

PHYS 405
GENERAL THEORY OF RELATIVITY
Homework I
SOLUTIONS
October 14 , 2016

PROBLEMS.

2.1. In three-dimensional Euclidean space $\mathbb{R}^3$, write down expressions for the change of coordinates from Cartesian coordinates $\{x^a\} = (x, y, z)$ to spherical polar coordinates $\{x'^a\} = (r, \phi, \theta)$. Obtain expressions for the transformation and inverse transformation matrices in terms of the primed coordinates. By calculating the Jacobians J and J' for the transformation and its inverse, find where the transformation is non-invertible.

2.2 Write down the line element for three-dimensional Euclidean space in spherical polar coordinates x^a and cylindrical polar coordinates x'^a . Hence identify the metric functions in each coordinate system and show that they obey

$$g'_{cd}(x') = g_{ab}(x) \frac{\partial x^a}{\partial x'^c} \frac{\partial x^b}{\partial x'^d}$$

2.3 In three-dimensional Euclidean space a coordinate system x'^a 's related to the Cartesian coordinates x^a by

$$x^1 = x'^1 + x'^2, \quad x^2 = x'^1 - x'^2, \quad x^3 = 2x'^1 x'^2 + x'^3$$

Describe the coordinate surfaces in the primed system. Obtain the metric functions g'_{ab} in the primed system and hence show that these coordinates are not orthogonal. Calculate the volume element dV in the primed coordinate system.

2.4 Consider the surface of a 2-sphere embedded in three-dimensional Euclidean space. In a stereographic projection, one assigns coordinates (ρ, ϕ) to each point on the surface of the sphere. The ϕ -coordinate is the standard azimuthal polar angle. The ρ -coordinate of each point is obtained by

drawing a straight line in three dimensions from the south pole of the sphere through the point in question and extending the line until it intersects the tangent plane to the north pole of the sphere; the ρ -coordinate is then the distance in the tangent plane from the north pole to the intersection point. Show that the line element for the surface of the sphere in these coordinates is

$$ds^2 = \frac{d\rho^2}{(1 + \rho^2/a^2)^2} + \frac{\rho^2}{1 + \rho^2/a^2} d\phi^2$$

At what point(s) on the sphere are these coordinates degenerate? If instead one works in terms of the Cartesian coordinates x and y in the tangent plane at the north pole, what is the corresponding form of the line element? At what point(s) on the sphere are these new coordinates degenerate?

2.12 Consider the two-dimensional space with line element

$$ds^2 = \frac{dr^2}{1 - 2V/r} + r^2 d\phi^2$$

Using the result (2.25), show that this geometry can be embedded in three dimensional Euclidean space, and find the equations for the corresponding two dimensional surface.

3.1 Show that, in general, $e_a = g_{ab} e^b$ and $e^a = g^{ab} e_b$. Show also that, under a coordinate transformation,

$$e'^a = \frac{\partial x'^a}{\partial x^b} e^b, \quad e'_a = \frac{\partial x^b}{\partial x'^a} e_b,$$

3.2 Calculate the coordinate basis vectors e'^a of the coordinates system x'^a in Exercise 2.3 in terms of the coordinate basis vectors e^a of the Cartesian system. Hence verify that the metric functions g'_{ab} agree with those found earlier. Calculate the dual basis vectors e'_a in the primed system and hence the quantities g'^{ab} . Find the contravariant and covariant components of e_1 in the primed basis. Hence verify that e_1 is of unit length.

3.3 For any metric g_{ab} show that $g^{ab} g_{ab} = N$, where N is the dimension of the manifold.

3.4 Show that the affine connection can be written as $\Gamma_{ac}^b = e^b \cdot \partial_c e_a$. Show further that, in a torsionless manifold, $\partial_a e_b = \partial_b e_a$.

3.5 Show that, under a coordinate transformation, the affine connection transforms as

$$\Gamma_{bc}^{'a} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^b} \frac{\partial x^f}{\partial x'^c} \Gamma_{ef}^d - \frac{\partial x^d}{\partial x'^b} \frac{\partial x^f}{\partial x'^c} \frac{\partial^2 x'^a}{\partial x^d \partial x^f}$$

3.6 For a diagonal metric g_{ab} , show that the connection coefficients are given by (with $a \neq b \neq c$ and no summation over repeated indices)

$$\Gamma_{bc}^a = 0, \quad \Gamma_{aa}^b = \frac{\partial_b g_{aa}}{2g_{bb}}, \quad \Gamma_{ba}^a = \partial_b \left(\ln(\sqrt{|g_{aa}|}) \right)$$

3.12 If a vector field v^a vanishes on a hypersurface S that bounds a region V of an N -dimensional manifold, show that

$$\int_V (\nabla_a v^a) \sqrt{|g|} d^N x = 0$$

3.14 If the two vectors with contravariant components v^a and w^a are each parallel transported along a curve, show that $v^a w_a$ remains constant along the curve. Hence show that if a geodesic is timelike (or null or spacelike) at some point, it is timelike (or null or spacelike) at all points.

