

PHYS 405: THEORY OF GENERAL RELATIVITY

Final Exam *solutions*

January 13, 2017; 09.00-11.00 (SB-Z10)

QUESTIONS: Choose any four of the following six problems

[25] 1. Show that the Einstein tensor $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}$ satisfies divergence free condition $\nabla_\mu G^{\mu\nu} = 0$.

[25] 2. Show that, on introduction of the retarded Eddington-Finkelstein timelike coordinate $ct^* = ct - 2\mu \ln |r/2\mu - 1|$, the Schwarzschild line element takes the form

$$ds^2 = c^2 \left(1 - \frac{2\mu}{r}\right) dt^{*2} + \frac{4\mu c}{r} dt^* dr - \left(1 + \frac{2\mu}{r}\right) dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

where $\mu = \frac{MG}{c^2}$. Hence find the equations for the worldlines of radially moving photons in retarded Eddington-Finkelstein coordinates. Use this result to sketch the spacetime diagram showing the light-cone structure in this coordinate system.

[25] 3. A particle at infinity in the Schwarzschild geometry is moving radially inwards with coordinate speed u_0 . Show that at any coordinate radius r the coordinate velocity is given by

$$\left(\frac{dr}{dt}\right)^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right)^2 \left[1 - \frac{1}{\gamma_0^2} \left(1 - \frac{2GM}{c^2 r}\right)\right]$$

where $\gamma_0 = (1 - u_0^2/c^2)^{-1/2}$. Determine the velocity relative to a stationary observer at r , and show that this velocity tends to c as r tends to $2GM/c^2$, irrespective of the value of u_0 .

[25] 4. Find all circular orbits of massive particles in Schwarzschild geometry

[25] 5. For a pure 4-force f acting on a particle of rest mass m_0 , show that the corresponding 3-force $\vec{f}$ satisfies

$$\vec{f} = \gamma_u m_0 \vec{a} + \left(\frac{\vec{f} \cdot \vec{u}}{c^2}\right) \vec{u}$$

Hence show that $\vec{a}$ is only parallel to $\vec{f}$ when $\vec{f}$ is either parallel or orthogonal to $\vec{u}$. Show that, in these two cases, one has $\vec{f} = (\gamma_u)^3 m_0 \vec{a}$ and $\vec{f} = \gamma_u m_0 \vec{a}$ respectively. Here $\gamma_u = (1 - u^2/c^2)^{-1/2}$

[25] 6. If $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$, (a) verify that, to the first order in $h_{\mu\nu}$

$$R_{\mu\nu} = \frac{1}{2}(\partial_\nu \partial_\mu h + \square^2 h_{\mu\nu} - \partial_\nu \partial_\rho h_\mu^\rho - \partial_\mu \partial_\rho h_\nu^\rho), \quad (1)$$

$$R = \square^2 h - \partial_\rho \partial_\mu h^{\rho\mu} \quad (2)$$

where $h = \eta^{\mu\nu} h_{\mu\nu}$ and $\square^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu$. (b) Let $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$ then show that the linearized Einstein field equations can be written as

$$\square^2 \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial_\alpha \partial_\beta \bar{h}^{\alpha\beta} - \partial_\nu \partial_\rho \bar{h}_\mu^\rho - \partial_\mu \partial_\rho \bar{h}_\nu^\rho = -2\kappa T_{\mu\nu} \quad (3)$$

(c) show that $\partial_\mu T^{\mu\nu} = 0$.

Appendix:

1. Christoffel symbol

$$\Gamma_{bc}^a = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{cb}) \quad (4)$$

2. Lorentz Transformations in matrix form

$$[\Lambda_\nu^\mu] = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (5)$$

3. The curvature tensor

$$R^d{}_{abc} = \partial_b \Gamma^d{}_{ac} - \partial_c \Gamma^d{}_{ab} + \Gamma^e{}_{ac} \Gamma^d{}_{eb} - \Gamma^e{}_{ab} \Gamma^d{}_{ec}$$

4. The Ricci tensor

$$R_{ab} \equiv R^d{}_{abd} = \partial_b \Gamma^d{}_{ad} - \partial_d \Gamma^d{}_{ab} + \Gamma^e{}_{ad} \Gamma^d{}_{eb} - \Gamma^e{}_{ab} \Gamma^d{}_{ed}$$

