PHYS 405
GENERAL THEORY OF RELATIVITY
Exercises

October 24 , 2016

EXERCISES.

1.(Exercise 4.4) If t_{ab} are the components of a symmetric tensor ad v_b the components of a vector show that if

$$v_a\, t_{bc} + v_c\, t_{ab} + v_b\, t_{ca} = 0 \tag{1}$$

then either $t_{ab} = 0$ or $v_a = 0$

Proof; Some information: Contraction of indices: Contraction of u^a with t_{ab} gives a vector,call it v_b

$$u^a\, t_{ab} = v_b$$

Contraction of u^a with t_{ba} gives the same vector v_b, because $t_{ab} = t_{ba}$ is a symmetric tensor.

(i) If $u^a = 0$ for all $a = 1, 2, \cdots, n$, then equation (1) is satisfied identically.
(ii) If $u^a \neq 0$ (at least one component is different than zero. Contracting equation (1) by u^a we get

$$\lambda t_{bc} + u_c\, v_b + u_b\, v_c = 0 \tag{2}$$

or

$$t_{bc} = -\frac{1}{\lambda}[u_c\, v_b + u_b\, v_c] \tag{3}$$

where $\lambda = u^a\, u_a \neq 0$. Contracting equation (3) by u^b we get

$$v_c = -\frac{1}{\lambda}[u_c\, (u^b\, v_b) + \lambda\, v_c] \tag{4}$$

or

$$v_c = -\frac{1}{2\,\lambda}\,(u^b\,v_b)\,u_c \tag{5}$$

Lastly contraction the las equation by u^b we obtain

$$\lambda\,u^b\,v_b = 0 \tag{6}$$

Since $\lambda \neq 0$ then $u^b\,v_b = 0$. This implies that, from equation (5) $v_c = 0$. This finally implies, from equation (3) that $t_{bc} = 0$.

2.(Exercise 4.8) If v_a are components of a vector and A_{ab} are the components of an antisymmetric tensor rank-2 tensor, show that

$$\nabla_a\,v_b - \nabla_b\,v_a = \partial_a\,v_b - \partial_b\,v_a, \tag{7}$$

$$\nabla_a A_{bc} + \nabla_c\,A_{ab} + \nabla_b\,A_{ca} = \partial_a A_{bc} + \partial_c\,A_{ab} + \partial_b\,A_{ca} \tag{8}$$

Determine the symmetry properties of the rank-3 tensor

$$B_{abc} = \partial_a A_{bc} + \partial_c\,A_{ab} + \partial_b\,A_{ca} \tag{9}$$

Proof. let us recall the covariant derivatives of a vector and a second rank tensor

$$\nabla_a\,v_b = \partial_a\,v_b - \Gamma^c_{ab}\,v_c, \tag{10}$$

$$\nabla_a\,A_{bc} = \partial_a\,A_{bc} - \Gamma^d_{ab}\,A_{dc} - \Gamma^d_{ac}\,A_{bd}. \tag{11}$$

(i) Using (10) we get

$$\nabla_a\,v_b - \nabla_b\,v_a = \partial_a\,v_b - \Gamma^c_{ab}\,v_c - \partial_a\,v_b + \Gamma^c_{ab}\,v_c \tag{12}$$

Terms with the Christoffel symbol cancel each other, then we obtain

$$\nabla_a\,v_b - \nabla_b\,v_a = \partial_a\,v_b - \partial_b\,v_a, \tag{13}$$

(ii) Using *11) we have

$$\nabla_a\,A_{bc} + \nabla_c\,A_{ab} + \nabla_b\,A_{ca} = \partial_a\,A_{bc} - \Gamma^d_{ab}\,A_{dc} - \Gamma^d_{ac}\,A_{bd} + \tag{14}$$

$$\partial_c\,A_{ab} - \Gamma^d_{ca}\,A_{db} - \Gamma^d_{cb}\,A_{ad} + \partial_b\,A_{ca} - \Gamma^d_{bc}\,A_{da} - \Gamma^d_{ba}\,A_{cd} \tag{15}$$

$$= \partial_a\,A_{bc} + \partial_c\,A_{ab} + \partial_b\,A_{ca} - \Gamma^d_{ab}\,(A_{dc} + A_{cd}) - \Gamma^d_{ac}\,(A_{bd} + A_{db}) - \Gamma^d_{bc}\,(A_{da} + A_{ad})$$

Due to the antisymmetry property of A_{ab}, ie., $A_{ab} = -A_{ba}$, terms with the Christoffel symbol cancel each other, then we obtain

$$\nabla_a A_{bc} + \nabla_c A_{ab} + \nabla_b A_{ca} = \partial_a A_{bc} + \partial_c A_{ab} + \partial_b A_{ca} \tag{16}$$

(iii) It can be shown easily that the tensor B_{abc} is antisymmetric with respect to any two indices,i.e., $B_{abc} = -B_{acb}$, $B_{abc} = -B_{bac}$,$B_{abc} = -B_{cba}$. For instance

$$B_{abc} + B_{acb} = \partial_a A_{bc} + \partial_c A_{ab} + \partial_b A_{ca} + \partial_a A_{cb} + \partial_c A_{ba} + \partial_b A_{ac} \tag{17}$$

$$= \partial_a (A_{bc} + A_{cb}) + \partial_c (A_{ab} + A_{ba}) + \partial_b (A_{ca} + A_{ac}) = 0 \tag{18}$$