Solutions of the first homework problems:

2.1) $x = r \cos \theta \sin \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \phi$

$T: (r, \theta, \phi) \rightarrow (x, y, z)$

$$\left(\frac{\partial x^a}{\partial x^b} \right) = \begin{pmatrix} \partial r / \partial x & \partial r / \partial y & \partial r / \partial z \\ \partial \theta / \partial x & \partial \theta / \partial y & \partial \theta / \partial z \\ \partial \phi / \partial x & \partial \phi / \partial y & \partial \phi / \partial z \end{pmatrix}$$

$$x^2 + y^2 + z^2 = r^2, \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{y}{x} = \tan \theta \Rightarrow \theta = \tan^{-1} y/x$$

$$\frac{z}{r} = \cos \phi \Rightarrow \phi = \cos^{-1}(z/r)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+y^2/x^2} \left(-\frac{y}{x^2} \right) = -\frac{y}{x^2+y^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1/x}{1+y^2/x^2} = \frac{x}{x^2+y^2}, \quad \frac{\partial \theta}{\partial z} = 0$$

$$\frac{\partial \phi}{\partial x} = \frac{-1}{\sqrt{1-z^2/r^2}} \left(\frac{-z}{r^3} x \right) = \frac{+xz}{r^2 \sqrt{x^2+y^2}}$$

$$\frac{\partial \phi}{\partial y} = + \frac{yz}{r^2 \sqrt{x^2+y^2}}$$

$$\frac{\partial \phi}{\partial z} = \frac{-1}{\sqrt{1-z^2/r^2}} \left(\frac{1}{r} - \frac{z^2}{r^3} \right) = -\frac{x^2+y^2}{r^2 \sqrt{x^2+y^2}} = -\frac{\sqrt{x^2+y^2}}{r^2}$$

(2)

$$\frac{\partial r}{\partial x} = \cos\theta \sin\phi, \quad \frac{\partial r}{\partial y} = \sin\theta \sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta$$

$$\frac{\partial \theta}{\partial x} = -\frac{1}{r} \frac{\sin\theta \sin\phi}{\sin\phi} = -\frac{\sin\theta}{r \sin\phi}$$

$$\frac{\partial \theta}{\partial y} = +\frac{\cos\theta}{r \sin\phi}, \quad \frac{\partial \theta}{\partial z} = 0$$

$$\frac{\partial \phi}{\partial x} = +\frac{1}{r} \frac{\cos\theta \sin\phi \cos\phi}{\sin\phi} = +\frac{1}{r} \cos\theta \cos\phi$$

$$\frac{\partial \phi}{\partial y} = +\frac{1}{r} \sin\theta \cos\phi, \quad \frac{\partial \phi}{\partial z} = -\frac{1}{r} \sin\phi$$

$$T = \left(\frac{\partial x'^a}{\partial x^b} \right) = \begin{pmatrix} \cos\theta \sin\phi & \sin\theta \sin\phi & \cos\theta \\ -\frac{\sin\theta}{r \sin\phi} & +\frac{\cos\theta}{r \sin\phi} & 0 \\ +\frac{1}{r} \cos\theta \cos\phi & +\frac{1}{r} \sin\theta \cos\phi & -\frac{1}{r} \sin\phi \end{pmatrix}$$

$$T^{-1} = \left(\frac{\partial x^b}{\partial x'^a} \right) = \begin{pmatrix} \cos\theta \sin\phi & -r \sin\theta \sin\phi & r \cos\theta \cos\phi \\ \sin\theta \sin\phi & r \cos\theta \sin\phi & r \sin\theta \cos\phi \\ \cos\phi & 0 & -r \sin\phi \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$J = \det T = \frac{\sin \theta}{r^2 \sin \phi} \left(\sin^2 \theta \sin^2 \phi + \sin^2 \theta \cos^2 \phi \right)$$

$$+ \frac{\cos \theta}{r^2 \sin \phi} \left(\cos^2 \theta \sin^2 \phi + \cos^2 \theta \cos^2 \phi \right)$$