5. The Ricci Scalar

$$R \equiv g^{ab} R_{ab}$$

6. The Schwarzschild Metric

$$ds^2 = c^2 \left(1 - \frac{2\mu}{r}\right) dt^2 - \frac{1}{1 - \frac{2\mu}{r}} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

where $\mu = \frac{MG}{c^2}$

7. Geodesic Equations for a massive particle at the equatorial plane $\theta = \frac{\pi}{2}$.

$$\left(1 - \frac{2\mu}{r}\right) \dot{t} = k, \quad (6)$$

$$c^2 \left(1 - \frac{2\mu}{r}\right) \dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2 = c^2, \quad (7)$$

$$r^2 \dot{\phi} = h \quad (8)$$

where $k = \frac{E}{m_0 c^2}$ and h is the angular velocity of the test particle

8. Geodesic Equations for a massless particle at the equatorial plane $\theta = \frac{\pi}{2}$.

$$\left(1 - \frac{2\mu}{r}\right) \dot{t} = k, \quad (9)$$

$$c^2 \left(1 - \frac{2\mu}{r}\right) \dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2 = 0, \quad (10)$$

$$r^2 \dot{\phi} = h \quad (11)$$

where $k = \frac{E}{m_0 c^2}$ and h is the angular velocity of the test particle

9. Symmetries of the Curvature tensor

$$R_{dabc} = -R_{adb c} = -R_{dac b} \quad (12)$$

$$R_{adbc} = R_{bcad}, \quad R_{dabc} + R_{dcab} + R_{dacb} = 0, \quad (13)$$

$$\nabla_a R_{bcde} + \nabla_b R_{caed} + \nabla_c R_{abed} = 0 \quad (14)$$

Solutions of the Final Exam Problems

1) please see the solution of the first problem of second Midterm Exam

2). The Schwarzschild Metric in (t, r, θ, ϕ) coordinates is.

$$ds^2 = c^2 \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

letting $ct^* = ct + 2M \ln \left| \frac{r}{2M} - 1 \right|$ or

$$ct = ct^* - 2M \ln \left| \frac{r}{2M} - 1 \right|$$

$$\Rightarrow c dt = c dt^* - \frac{dr}{r/2M - 1}$$

Hence

$$ds^2 = \left(1 - \frac{2M}{r}\right) \left[c dt^* - \frac{dr}{r/2M - 1} \right]^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

$$= c^2 \left(1 - \frac{2M}{r}\right) dt^{*2} - \frac{4Mc}{r} dt^* dr - \left(1 + \frac{2M}{r}\right) dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

radial null geodesics ($\theta = \pi/2$, $\phi = \text{constant}$)

$$ds^2 = c^2 \left(1 - \frac{2M}{r}\right) dt^{*2} + \frac{4Mc}{r} dt^{*} dr - \left(1 + \frac{2M}{r}\right) dr^2 = 0$$

or

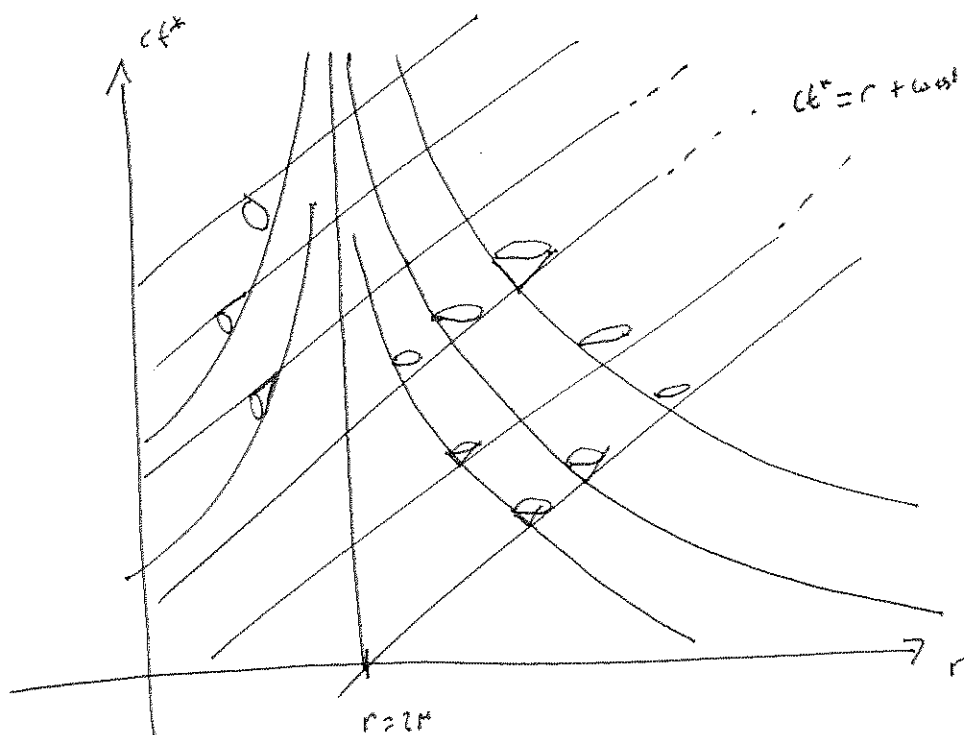
$$c^2 \left(1 - \frac{2M}{r}\right) \left(\frac{dt^{*}}{dr}\right)^2 + \frac{4Mc}{r} \frac{dt^{*}}{dr} - \left(1 + \frac{2M}{r}\right) = 0$$

