satisfies

$$g_{ac}\partial_b v^c + g_{cb}\partial_a v^c + v^c\partial_c g_{ab} = 0.$$

Show further that this expression can be written as

$$\nabla_a v_b + \nabla_b v_a = 0.$$

This is *Killing's equation* and any vector satisfying it is known as a *Killing vector* of the metric g_{ab} . Show that if v^a and w^a are both Killing vectors then so too is any linear combination $\lambda v^a + \mu w^a$, where λ and μ are constants.

Solutions of the second homework problems:

4.1. From linearity

$$t(u+v, w+z) = t(u, w) + t(u, z) + t(v, w) + t(v, z)$$

since t is a tensor of second rank then

$$t(u, v) = t(e_a u^a, e_b v^b) = t(e_a, e_b) u^a v^b$$

$$\text{but } t(e_a, e_b) = t_{ab} \Rightarrow$$

$$t(u, v) = t_{ab} u^a v^b$$

using this result we can write

$$\begin{aligned} t(u+v, w+z) &= t_{ab} u^a w^b + t_{ab} u^a z^b + t_{ab} v^a w^b + t_{ab} v^a z^b \\ &= t_{ab} u^a (w^b + z^b) + t_{ab} v^a (w^b + z^b) \\ &= t_{ab} (u^a + v^a) (w^b + z^b) \quad \checkmark \end{aligned}$$

$$4.2 \quad S_{ab} t^{ab} = S_{ab} \frac{1}{2} (t^{ab} + t^{ba}) = \frac{1}{2} S_{ab} t^{ab} + \frac{1}{2} S_{ab} t^{ba}$$

$$\text{since } t^{ab} = -t^{ba} \Rightarrow$$

$$S_{ab} t^{ab} = \frac{1}{2} S_{ab} t^{ab} - \frac{1}{2} S_{ab} t^{ba}$$

$$\text{since } S_{ab} = S_{ba} \Rightarrow$$

$$S_{ab} t^{ab} = \frac{1}{2} S_{ab} t^{ab} - \frac{1}{2} S_{ba} t^{ba}$$

In the RHS second term change $a \leftrightarrow b$ we get

$$S_{ab} t^{ab} = \frac{1}{2} S_{ab} t^{ab} - \frac{1}{2} S_{ba} t^{ab} = 0$$

4.3. See the definition of the bracket []

in page 95

$$t[abc] = \frac{1}{6} (t_{abc} - t_{acb} + t_{cab} - t_{cba} + t_{bca} - t_{bac})$$

let $t_{abc} = V_a S_{bc}$ where $S_{ab} = -S_{ba}$ antisymmetric tensor then

$$\begin{aligned} V[a S_{bc}] &= \frac{1}{6} [V_a S_{bc} - V_a S_{cb} + V_c S_{ab} - V_c S_{ba} + V_b S_{ca} - V_b S_{ac}] \\ &= \frac{1}{6} [2 V_a S_{bc} + 2 V_c S_{ab} + 2 V_b S_{ca}] \\ &= \frac{1}{3} [V_a S_{bc} + V_c S_{ab} + V_b S_{ca}] \end{aligned}$$

4.4 solved in exercise set.

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4.5 Since $u^a u^c$ and $w^b w^d$ are symmetric tensors we have

$$\begin{aligned} t_{abcd} u^a u^c w^b w^d &= \frac{1}{2} (t_{abcd} + t_{cbad}) u^a u^c w^b w^d \\ &= \frac{1}{2} \left[\frac{1}{2} (t_{abcd} + t_{cbad}) + \frac{1}{2} (t_{adcb} + t_{cdab}) \right] u^a u^c w^b w^d \\ &= \frac{1}{4} (t_{abcd} + t_{cbad} + t_{adcb} + t_{cdab}) u^a u^c w^b w^d \Rightarrow \end{aligned}$$

Now, since u and w are arbitrary then the coefficient tensor must vanish

$$t_{abcd} + t_{cbad} + t_{adcb} + t_{cdab} = 0$$

4.6 we know that

$$g'_{ab}(x') = \frac{\partial x^a}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} g_{ad}$$

$$\text{or } g_{ab}(x) = \frac{\partial x'^c}{\partial x^a} \frac{\partial x'^d}{\partial x^b} g'_{cd}(x')$$

$$\text{but } x'^a = x^a + \epsilon V^a(x)$$

$$\frac{\partial x'^a}{\partial x^b} = \delta^a_b + \epsilon \partial_b V^a$$

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Then

$$\begin{aligned}
 g_{ab}(x) &= (\delta_a^c + \varepsilon \partial_a v^c) (\delta_b^d + \varepsilon \partial_b v^d) g'_{cd} \\
 &= g'_{cd}(x') + \varepsilon \partial_a v^c g'_{cb} + \varepsilon \partial_b v^d g'_{ad}.
 \end{aligned}$$

in ε order $\varepsilon g' = \varepsilon g$.

$$\Rightarrow g'_{cd}(x') = g_{cb}(x) - \varepsilon (\partial_a v^c g_{cb} + \partial_b v^c g_{ca})$$

4.7 $\partial_a v = (\nabla_a v^b) e_b$ where $v = v^b e_b$

In the primed system

$$\partial_a' v = (\nabla_a' v'^b) \bar{e}_b'$$

$$\frac{\partial x^b}{\partial x'^a} \partial_b v = (\nabla_a' v'^b) \frac{\partial x^c}{\partial x'^b} \bar{e}_c$$

multiply both sides by $\frac{\partial x'^a}{\partial x^d}$ we get

$$\frac{\partial x'^a}{\partial x^d} \frac{\partial x^b}{\partial x'^c} \partial_b v = \left(\frac{\partial x'^a}{\partial x^d} \frac{\partial x^c}{\partial x'^b} \nabla_a' v'^b \right) \bar{e}_c$$

$$\partial_d v = \left(\frac{\partial x'^a}{\partial x^d} \frac{\partial x^c}{\partial x'^b} \nabla_a' v'^b \right) \bar{e}_c$$

$$\Rightarrow \nabla_d v^c = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^c}{\partial x'^b} \nabla_a' v'^b \Rightarrow$$