$$= \frac{\sin^2 \theta}{r^2 \sin \phi} + \frac{\cos^2 \theta}{r^2 \sin \phi} = \frac{1}{r^2 \sin \phi}$$

$$J' = \det T^{-1} = r^2 \sin \phi$$

2.2) line element in spherical coordinates

$$ds^2 = dx^2 + dy^2 + dz^2 = d\rho^2 + \rho^2 d\phi^2 + \rho^2 \sin^2\phi d\theta^2$$

$$\begin{cases} dx = d\rho \cos\theta \sin\phi + \rho \sin\theta \sin\phi d\theta + \rho \cos\theta \cos\phi d\phi \\ dy = d\rho \sin\theta \sin\phi + \rho \cos\theta \sin\phi d\theta + \rho \sin\theta \cos\phi d\phi \\ dz = d\rho \cos\phi - \rho \sin\phi d\phi \end{cases}$$

line element in cylindrical coordinates

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + dz^2$$

$$x = r \cos\theta, \quad y = r \sin\theta, \quad z = z$$

$$dx = dr \cos\theta - r \sin\theta d\theta$$

$$dy = dr \sin\theta + r \cos\theta d\theta$$

$$dz = dz$$

$$\rho \cos\theta \sin\phi = r \cos\theta \Rightarrow \rho \sin\phi = r$$

$$\rho \sin\theta \sin\phi = r \sin\theta \Rightarrow \rho \sin\phi = r$$

$$\rho \cos\phi = z$$

$$\phi = \theta$$

$$\theta = \theta$$

(5)

$$g_{sph} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 \sin^2 \phi & 0 \\ 0 & 0 & r^2 \cos^2 \phi \end{pmatrix}$$

$$g_{cyl} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T = \begin{pmatrix} \frac{\partial r}{\partial \rho} & \frac{\partial r}{\partial \sigma} & \frac{\partial r}{\partial \phi} \\ \frac{\partial \theta}{\partial \rho} & \frac{\partial \theta}{\partial \sigma} & \frac{\partial \theta}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \sigma} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin \phi & 0 & r \cos \phi \\ 0 & 1 & 0 \\ \cos \phi & 0 & -r \sin \phi \end{pmatrix}$$

$$g_{sph} = \frac{\partial x^c}{\partial x^a} \frac{\partial x^d}{\partial x^b} g_{cd}$$

$$g_{sph} = T^t g_{cyl} T$$

$$= \begin{pmatrix} \sin \phi & 0 & r \cos \phi \\ 0 & 1 & 0 \\ r \cos \phi & 0 & -r \sin \phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin \phi & 0 & r \cos \phi \\ 0 & 1 & 0 \\ r \cos \phi & 0 & -r \sin \phi \end{pmatrix}$$

$$= \begin{pmatrix} \sin \phi & 0 & r \cos \phi \\ 0 & r^2 & 0 \\ r \cos \phi & 0 & -r \sin \phi \end{pmatrix} \begin{pmatrix} r \sin \phi & 0 & r \cos \phi \\ 0 & 1 & 0 \\ r \cos \phi & 0 & -r \sin \phi \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 \sin^2 \phi & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$

2.3)

$$X^1 = X'^1 + X'^2, \quad X^2 = X'^1 - X'^2, \quad X^3 = 2X'^1 X'^2 + X'^3$$

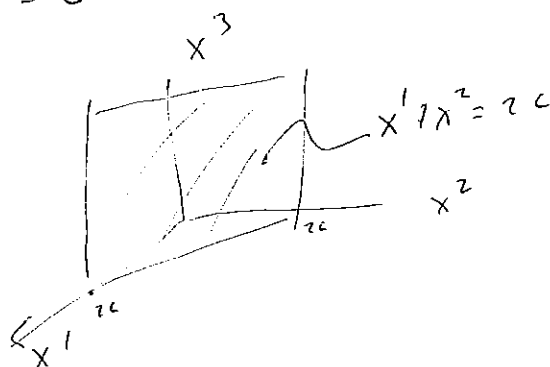
$$X'^1 = \frac{1}{2}(X^1 + X^2)$$

$$X'^2 = \frac{1}{2}(X^1 - X^2)$$

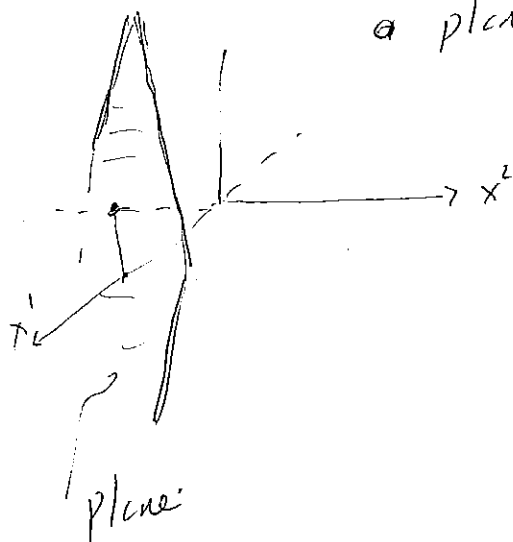
$$X'^3 = X^3 - \frac{1}{2}(X^1 + X^2)(X^1 - X^2) = X^3 - \frac{1}{2}(X'^1 - X'^2)^2$$