We get

$$\frac{dt^{*}}{dr} = + \frac{1}{c} \quad \text{and} \quad \frac{dt^{*}}{dr} = - \frac{1}{c} \frac{1 + 2M/r}{1 - 2M/r}$$

Solutions

$$ct^{*} = +r + \text{constant}, \quad ct^{*} = -r - 4M \ln \left| \frac{r}{2M} - 1 \right| + \text{const.}$$



Lightcone structure in retarded Eddington-Finkelstein coordinates.

(3)

(3) Radial motion: $\dot{\phi} = 0, \dot{\theta} = 0$

$$c^2 \left(1 - \frac{2M}{r}\right) \dot{t}^2 - \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 = c^2$$

$$\left(1 - \frac{2M}{r}\right) \dot{t} = k.$$

$$\dot{r} = \frac{dr}{dt} \cdot \frac{dt}{d\tau} = \frac{dr}{d\tau} \cdot \dot{t}$$

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{\dot{r}^2}{\dot{t}^2} = \frac{\left(1 - \frac{2M}{r}\right)^2}{k^2} \left[c^2 \left(1 - \frac{2M}{r}\right) \dot{t}^2 - \dot{r}^2 \right]$$

$$= \frac{\left(1 - \frac{2M}{r}\right)^2}{k^2} \left[c^2 k^2 - c^2 + \frac{2Mc^2}{r} \right]$$

$$= \frac{\left(1 - \frac{2M}{r}\right)^2}{k^2} \left[c^2 - \frac{1}{k^2} \left(c^2 - \frac{2Mc^2}{r} \right) \right]$$

as $r \rightarrow \infty \quad \frac{dr}{d\tau} \rightarrow u_0$

$$u_0^2 = c^2 - c^2/k^2 \Rightarrow c^2/k^2 = c^2 - u_0^2$$

$$\Rightarrow \frac{1}{k^2} = 1 - u_0^2/c^2 \Rightarrow k = \gamma_{u_0}$$

$$\Rightarrow \left(\frac{dr}{d\tau}\right)^2 = c^2 \left(1 - \frac{2M}{r}\right)^2 \left[1 - \gamma^{-2} \left(1 - \frac{2M}{r}\right) \right].$$

4) Done in class and see also the text book

(4)

5) For a pure 4-force $\vec{f}$ acting on a particle of rest mass m_0 , show that the corresponding 3-force $\vec{f}$ satisfies

$$\vec{f} = \gamma_u m_0 \vec{a} + \frac{\vec{f} \cdot \vec{u}}{c^2} \vec{u}$$

Hence show that $\vec{a}$ is only parallel to $\vec{f}$ when $\vec{f}$ is either parallel or orthogonal to $\vec{u}$. Show that, in these two cases, one has $\vec{f} = \gamma_u^3 m_0 \vec{a}$ and $\vec{f} = \gamma_u m_0 \vec{a}$ respectively.

