QUESTIONs: Choose any four of the following six problems

[25] 1. Show that the Einstein tensor \( G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \) satisfies divergence free condition \( \nabla_\mu G^{\mu\nu} = 0 \).

[25] 2. Show that, on introduction of the retarded Eddington-Finkelstein timelike coordinate \( ct^\ast = ct - 2\mu \ln |r/2\mu - 1| \), the Schwarzschild line element takes the form

\[
ds^2 = c^2 \left( 1 - \frac{2\mu}{r} \right) dt^\ast^2 + \frac{4\mu c}{r} dt^\ast dr - \left( 1 + \frac{2\mu}{r} \right) dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)\]

where \( \mu = \frac{MG}{c^2} \). Hence find the equations for the worldlines of radially moving photons in retarded Eddington-Finkelstein coordinates. Use this result to sketch the spacetime diagram showing the light-cone structure in this coordinate system.

[25] 3. A particle at infinity in the Schwarzschild geometry is moving radially inwards with coordinate speed \( u_0 \). Show that at any coordinate radius \( r \) the coordinate velocity is given by

\[
\left( \frac{dr}{dt} \right)^2 = c^2 \left( 1 - \frac{2GM}{c^2r} \right)^2 \left[ 1 - \frac{1}{\gamma_0^2} \left( 1 - \frac{2GM}{c^2r} \right) \right]
\]

where \( \gamma_0 = (1 - u_0^2/c^2)^{-1/2} \). Determine the velocity relative to a stationary observer at \( r \), and show that this velocity tends to \( c \) as \( r \) tends to \( 2GM/c^2 \), irrespective of the value of \( u_0 \).

[25] 4. Find all circular orbits of massive particles in Schwarzschild geometry

[25] 5. For a pure 4-force \( \mathbf{f} \) acting on a particle of rest mass \( m_0 \), show that the corresponding 3-force \( \vec{f} \) satisfies

\[
\vec{f} = \gamma_u m_0 \vec{a} + \left( \frac{\vec{f} \cdot \vec{u}}{c^2} \right) \vec{u}
\]
Hence show that $\vec{a}$ is only parallel to $\vec{f}$ when $\vec{f}$ is either parallel or orthogonal to $\vec{u}$. Show that, in these two cases, one has $\vec{f} = (\gamma_u)^3 m_0 \vec{a}$ and $\vec{f} = \gamma_u m_0 \vec{a}$ respectively. Here $\gamma_u = (1 - u^2/c^2)^{-1/2}$.

[25] 6. If $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| << 1$, (a) verify that, to the first order in $h_{\mu\nu}$

\[
\begin{align*}
R_{\mu\nu} &= \frac{1}{2} (\partial_\nu \partial_\mu h + \Box^2 h_{\mu\nu} - \partial_\nu \partial_\rho h^\rho_\mu - \partial_\mu \partial_\rho h^\rho_\nu, \quad (1) \\
R &= \Box^2 h - \partial_\rho \partial_\mu h^\rho_\mu \quad (2)
\end{align*}
\]

where $h = \eta^{\mu\nu} h_{\mu\nu}$ and $\Box^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu$. (b) Let $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$ then show that the linearized Einstein filed equations can be written as

\[
\Box^2 \tilde{h}_{\mu\nu} + \eta_{\mu\nu} \partial_\alpha \partial_\beta \tilde{h}^{\alpha\beta} - \partial_\nu \partial_\rho \tilde{h}^\rho_\mu - \partial_\mu \partial_\rho \tilde{h}^\rho_\nu = -2\kappa T_{\mu\nu} \quad (3)
\]

(c) show that $\partial_\mu T^{\mu\nu} = 0$.

Appendix:

1. Christoffel symbol

\[ \Gamma_{bc}^a = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}) \quad (4) \]

2. Lorentz Transformations in matrix form

\[ [\Lambda^\mu_\nu] = \begin{pmatrix} \gamma & -\beta \gamma & 0 & 0 \\ -\beta \gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (5) \]

3. The curvature tensor

\[ R^d_{\ abc} = \partial_b \Gamma^d_{\ ac} - \partial_c \Gamma^d_{\ ab} + \Gamma^e_{\ ac} \Gamma^d_{\ eb} - \Gamma^e_{\ ab} \Gamma^d_{\ ec} \]

4. The Ricci tensor

\[ R_{ab} \equiv R^d_{\ abd} = \partial_b \Gamma^d_{\ ac} - \partial_c \Gamma^d_{\ ab} + \Gamma^e_{\ ac} \Gamma^d_{\ eb} - \Gamma^e_{\ ab} \Gamma^d_{\ ec} \]

2
5. The Ricci Scalar

\[ R \equiv g^{ab} R_{ab} \]

6. The Schwarzschild Metric

\[ ds^2 = c^2 \left( 1 - \frac{2\mu}{r} \right) dt^2 - \frac{1}{1 - \frac{2\mu}{r}} dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \]

where \( \mu = \frac{MG}{c^2} \)

7. Geodesic Equations for a massive particle at the equatorial plane \( \theta = \frac{\pi}{2} \).

\[ (1 - \frac{2\mu}{r}) \dot{t} = k, \quad c^2 \left( 1 - \frac{2\mu}{r} \right) \dot{t}^2 - \frac{1}{1 - \frac{2\mu}{r}} \dot{r}^2 - r^2 \dot{\phi}^2 = c^2, \]

\[ r^2 \dot{\phi} = h \]

where \( k = \frac{E}{m_0 c^2} \) and \( h \) is the angular velocity of the test particle

8. Geodesic Equations for a massless particle at the equatorial plane \( \theta = \frac{\pi}{2} \).

\[ (1 - \frac{2\mu}{r}) \dot{t} = k, \quad c^2 \left( 1 - \frac{2\mu}{r} \right) \dot{t}^2 - \frac{1}{1 - \frac{2\mu}{r}} \dot{r}^2 - r^2 \dot{\phi}^2 = 0, \]

\[ r^2 \dot{\phi} = h \]

where \( k = \frac{E}{m_0 c^2} \) and \( h \) is the angular velocity of the test particle

9. Symmetries of the Curvature tensor

\[ R_{dabc} = -R_{adbc} = -R_{dabc} \]

\[ R_{adbc} = R_{bcad}, \quad R_{dabc} + R_{dacb} + R_{dcab} = 0, \]

\[ \nabla_a R_{bcd} + \nabla_b R_{cad} + \nabla_c R_{abcd} = 0 \]