coordinate surfaces

a) $X'^1 = \text{constant} = C$ $X^1 + X^2 = 2C$
a plane $\perp$ xy plane

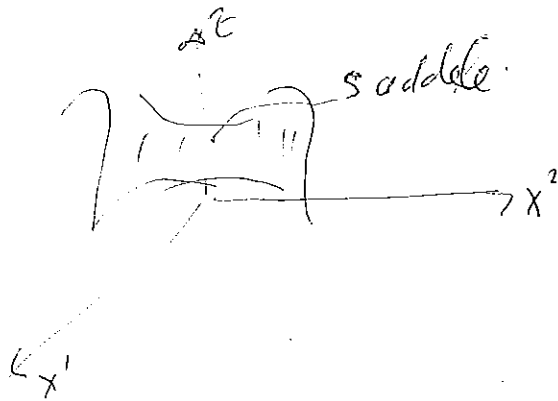


b) $X'^2 = \text{constant} = C$ $X^1 - X^2 = 2C$
a plane $\perp$ xy plane



c) $x'^3 = \text{constant} = c$

$$x'^3 - \frac{1}{2} (x'^2 - x'^2) = c$$



hyperbolic paraboloid

metric

$$ds^2 = dx'^1{}^2 + dx'^2{}^2 + dx'^3{}^2$$

$$= (dx'^1 + dx'^2)^2 + (dx'^1 - dx'^2)^2 + [2x'^1 dx'^2 + 2x'^2 dx'^1 + dx'^3]^2$$

$$= 2(dx'^1)^2 + 2(dx'^2)^2 + 4(x'^2)^2(dx'^1)^2 + 4(x'^1)^2(dx'^2)^2 + 8x'^1 x'^2 dx'^1 dx'^2 + (dx'^3)^2$$

$$+ 4(x'^1 dx'^2 + x'^2 dx'^1) dx'^3$$

$$ds^2 = (2 + 4(x'^2)^2)(dx'^1)^2 + (2 + 4(x'^1)^2)(dx'^2)^2 + 8x'^1 x'^2 dx'^1 dx'^2 + 4(x'^1 dx'^2 + x'^2 dx'^1) dx'^3 + (dx'^3)^2$$

coordinate directions:

$$\bar{\nabla}' x'^1 = \langle 1, 0, 0 \rangle = e_1$$

$$\bar{\nabla}' x'^2 = \langle 0, 1, 0 \rangle = e_2$$

$$\bar{\nabla}' x'^3 = \langle 0, 0, 1 \rangle = e_3$$

$$e'_1 \cdot e'_2 = g'_{12} \neq 0$$

or

$$\begin{aligned} d\vec{s} &= \bar{e}_a dx^a = \bar{e}_1 dx^1 + \bar{e}_2 dx^2 + \bar{e}_3 dx^3 \\ &= \bar{e}_1 (dx'^1 + dx'^2) + \bar{e}_2 (dx'^1 - dx'^2) \\ &\quad + \bar{e}_3 (dx'^3 + 2dx'^1 x'^2 + 2x'^1 dx'^2) \end{aligned}$$