Solution: 4-velocity $u^\mu = (\gamma_u c, \gamma_u \vec{u})$

where $\vec{u} = \frac{d\vec{x}}{dt}$. 4-acceleration $a^\mu = \frac{du^\mu}{d\tau}$

$$a^\mu = \gamma_u \frac{d}{dt} (\gamma_u c, \gamma_u \vec{u}) = \gamma_u \left(c \frac{d\gamma_u}{dt}, \gamma_u \vec{a} + \vec{u} \frac{d\gamma_u}{dt} \right)$$

where $\vec{a} = \frac{d\vec{u}}{dt}$. 4-momentum $p^\mu = m$

Here we have the relation

$$u^\mu a_\mu = 0 = c \frac{d\gamma_4}{dt} - \gamma_4 \bar{a} \cdot \bar{u} - u^2 \frac{d\gamma_4}{dt}$$

or $\gamma_4 \bar{a} \cdot \bar{u} + u^2 \frac{d\gamma_4}{dt} = c^2 \frac{d\gamma_4}{dt}$ (1)

4-force vector $f^\mu = \frac{dp^\mu}{d\tau} = \gamma_4 \left(\frac{\bar{f} \cdot \bar{u}}{c}, \bar{f} \right)$

where $\bar{f} = \frac{d\bar{p}}{dt}$: on the other hand

$$f^\mu = m_0 a^\mu \Rightarrow$$

$$\bar{f} = m_0 \gamma_4 \bar{a} + m_0 \frac{d\gamma_4}{dt} \bar{u} \quad (2)$$

Taking dot product with $\bar{u}$ we get

$$\bar{f} \cdot \bar{u} = m_0 \gamma_4 \bar{a} \cdot \bar{u} + m_0 u^2 \frac{d\gamma_4}{dt} \quad (3)$$

using (1) we get $\bar{f} \cdot \bar{u} = m_0 c^2 \frac{d\gamma_4}{dt}$ (4)

Therefore (2) becomes

$$\bar{f} = m_0 \gamma_4 \bar{a} + \frac{\bar{f} \cdot \bar{u}}{c^2} \bar{u}$$

$$(i) \text{ if } f \perp \bar{u} \Rightarrow \bar{f} \cdot \bar{u} = 0$$

$$\Rightarrow \bar{f} = m_0 \gamma_u \bar{a}$$

$$(ii) \text{ if } f \parallel u \Rightarrow \bar{f} = \lambda \bar{u}$$

$$\Rightarrow \text{from (4)} \quad \lambda u^2 = m_0 c^2 \frac{d\gamma_u}{dt} \Rightarrow \lambda = m_0 \frac{c^2}{u^2} \frac{d\gamma_u}{dt}$$

$$\text{or } \lambda = \frac{\gamma_u^2}{\gamma_u^2 - 1} \cdot m_0 \frac{d\gamma_u}{dt}$$

$$\Rightarrow \text{from (2)} \Rightarrow$$

$$\frac{\gamma_u^2}{\gamma_u^2 - 1} m_0 \frac{d\gamma_u}{dt} \bar{u} = m_0 \gamma_u \bar{a} + m_0 \frac{d\gamma_u}{dt} \bar{u}$$

$$- m_0 \frac{d\gamma_u}{dt} \bar{u} \left(1 - \frac{\gamma_u^2}{\gamma_u^2 - 1} \right) = m_0 \gamma_u \bar{a}$$

$$- m_0 \frac{d\gamma_u}{dt} \bar{u} \left(\frac{-1}{1 - \gamma_u^2} \right) = m_0 \gamma_u \bar{a}$$

$$m_0 \frac{d\gamma_u}{dt} \bar{u} = -m_0 \gamma_u (1 - \gamma_u^2) \bar{a}$$

$$\Rightarrow \bar{f} = m_0 \gamma_u \bar{a} + \frac{\lambda u^2}{c^2} \bar{u}$$

$$\hat{f} = m_0 \gamma_u \bar{a} + \frac{u^2}{c^2} \cdot m_0 \frac{c^2}{u^2} \frac{d\gamma_u}{dt} \bar{u}$$

$$= m_0 \gamma_u \bar{a} + m_0 \gamma_u (-1 + \gamma_u^2) \bar{a} = m_0 \gamma_u^3 \bar{a}$$

$$\underline{\hat{f} = m_0 \gamma_u^3 \bar{a}}$$

(8)

$$6) g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{where } \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

