PHYS 405
GENERAL THEORY OF RELATIVITY
Exercises set 2

November 24 , 2016

EXERCISES.

1.(Exercise 7.13) Show that for any two-dimensional manifold the covariant curvature tensor has the form

$$R_{abcd} = K\,(g_{ac}\,g_{bd} - g_{ad}\,g_{cb}) \tag{1}$$

Proof of (a): The curvature tensor R_{abcd} and the tensor $K_{abcd} = g_{ac}\,g_{bd} - g_{ad}\,g_{cb}$ have the symmetries and hence only has one independent componenets R_{1212} and K_{1212} respectively.

$$K_{abcd} = -K_{bacd}, \tag{2}$$
$$K_{abcd} = -K_{abdc}, \tag{3}$$
$$K_{abcd} + K_{acdb} + K_{adbc} = 0 \tag{4}$$

Hence

$$R_{abcd} = K\,(g_{ac}\,g_{bd} - g_{ad}\,g_{cb}) \tag{5}$$

where K is given in terms of the Ricci scalar. By contracting the indices we get

$$R_{bc} = -K\,g_{bc} \tag{6}$$

Hence $R = -2K$. We find that

$$R_{abcd} = -\frac{1}{2}\,R\,(g_{ac}\,g_{bd} - g_{ad}\,g_{cb}) \tag{7}$$

So in two dimensions if the Ricci scalar vanishes at all points then the curvature vanishes at all points of the manifold. Hence the geometry is flat Euclidean space.

2.(Exercise 7.10) (a) Show that the line element $ds^2 = y^2dx^2 + x^2dy^2$ represents the Euclidean plane, (b) but the line element $ds^2 = ydx^2 + xdy^2$ represents a curved two-dimensional manifold.

Proof of (a): As we have shown in the above problem in two dimensions it is enough to find the Ricci scalar to understand whether a 2- dimensional manifold is curved or not. Let us calculate the Ricci tensor. For this purpose we first calculate the christoffel symbol with $g_{11} = y^2,\ g_{22} = x^2$

$$\Gamma^1{}_{11} = 0,\ \Gamma^1{}_{12} = \Gamma^1{}_{21} = \frac{1}{y},\ \gamma^1{}_{22} = -\frac{x}{y^2}, \tag{8}$$

$$\Gamma^2{}_{11} = -\frac{y}{x^2},\ \Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{1}{x},\ \Gamma^2{}_{22} = 0, \tag{9}$$

Hence

$$\begin{aligned} R_{11} &= \partial_2 \Gamma^2{}_{11} - \partial_1 \Gamma^2{}_{21} + \Gamma^1{}_{12}\Gamma^2{}_{11} && (10)\\ &\quad -\Gamma^1{}_{12}\Gamma^2{}_{11} - \Gamma^2{}_{11}\Gamma^1{}_{12} - \Gamma^2{}_{12}\Gamma^2{}_{12}, && (11)\\ &= -\partial_y (y/x^2) + \frac{1}{x^2} - (y/x^2)\frac{1}{y} + \frac{2}{y}(y/x^2) - \frac{1}{x^2} = 0 && (12) \end{aligned}$$

Hence $R_{11} = \frac{1}{2}R\, g_{11} = 0$ since $g_{11} = y^2 \neq 0$ then $R = 0$ everywhere except at $y = 0$. By calculating R_{22} we find that it also vanishes. Hence $R_{22} = \frac{1}{2}Rg_{22} = 0$. So $R = 0$ everywhere except $x^2 \neq 0$. Combining these two results we conclude that $R = 0$ everywhere in the manifold. Hence the geometry is a plane Euclidean geometry.

Proof of (b): In this case we have $g_{11} = y,\ g_{22} = x$ and

$$\Gamma^1{}_{11} = 0,\ \Gamma^1{}_{12} = \Gamma^1{}_{21} = \frac{1}{2y},\ \gamma^1{}_{22} = -\frac{1}{2y}, \tag{13}$$

$$\Gamma^2{}_{11} = -\frac{1}{2x},\ \Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{1}{2x},\ \Gamma^2{}_{22} = 0, \tag{14}$$

Hence

$$\begin{aligned} R_{11} &= \partial_2 \Gamma^2{}_{11} - \partial_1 \Gamma^2{}_{21} + \Gamma^1{}_{12}\Gamma^2{}_{11} && (15)\\ &\quad -\Gamma^1{}_{12}\Gamma^2{}_{11} - \Gamma^2{}_{11}\Gamma^1{}_{12} - \Gamma^2{}_{12}\Gamma^2{}_{12}, && (16)\\ &= \frac{1}{4xy} - \frac{1}{4x^2} && (17) \end{aligned}$$

Hence $R = \frac{1}{2yx}\,(\frac{1}{y} - \frac{1}{x})$. This result says the manifold is not flat it is a curved 2-dimensional Riemannian space

3.(Exercise 7.17) Show that any Killing vector v^μ, as defined in Exercise 4.11, satisfies the relations

$$\begin{aligned} (a) \quad \nabla_c \nabla_b v^a &= R^a{}_{bcd}\, v^d, & (18)\\ (b) \quad v^a \nabla_a R &= 0 & (19) \end{aligned}$$