$$\bar{e}'_1 = \bar{e}_1 + \bar{e}_2 + 2x'^2 \bar{e}_3$$

$$e'_2 = \bar{e}_1 - \bar{e}_2 + 2x'^1 \bar{e}_3$$

$$\bar{e}'_3 = \bar{e}_3$$

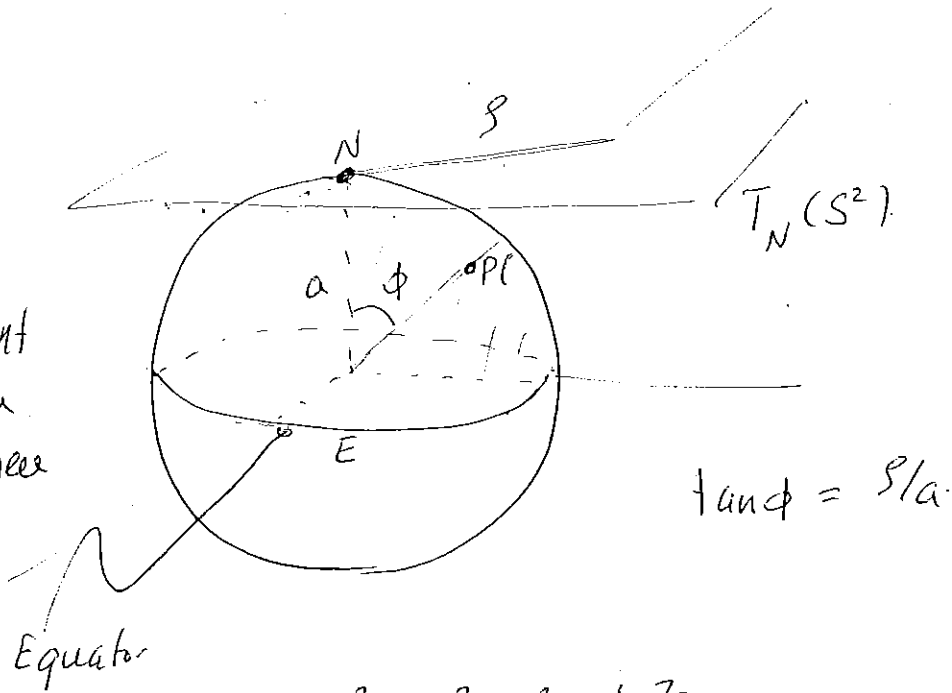
$$e'_1 \cdot e'_2 = g'_{12} = 1 + 1 + 4x'^1 x'^2 \neq 0$$

etc.

2.4)

a)

Reference point
is the center
of the sphere



$$ds^2 = a^2 d\phi^2 + a^2 \sin^2 \phi d\theta^2$$

(s, θ)
coordinate
system doesn't
cover the
point of
the equator E

$$\sin \phi = \frac{s}{\sqrt{s^2 + a^2}}, \quad \cos \phi = \frac{a}{\sqrt{s^2 + a^2}}$$

$$\frac{d\phi}{\cos^3 \phi} = \frac{1}{a} ds \Rightarrow d\phi = \frac{a ds}{s^2 + a^2}$$

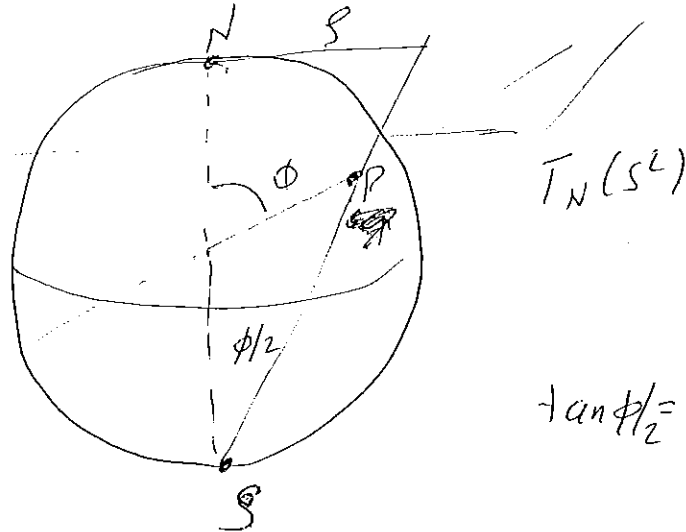
$$ds^2 = a^2 \frac{ds^2}{(s^2 + a^2)^2} + a^2 \frac{s^2}{s^2 + a^2} d\theta^2$$

$$= a^2 \left[\frac{ds^2}{(s^2 + a^2)^2} + \frac{s^2}{(s^2 + a^2)} d\theta^2 \right]$$

$$= \frac{ds^2}{(1 + s^2/a^2)^2} + \frac{s^2}{1 + s^2/a^2} d\theta^2$$

b)