Then

$$\Gamma_{\alpha\beta}^{\mu} = \frac{1}{2} g^{\mu\sigma} (\partial_{\alpha} g_{\sigma\beta} + \partial_{\beta} g_{\sigma\alpha} - \partial_{\sigma} g_{\alpha\beta})$$

$$= \frac{1}{2} \eta^{\mu\sigma} (\partial_{\alpha} h_{\sigma\beta} + \partial_{\beta} h_{\sigma\alpha} - \partial_{\sigma} h_{\alpha\beta})$$

$$\Gamma_{\alpha\beta}^{\mu} = \frac{1}{2} (\partial_{\alpha} h^{\mu}_{\beta} + \partial_{\beta} h^{\mu}_{\alpha} - \eta^{\mu\sigma} \partial_{\sigma} h_{\alpha\beta})$$

$$\Gamma_{\alpha\beta}^{\alpha} = \frac{1}{2} \partial_{\alpha} h^{\alpha}_{\beta} + \frac{1}{2} \partial_{\beta} h - \frac{1}{2} \partial_{\alpha} h^{\alpha}_{\beta} = \frac{1}{2} \partial_{\beta} h$$

$$R_{\mu\nu} = \partial_{\mu} \Gamma_{\alpha\nu}^{\alpha} - \partial_{\alpha} \Gamma_{\mu\nu}^{\alpha} + \text{small terms}$$

$$= \frac{1}{2} \partial_{\mu} \partial_{\nu} h - \frac{1}{2} \partial_{\mu} \partial_{\alpha} h^{\alpha}_{\nu} - \frac{1}{2} \partial_{\nu} \partial_{\alpha} h^{\alpha}_{\mu}$$

$$+ \frac{1}{2} \square^2 h_{\mu\nu}$$

$$R = \square^2 h - \partial_{\alpha} \partial_{\beta} h^{\alpha\beta}$$

Linearized Einstein Eqs.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

(9)

$$G_{\mu\nu} = \frac{1}{2} \square^2 h_{\mu\nu} + \frac{1}{2} \partial_\mu \partial_\nu h - \frac{1}{2} \partial_\mu \partial_\beta h^\beta_\nu - \frac{1}{2} \partial_\nu \partial_\beta h^\beta_\mu \\ - \frac{1}{2} (\square^2 h - \partial_\alpha \partial_\beta h^{\alpha\beta}) \gamma_{\mu\nu} = -\kappa T_{\mu\nu}$$

$$b) \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \gamma_{\mu\nu}$$

$$\bar{h} = -h \quad \Rightarrow \quad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \bar{h} \gamma_{\mu\nu}$$

$$\Rightarrow G_{\mu\nu} = \frac{1}{2} \square^2 (\bar{h}_{\mu\nu} - \frac{1}{2} \bar{h} \gamma_{\mu\nu}) - \frac{1}{2} \partial_\mu \partial_\nu \bar{h} \\ - \frac{1}{2} \partial_\mu \partial_\beta (\bar{h}^\beta_\nu - \frac{1}{2} \bar{h} \delta^\beta_\nu) - \frac{1}{2} \partial_\nu \partial_\beta (\bar{h}^\beta_\mu - \frac{1}{2} \bar{h} \delta^\beta_\mu) \\ - \frac{1}{2} (-\square^2 \bar{h} - \partial_\alpha \partial_\beta \bar{h}^{\alpha\beta} + \frac{1}{2} \square^2 \bar{h}) \gamma_{\mu\nu} \\ = \frac{1}{2} \square^2 \bar{h}_{\mu\nu} - \frac{1}{2} \partial_\mu \partial_\beta \bar{h}^\beta_\nu - \frac{1}{2} \partial_\nu \partial_\beta \bar{h}^\beta_\mu \\ + \frac{1}{2} (\partial_\alpha \partial_\beta \bar{h}^{\alpha\beta}) \gamma_{\mu\nu} = -\kappa \bar{T}_{\mu\nu}$$

c) Apply ∂_μ both sides

$$\frac{1}{2} \square^2 (\partial_\mu \bar{h}^\mu_\nu) - \frac{1}{2} \square^2 (\partial_\beta \bar{h}^\beta_\nu) - \frac{1}{2} \partial_\nu \partial_\beta \partial_\mu \bar{h}^\beta_\mu \\ + \frac{1}{2} \partial_\mu (\partial_\nu \partial_\beta \bar{h}^{\alpha\beta}) = -\kappa \partial_\mu \bar{T}^\mu_\nu$$

$$\Rightarrow \partial_\mu \bar{T}^\mu_\nu = 0 \quad \checkmark$$