Proof of (a): Using the Ricci identity

$$\nabla_c \nabla_b v_a - \nabla_b \nabla_c v_a = -R_{bcad}\, v^d \tag{20}$$

Writing the above equation three times permutating the indices $\{abc\}$

$$\begin{aligned} \nabla_c \nabla_b v_a - \nabla_b \nabla_c v_a &= -R_{bcad}\, v^d, & (21)\\ \nabla_a \nabla_c v_b - \nabla_c \nabla_a v_b &= -R_{cabd}\, v^d, & (22)\\ \nabla_b \nabla_a v_c - \nabla_a \nabla_b v_c &= -R_{abcd}\, v^d, & (23) \end{aligned}$$

adding these equations and using the cyclic identity for the curvature tensor

$$\nabla_c \nabla_b v_a - \nabla_b \nabla_c v_a + \nabla_a \nabla_c v_b - \nabla_c \nabla_a v_b + \nabla_b \nabla_a v_c - \nabla_a \nabla_b v_c = 0 \tag{24}$$

This equation can be simplified

$$2(\nabla_c \nabla_b v_a + \nabla_a \nabla_c v_b + \nabla_b \nabla_a v_c) = 0 \tag{25}$$

The second term can be re-written as

$$\nabla_c \nabla_b v_a - \nabla_a \nabla_b v_c + \nabla_b \nabla_a v_c = 0 \tag{26}$$

For the last two terms we can use the Ricci identity once more and obtain

$$\nabla_c \nabla_b v_a - R_{abcd}\, v^d = 0 \tag{27}$$

or

$$\nabla_c \nabla_b v_a = R_{abcd}\, v^d \tag{28}$$

Proof of (b): From the Ricci identity for a second rank tensor T^{ab}

$$\nabla_d \nabla_c T^{ab} - \nabla_c \nabla_d T^{ab} = -R^a{}_{ecd}\, T^{eb} - R^b{}_{ecd}\, T^{ae} \tag{29}$$

Contraction the indices a with d and c with b we get

$$\nabla_a \nabla_b T^{ab} - \nabla_b \nabla_a T^{ab} = -R_{eb}\, T^{eb} + R_{eb}\, T^{be} \tag{30}$$

or

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) T^{ab} = R_{eb}\, (-T^{eb} + T^{be}) \tag{31}$$

The right hand side is identically zero because the Ricci tensor R_{eb} is symmetric and $(-T^{eb} + T^{be})$ with respect the indices (eb). Letting $T^{ab} = \nabla^a v^b$ where v^a is a Killing vector then

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \nabla^a v^b = 2 \nabla_a \nabla_b (\nabla^a v^b) = 0 \tag{32}$$

From part (a) we can get

$$\nabla_b (\nabla^a v^b) = -R^a{}_b\, v^b \tag{33}$$

Hence from (32)

$$\nabla_a (R^a{}_b\, v^b) = 0 \tag{34}$$

This equation leads to

$$\nabla_a (R^a{}_b)\, v^b + R^{ab} \nabla_a v_b = 0 \tag{35}$$

the last term is identically zero (because Ricci tensor is symmetric and $\nabla_a v_b$ is anti symmetric tensor) and since $\nabla^a (R_{ab} - \frac{1}{2} R\, g_{ab}) = 0$ we get the result

$$v^a\, \partial_b R = 0 \tag{36}$$

4.(Exercise 7.18) Calculate explicit forms for the Ricci tensor R_{ab}, and the Ricci scalar R interms of the metric, the connection and its partial derivatives.

Proof. Remember the defintion of the curvature tensor.

$$R^d{}_{abc} = \partial_b \Gamma^d{}_{ac} - \partial_c \Gamma^d{}_{ab} + \Gamma^e{}_{ac} \Gamma^d{}_{eb} - \Gamma^e{}_{ab} \Gamma^d{}_{ec} \tag{37}$$

Contarcting the indices d and c we get the Ricci tensor

$$R^c{}_{abc} = R_{ab} = \partial_b \Gamma^d{}_{ad} - \partial_d \Gamma^d{}_{ab} + \Gamma^e{}_{ac} \Gamma^d{}_{eb} - \Gamma^e{}_{ab} \Gamma^d{}_{ed} \tag{38}$$

contracting the indices a and b we the Ricci scalar R

$$g^{ab} R_{ab} = R = g^{ab} \left(\partial_b \Gamma^d{}_{ad} - \partial_d \Gamma^d{}_{ab} + \Gamma^e{}_{ac} \Gamma^d{}_{eb} - \Gamma^e{}_{ab} \Gamma^d{}_{ed}\right) \tag{39}$$

5.(Exercise 7.19) Prove that the Ricci tensor is symmetric

Proof. The cyclic identity of the curvature tensor

$$R_{abcd} + R_{acdb} + R_{adbc} = 0 \tag{40}$$

Contract the indices a and d and remember the definition of the Ricci tensor $R_{bc} = R^{a}{}_{bca}$

$$R_{bc} + R^{a}{}_{cab} + R^{a}{}_{abc} = 0 \tag{41}$$

The last term vanishes identically and second tensor gives $R^{a}{}_{cab} = -R^{a}{}_{cba} = -R_{cb}$. Hence

$$R_{bc} - R_{cb} = 0 \tag{42}$$

or $R_{bc} = R_{cb}$. Hence the Ricci tensor is symmetric.