Reference point
is the ~~South~~
pole (θ)



$$\tan \phi/2 = \frac{r}{2a}$$

(r, θ) coordinate
system does not
cover the south
pole

$$\sin \phi/2 = \frac{r/2a}{\sqrt{1 + r^2/4a^2}} = \frac{r}{\sqrt{r^2 + 4a^2}}$$

$$\cos \phi/2 = \frac{2a}{\sqrt{r^2 + 4a^2}}$$

$$\frac{1}{2} \frac{d\phi}{\cos^3 \phi/2} = \frac{1}{2a} dr \Rightarrow d\phi = \frac{1}{a} \cos^3 \phi/2 dr$$

$$d\phi = \frac{4a}{r^2 + 4a^2} dr$$

$$ds^2 = a^2 d\phi^2 + a^2 \sin^2 \phi d\theta^2$$

$$= \frac{16a^4}{(r^2 + 4a^2)^2} dr^2 + 4a^2 \frac{4a^2 r^2}{(r^2 + 4a^2)^2} d\theta^2$$

$$= \frac{dr^2}{(1 + r^2/4a^2)} + \frac{r^2 d\theta^2}{(1 + r^2/4a^2)} = \frac{1}{(1 + r^2/4a^2)} [dr^2 + r^2 d\theta^2]$$

2.12)

$$ds^2 = \frac{dr^2}{1-2v/r} + r^2 d\phi^2$$

Let

$$x = r \cos \phi, \quad y = r \sin \phi, \quad z = f(r)$$

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= dr^2 + r^2 d\phi^2 + f'^2 dr^2$$

$$\Rightarrow 1 + f'^2 = \frac{1}{1-2v/r} \Rightarrow f'^2 = \frac{1}{1-2v/r} - 1$$

$$f'^2 = \frac{2v/r}{1-2v/r} = \frac{2v}{r-2v}$$

 $v = \text{constant} > 0$

$$f' = \pm \sqrt{2v} \frac{1}{\sqrt{r-2v}} \Rightarrow f(r) = \pm \sqrt{2v} \sqrt{r-2v} + f_0$$

$$\begin{aligned} f(r) &= \pm 2\sqrt{2v} \sqrt{r-2v} + f_0 \\ &= \pm 2\sqrt{2vr-4v^2} + f_0 \end{aligned}$$

$$\Rightarrow ds^2 = \frac{dr^2}{1-2v/r} + r^2 d\phi^2$$

$$h_{ab} = \begin{pmatrix} 1/(1-2v/r) & 0 \\ 0 & r^2 \end{pmatrix}$$

(12)

$$x = r \cos \phi, \quad y = r \sin \phi$$

$$z = \pm 2 \sqrt{2vr - 4v^2} + f_0$$

$$\begin{aligned} \Rightarrow (z - f_0)^2 &= 4 [2vr - 4v^2] \\ &= 8v \sqrt{x^2 + y^2} - 16v^2 \end{aligned}$$

This is the equation of the surface

$r = 2v$ is not covered by
(r, ϕ) coordinate system.

3.1) Let V be a vector field, then

$$\vec{V} = \bar{e}_a V^a \quad \text{or} \quad V = \bar{e}^a V_a$$

where $\bar{e}_a \cdot \bar{e}_b = g_{ab}$

$$\bar{e}^a \cdot \bar{e}^b = g^{ab}$$

on the other hand

i) $\bar{e}^a \cdot \bar{e}_b = \delta^a_b$ (definition of $\bar{e}^a$)

$$g_{ca} \bar{e}^a \cdot \bar{e}_b = g_{ca} \delta^a_b = g_{bc}$$

$$\Rightarrow (g_{ca} \bar{e}^a) \cdot \bar{e}_b = g_{bc} = \bar{e}_b \cdot \bar{e}_c$$

$$(g_{ca} \bar{e}^a - \bar{e}_c) \cdot \bar{e}_b = 0$$

$$\Rightarrow \bar{e}_c = g_{ca} \bar{e}^a$$

ii) $\bar{e}^a \bar{e}_b g^{bc} = g^{bc} \delta^a_b = g^{ac}$
 $= \bar{e}^a \cdot \bar{e}^c$

$$\Rightarrow \bar{e}^a [\bar{e}^c - g^{bc} \bar{e}_b] = 0$$

$$\Rightarrow \bar{e}^c = g^{bc} \bar{e}_b$$

iii)

$$d\vec{s} = \vec{e}_a \cdot dx^a = \vec{e}'_a dx'^a$$

$$dx'^a = \frac{\partial x'^a}{\partial x^b} dx^b$$

$$d\vec{s} = \vec{e}_a dx^a = \vec{e}'_a \frac{\partial x'^a}{\partial x^b} dx^b$$

$$\Rightarrow \vec{e}_c dx^c - \vec{e}'_a \frac{\partial x'^a}{\partial x^c} dx^c = 0$$

$$(\vec{e}_c - \vec{e}'_a \frac{\partial x'^a}{\partial x^c}) dx^c = 0$$

$$\Rightarrow \vec{e}_c = \frac{\partial x'^a}{\partial x^c} \vec{e}'_a$$

Since $\frac{\partial x'^a}{\partial x^c} \frac{\partial x^c}{\partial x'^b} = \delta^a_b$

$$\vec{e}'_a = \frac{\partial x^b}{\partial x'^a} \vec{e}_b$$

iv)