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Hence $\nabla_a v^b$ is a tensor of rank-2 of mixed type

$$\nabla_d v^c = \frac{\partial x'^c}{\partial x^d} \cdot \frac{\partial x'^c}{\partial x'^b} \nabla'_b v'^c$$

4.8

$$\begin{aligned} a) \quad \nabla_a t_b - \nabla_b t_a &= \partial_a t_b - \Gamma_{ab}^c t_c - (\partial_b t_a - \Gamma_{ba}^c t_c) \\ &= \partial_a t_b - \partial_b t_a - (\Gamma_{ab}^c - \Gamma_{ba}^c) t_c \end{aligned}$$

$$\text{Since } \Gamma_{ab}^c = \Gamma_{ba}^c \Rightarrow$$

$$\nabla_a t_b - \nabla_b t_a = \partial_a t_b - \partial_b t_a$$

$$b) \text{ let } A_{ab} = -A_{ba} \Rightarrow$$

$$\begin{aligned} \nabla_a A_{bc} + \nabla_c A_{ab} + \nabla_b A_{ca} &= \partial_a A_{bc} - \Gamma_{ab}^d A_{dc} - \Gamma_{ac}^d A_{bd} \\ &\quad + \partial_c A_{ab} - \Gamma_{ca}^d A_{db} - \Gamma_{cb}^d A_{ad} \\ &\quad + \partial_b A_{ca} - \Gamma_{bc}^d A_{da} - \Gamma_{ba}^d A_{cd} \end{aligned}$$

$$\begin{aligned} &= \partial_a A_{bc} + \partial_c A_{ab} + \partial_b A_{ca} - \Gamma_{ab}^d (A_{dc} + \cancel{A_{ca}}) \\ &\quad - \Gamma_{ac}^d (A_{bd} + \cancel{A_{db}}) - \Gamma_{bc}^d (A_{ad} + \cancel{A_{cb}}) \end{aligned}$$

$$= \partial_a A_{bc} + \partial_c A_{ab} + \partial_b A_{ca}$$

c)

$$c) B_{abc} = \partial_a A_{bc} + \partial_c A_{ab} + \partial_b A_{ca}$$

$$B_{acb} = \partial_a A_{cb} + \partial_b A_{ac} + \partial_c A_{ba}$$

$$= -\partial_a A_{bc} - \partial_b A_{ca} - \partial_c A_{ab} = -B_{abc}$$

$$B_{acb} = -B_{abc}$$

one can also show that

$$B_{acb} = -B_{cab}$$

Hence B is an antisymmetric tensor of rank-3.

4.9. In local cartesian coordinates $\nabla \rightarrow \partial$
and ∂ satisfies the Leibnitz rule $\Rightarrow$

$$\nabla_a (A_{bc} B^{cd}) = (\nabla_a A_{bc}) B^{cd} + A_{bc} \nabla_a B^{cd}$$

$$4.10 \quad a) \quad \nabla_c T^{ab} = \partial_c T^{ab} + \Gamma_{cd}^a T^{db} + \Gamma_{cd}^b T^{ad}$$

$$\nabla_a T^{ab} = \partial_a T^{ab} + \Gamma_{ad}^a T^{db} + \Gamma_{cd}^b T^{cd}$$

$$\Gamma_{cd}^a = \frac{\partial_a \sqrt{|g|}}{\sqrt{|g|}} \Rightarrow$$

$$\nabla_a T^{ab} = \frac{1}{\sqrt{|g|}} \partial_a (\sqrt{|g|} T^{ab}) + \Gamma_{cd}^b T^{cd}$$

b) If $A_{ab} = -A_{ba}$ and using 4.2

$$\Gamma_{ab}^{ab} A^{ab} = 0 \Rightarrow$$

$$\nabla_a A^{ab} = \frac{1}{\sqrt{|g|}} \partial_a (\sqrt{|g|} A^{ab})$$

c) if the tensor A^{ab} vanishes on a hypersurface S ~~covering~~ bounding a region V then

$$\begin{aligned} \oint_V (\nabla_a A^{ab}) dV &= \int_V (\nabla_a A^{ab}) \sqrt{|g|} d^4x \\ &= \oint_V \partial_a (\sqrt{|g|} A^{ab}) d^4x = \oint_S \cancel{\sqrt{|g|} A^{ab} n_a} ds = 0 \end{aligned}$$

4.11 If there is an isometry $x' \rightarrow x''$ then first

a) $g'(x) = g(x)$

then by 4.6 we get

$$g'_{ab}(x') = g_{ab}(x') = g_{ab}(x) - \varepsilon (g_{ac} \partial_b v^c + g_{cb} \partial_a v^c)$$

but $g_{ab}(x') = g_{ab}(x + \varepsilon v) = g_{ab}(x) + \varepsilon v^c g_{ab,c}$

$$\Rightarrow v^c g_{ab,c} + g_{ac} \partial_b v^c + g_{cb} \partial_a v^c = 0$$

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The last equation is identical with

$$\nabla_a V_b + \nabla_b V_a = 0$$

which is called the Killing's equation and the vector V_a is called the Killing vector.

————— 0 —————

b) easy.