PHYS 405: THEORY OF GENERAL RELATIVITY
Final Exam
January 13, 2017; 09.00-11.00 (SB-Z10)

QUESTIONS: Choose any four of the following six problems

[25] **1**. Show that the Einstein tensor $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}$ satisfies divergence free condition $\nabla_\mu G^{\mu\nu} = 0$.

[25] **2**. Show that, on introduction of the retarded Eddington-Finkelstein timelike coordinate $ct^* = ct - 2\mu \ln|r/2\mu - 1|$, the Schwarzschild line element takes the form

$$ds^2 = c^2\,(1 - \frac{2\mu}{r})dt^{*2} + \frac{4\mu c}{r} dt^*\,dr - (1 + \frac{2\mu}{r})\,dr^2 - r^2\,(d\theta^2 + \sin^2\theta\,d\phi^2)$$

where $\mu = \frac{MG}{c^2}$. Hence find the equations for the worldlines of radially moving photons in retarded Eddington-Finkelstein coordinates. Use this result to sketch the spacetime diagram showing the light-cone structure in this coordinate system.

[25] **3**. A particle at infinity in the Schwarzschild geometry is moving radially inwards with coordinate speed u_0. Show that at any coordinate radius r the coordinate velocity is given by

$$\left(\frac{dr}{dt}\right)^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right)^2 \left[1 - \frac{1}{\gamma_0^2}\left(1 - \frac{2GM}{c^2 r}\right)\right]$$

where $\gamma_0 = (1 - u_0^2/c^2)^{-1/2}$. Determine the velocity relative to a stationary observer at r, and show that this velocity tends to c as r tends to $2GM/c^2$, irrespective of the value of u_0.

[25] **4**. Find all circular orbits of massive particles in Schwarzschild geometry

[25] **5**. For a pure 4-force $\mathbf{f}$ acting on a particle of rest mass m_0, show that the corresponding 3-force $\vec{f}$ satisfies

$$\vec{f} = \gamma_u\, m_0\, \vec{a} + \left(\frac{\vec{f}\cdot\vec{u}}{c^2}\right)\vec{u}$$

Hence show that $\vec{a}$ is only parallel to $\vec{f}$ when $\vec{f}$ is either parallel or orthogonal to $\vec{u}$. Show that, in these two cases, one has $\vec{f} = (\gamma_u)^3\, m_0\, \vec{a}$ and $\vec{f} = \gamma_u\, m_0\, \vec{a}$ respectively. Here $\gamma_u = (1 - u^2/c^2)^{-1/2}$

[25] **6**. If $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| << 1$, **(a)** verify that, to the first order in $h_{\mu\nu}$

$$R_{\mu\nu} = \frac{1}{2}(\partial_\nu\, \partial_\mu h + \Box^2\, h_{\mu\nu} - \partial_\nu\, \partial_\rho\, h^\rho_\mu - \partial_\mu\, \partial_\rho\, h^\rho_\nu, \tag{1}$$

$$R = \Box^2 h - \partial_\rho\, \partial_\mu\, h^{\rho\mu} \tag{2}$$

where $h = \eta^{\mu\nu}\, h_{\mu\nu}$ and $\Box^2 = \eta^{\mu\nu}\, \partial_\mu\, \partial_\nu$. **(b)** Let $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$ then show that the linearized Einstein filed equations can be written as

$$\Box^2\, \bar{h}_{\mu\nu} + \eta_{\mu\nu}\, \partial_\alpha \partial_\beta\, \bar{h}^{\alpha\beta} - \partial_\nu\, \partial_\rho\, \bar{h}^\rho_\mu - \partial_\mu\, \partial_\rho\, \bar{h}^\rho_\nu = -2\kappa\, T_{\mu\nu} \tag{3}$$

(c) show that $\partial_\mu\, T^{\mu\nu} = 0$.

Appendix:

1. Christoffel symbol

$$\Gamma^a_{bc} = \frac{1}{2}\, g^{ad}\, (\partial_b\, g_{cd} + \partial_c\, g_{bd} - \partial_d\, g_{cb}) \tag{4}$$

2. Lorentz Transformations in matrix form

$$[\Lambda^\mu_\nu] = \begin{pmatrix} \gamma & -\beta\,\gamma & 0 & 0 \\ -\beta\,\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

3. The curvature tensor

$$R^d{}_{abc} = \partial_b\, \Gamma^d{}_{ac} - \partial_c\, \Gamma^d{}_{ab} + \Gamma^e{}_{ac}\, \Gamma^d{}_{eb} - \Gamma^e{}_{ab}\, \Gamma^d_{ec}$$

4. The Ricci tensor

$$R_{ab} \equiv R^d{}_{abd} = \partial_b\, \Gamma^d{}_{ad} - \partial_d\, \Gamma^d{}_{ab} + \Gamma^e{}_{ad}\, \Gamma^d{}_{eb} - \Gamma^e{}_{ab}\, \Gamma^d_{ed}$$

5. The Ricci Scalar

$$R \equiv g^{ab}\, R_{ab}$$

6. The Scwarschild Metric

$$ds^2 = c^2\,(1 - \frac{2\mu}{r})dt^2 - \frac{1}{1 - \frac{2\mu}{r}}\, dr^2 - r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

where $\mu = \frac{MG}{c^2}$

7. Geodesic Equations for a massive particle at the equatorial plane $\theta = \frac{\pi}{2}$.

$$(1 - \frac{2\mu}{r})\,\dot{t} = k, \tag{6}$$

$$c^2(1 - \frac{2\mu}{r})\dot{t}^2 - (1 - \frac{2\mu}{r})^{-1}\dot{r}^2 - r^2\,\dot{\phi}^2 = c^2, \tag{7}$$

$$r^2\,\dot{\phi} = h \tag{8}$$

where $k = \frac{E}{m_0 c^2}$ and h is the angular velocity of the test particle

8. Geodesic Equations for a massless particle at the equatorial plane $\theta = \frac{\pi}{2}$.

$$(1 - \frac{2\mu}{r})\,\dot{t} = k, \tag{9}$$

$$c^2(1 - \frac{2\mu}{r})\dot{t}^2 - (1 - \frac{2\mu}{r})^{-1}\dot{r}^2 - r^2\,\dot{\phi}^2 = 0, \tag{10}$$

$$r^2\,\dot{\phi} = h \tag{11}$$

where $k = \frac{E}{m_0 c^2}$ and h is the angular velocity of the test particle

9. Symmetries of the Curvature tensor

$$R_{dabc} = -R_{adbc} = -R_{dacb} \tag{12}$$

$$R_{adbc} = R_{bcad}, \quad R_{dabc} + R_{dcab} + R_{dcab} = 0, \tag{13}$$

$$\nabla_a\, R_{bced} + \nabla_b\, R_{caed} + \nabla_c\, R_{abed} = 0 \tag{14}$$