~~$$e'^a \frac{\partial x^c}{\partial x'^b} \vec{e}_c = \vec{e}'_b \delta^a_b = \vec{e}'_a$$

$$\Rightarrow \vec{e}'_a = \frac{\partial x^b}{\partial x'^a} \vec{e}_b$$~~

$$iv) \quad \bar{e}^{1a} \cdot e'_a = \bar{e}^a \cdot \bar{e}_a = n \quad (\text{dimension})$$

$$\bar{e}^{1a} \frac{\partial x^b}{\partial x'^a} \cdot \bar{e}_b = \bar{e}^b \cdot \bar{e}_b$$

$$\left(\bar{e}^{1a} \frac{\partial x^b}{\partial x'^a} \right) \bar{e}_b = \bar{e}^b \cdot \bar{e}_b$$

$$\left(\bar{e}^{1a} \frac{\partial x^b}{\partial x'^a} - \bar{e}^b \right) \cdot \bar{e}_b = 0$$

$$\Rightarrow \bar{e}^{1a} \frac{\partial x^b}{\partial x'^a} - \bar{e}^b = 0$$

$$\text{or } \Rightarrow \bar{e}^{1a} = \frac{\partial x'^a}{\partial x^b} \bar{e}^b$$

3.3) Since

$$g^{ac} g_{cb} = \delta^a_b$$

$$\Rightarrow g^{ac} g_{ca} = \delta^a_a = \underbrace{\delta^1_1 + \delta^2_2 + \dots + \delta^N_N}_{N\text{-times}}$$

$$= 1 + 1 + \dots + 1 = N$$

= dimension of the manifold

$$\underline{3.4}) \quad \partial_b \bar{e}_a = \Gamma_{ba}^c \bar{e}_c$$

$$\text{or} \quad e^c \cdot (\partial_b \bar{e}_a) = \Gamma_{ba}^c$$

Since the connection is torsionless

$$\Gamma_{ba}^c = \Gamma_{ab}^c$$

$$\Rightarrow \partial_b \bar{e}_a = \partial_a \bar{e}_b$$

3.5) in (3.11b) it is shown that

$$\Gamma_{bc}^{1a} = \frac{\partial x^{1a}}{\partial x^d} \frac{\partial x^f}{\partial x^{1b}} \frac{\partial x^g}{\partial x^{1c}} \Gamma_{fg}^d + \frac{\partial x^{1a}}{\partial x^d} \frac{\partial^2 x^d}{\partial x^{1c} \partial x^{1b}}$$

$$\text{Since} \quad \frac{\partial x^{1a}}{\partial x^d} \frac{\partial x^d}{\partial x^{1b}} = \delta_b^a$$

$$\Rightarrow \frac{\partial^2 x^{1a}}{\partial x^{1c} \partial x^d} \frac{\partial x^d}{\partial x^{1b}} + \frac{\partial x^{1a}}{\partial x^d} \frac{\partial^2 x^d}{\partial x^{1c} \partial x^{1b}} = 0$$

$$\begin{aligned} \Rightarrow \frac{\partial x^{1a}}{\partial x^d} \frac{\partial^2 x^d}{\partial x^{1c} \partial x^{1b}} &= - \frac{\partial}{\partial x^{1c}} \left(\frac{\partial x^{1a}}{\partial x^d} \right) \frac{\partial x^d}{\partial x^{1b}} \\ &= - \frac{\partial x^f}{\partial x^{1c}} \left(\frac{\partial^2 x^{1a}}{\partial x^f \partial x^d} \right) \frac{\partial x^d}{\partial x^{1b}} \end{aligned}$$

$\Rightarrow$

$$\Gamma_{bc}^{a} = \frac{\partial x^a}{\partial x^d} \frac{\partial x^f}{\partial x^b} \frac{\partial x^g}{\partial x^c} \Gamma_{fg}^d - \frac{\partial x^f}{\partial x^c} \frac{\partial x^d}{\partial x^b} \frac{\partial^2 x^a}{\partial x^f \partial x^d}$$

3.6) Christoffel symbol Γ_{bc}^a

(18)

$$\Gamma_{bc}^a = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc})$$

if g_{ab} is diagonal, i.e. $g_{ab} = 0$ if $a \neq b$

i) if all $a \neq b \neq c$

$$\Gamma_{bc}^a = \frac{1}{2} g^{aa} (\partial_b g_{ac} + \partial_c g_{ab} - \partial_a g_{bc})$$

no sum on a .

$$g_{ac} = 0, g_{ab} = 0, g_{bc} = 0$$

$$\Rightarrow \Gamma_{bc}^a = 0$$

$$\text{ii) } \Gamma_{aa}^b = \frac{1}{2} g^{bd} (\partial_a g_{da} + \partial_a g_{da} - \partial_d g_{aa})$$

$$= \frac{1}{2} g^{bb} (\partial_a g_{ba} + \partial_a g_{ba} - \partial_b g_{aa})$$

$$g_{ab} = 0$$

$$= -\frac{1}{2} g^{bb} \partial_b g_{aa} = -\frac{1}{2 g_{bb}} \partial_b g_{aa}$$

$$\text{iii) } \Gamma_{ba}^a = \frac{1}{2} g^{ad} (\partial_b g_{da} + \partial_a g_{db} - \partial_d g_{ba})$$

$$= \frac{1}{2} g^{aa} (\partial_b g_{aa} + \cancel{\partial_a g_{ab}} - \cancel{\partial_a g_{ba}}) = \frac{1}{2} \frac{\partial_b g_{aa}}{g_{aa}}$$

$$= \partial_b \ln(\sqrt{g_{aa}})$$

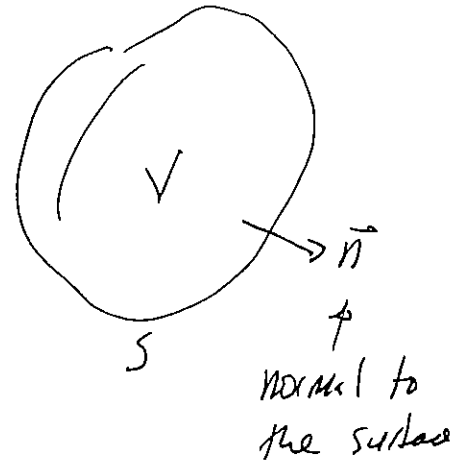
3.12) Let $v^a = 0$ on S which is the boundary of a region V of an N -dimensional manifold, then

(19)

$$\int_V (\nabla_a v^a) \sqrt{|g|} d^N x = \int_V (\nabla_a v^a) dV$$

$$= \oint_S v^a \cdot n_a ds$$

(divergence thm)



Since $v^a = 0$ on S .

the surface integral vanishes

identically $\Rightarrow$

$$\int_V (\nabla_a v^a) \sqrt{|g|} d^N x = 0$$

3.14) Let C be a curve with parameter $u \in I$ i.e. $\{x^a(u)\}$. let $\bar{v}$ and $\bar{w}$ be parallelly transported vector field along C . i.e.

$$\frac{d}{du} \bar{v} = \frac{Dv^a}{du} \cdot e_a = 0 \Rightarrow \frac{Dv^a}{du} = \frac{dv^a}{du} + \Gamma_{bc}^a v^b \frac{dx^c}{du} = 0$$

$$\frac{d}{du} \bar{w} = \frac{Dw^a}{du} \cdot e_a = 0 \Rightarrow \frac{Dw^a}{du} = \frac{dw^a}{du} + \Gamma_{bc}^a w^b \frac{dx^c}{du} = 0$$

$$i) \frac{d}{du} (\bar{v} \cdot \bar{w}) = \frac{d\bar{v}}{du} \cdot \bar{w} + \bar{v} \cdot \frac{d\bar{w}}{du} = 0 \text{ on } C$$

but $\bar{v} \cdot \bar{w} = v^a w_a = \text{constant}$ along the curve

This means that timelike, spacelike, and null at

ii) if C is a geodesic then

it's tangent vector is parallelly transported

$$\frac{d\bar{t}}{du} = 0 \quad \text{along the curve (for all } u\text{)}$$

$$\Rightarrow \frac{d}{du} (\bar{t} \cdot \bar{t}) = 0 \Rightarrow \bar{t} \cdot \bar{t} = \text{constant}$$

- Hence if $\bar{t} \cdot \bar{t} > 0$ at a point on C
it $\bar{t} \cdot \bar{t} > 0$ (space like) at all points
- if $\bar{t} \cdot \bar{t} < 0$ at a point on C it is
 $\bar{t} \cdot \bar{t} < 0$ (time like) at all points.
- if $\bar{t} \cdot \bar{t} = 0$ at a point on C it is
 $\bar{t} \cdot \bar{t} = 0$ (null) at all points of C .