

Solutions of the second Midterm Exam

① let us start with the Bianchi identity.

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0 \quad (1)$$

First contract the indices e and a .

$$\nabla_a R^a_{bcd} + \nabla_c R^a_{bda} + \nabla_d R^a_{bac} = 0 \quad (2)$$

or

$$\nabla_a R^a_{bcd} + \nabla_c R_{bd} - \nabla_d R_{bc} = 0 \quad (3)$$

where $R_{bc} \equiv R^a_{bca}$ is the Ricci tensor. Contracting the indices b and d in Eq. (3) we get

$$-\nabla_a R^a_c + \nabla_c R - \nabla_d R^d_c = 0 \quad (4)$$

$$\text{or} \quad \cdot \cdot 2 \nabla_a R^a_c - \nabla_c R = 0 \quad (5)$$

where $R = R^a_a$ is the Ricci scalar

Eq. (5) can be also written as

$$\nabla_a \left(R^a_c - \frac{1}{2} R \delta^a_c \right) = 0$$

or $\nabla_a G^a_c = 0$ where $G^a_c = R^a_c - \frac{1}{2} R \delta^a_c$ is the Einstein tensor.

② Done in the class and also in the text book
Page 165-166

②

③ A curve C defined as $x^H(u)$ in a manifold with metric is a geodesic if

$$\frac{d^2 x^H}{du^2} + \Gamma_{\alpha\beta}^H \frac{dx^\alpha}{du} \frac{dx^\beta}{du} = 0, \quad H=0,1,2,3$$

let $p^H = \frac{dx^H}{du}$, then

$$\dot{p}^H + \Gamma_{\alpha\beta}^H p^\alpha p^\beta = 0$$

Since $p_H = g_{H\beta} p^\beta$ then

$$\dot{p}_H = \frac{d}{du} (g_{H\beta}) p^\beta + g_{H\beta} \dot{p}^\beta$$

$$= (\partial_\sigma g_{H\beta}) \frac{dx^\sigma}{du} p^\beta + g_{H\beta} (-\Gamma_{\alpha\beta}^\sigma p^\alpha p^\beta)$$

$$= (\partial_\sigma g_{H\beta}) p^\sigma p^\beta - \Gamma_{\alpha\beta}^\sigma g_{H\beta} p^\alpha p^\beta$$

$$= (\partial_\sigma g_{H\beta}) p^\sigma p^\beta - \frac{1}{2} g^{\gamma\omega} (\partial_\alpha g_{\omega\beta} + \partial_\beta g_{\omega\alpha} - \partial_\omega g_{\alpha\beta})$$

$$g_{H\beta} p^\alpha p^\beta$$

$$= (\partial_\sigma g_{H\beta}) p^\sigma p^\beta - \frac{1}{2} (\partial_\alpha g_{H\beta} + \partial_\beta g_{H\alpha} - \partial_H g_{\alpha\beta}) p^\alpha p^\beta$$

$$= (\partial_\sigma g_{H\beta}) p^\sigma p^\beta - (\partial_\alpha g_{H\beta}) p^\alpha p^\beta + \frac{1}{2} (\partial_H g_{\alpha\beta}) p^\alpha p^\beta$$

$$\dot{p}_H = \frac{1}{2} (\partial_H g_{\alpha\beta}) p^\alpha p^\beta$$

- ④ a) a stationary observer at r sees a coordinate time interval dt corresponds to a proper time interval

$$dt' = \left(1 - \frac{2M}{r}\right)^{1/2} dt. \quad (\text{text book page 211})$$

Then photon trajectory $\theta = \pi/2$, $r = 3M$.

$$c^2 dt'^2 - r^2 d\phi^2 = 0$$

$$c dt' = 3M d\phi$$

$$c \int_0^\tau dt' = 3M \int_0^{2\pi} d\phi \Rightarrow \tau = \frac{6\pi M}{c}$$

- b) For a stationary observer at infinity ($\theta = \pi/2$, $r = 3M$)

$$c^2 \left(1 - \frac{2M}{r}\right) dt^2 - r^2 d\phi^2 = 0$$

$$c^2 \frac{1}{3} dt^2 = r^2 d\phi^2$$

$$c dt = \sqrt{3} 3M d\phi \Rightarrow \tau = \frac{\sqrt{3} \pi M}{c}$$

- c) For a particle in circular orbit

we have

$$\frac{1}{r} = \frac{GM}{h^2} + 3M \frac{1}{r^2} \Rightarrow h^2 = \frac{GM r^2}{r - 3M}$$

hence,

$$\begin{aligned}\frac{d\phi}{dt} &= \frac{d\phi}{dz} \cdot \frac{dz}{dt} = \frac{1}{\dot{t}} \dot{\phi} = \frac{h}{r^2 \dot{t}} = \frac{1}{r^2} \left(\frac{GM r^2}{r-3M} \right)^{1/2} \frac{1}{\dot{t}} \\ &= \left(\frac{GM}{r^2(r-3M)} \right)^{1/2} \frac{(1-\frac{2M}{r})}{k}.\end{aligned}$$

From 7.3 and 7.4 we get.

$$c^2 \left(1 - \frac{2M}{r}\right) \frac{k^2}{(1-\frac{2M}{r})^2} - \frac{h^2}{r^2} = c^2.$$

$$c^2 \left[\frac{k^2}{1-\frac{2M}{r}} - 1 \right] = \frac{h^2}{r^2} = \frac{GM}{(r-3M)}$$

$$\cancel{GM}^2 \quad c^2 k^2 = \left(c^2 + \frac{GM}{r-3M} \right) \left(1 - \frac{2M}{r}\right)$$

$$\Rightarrow c^2 k^2 = c^2 \left(1 + \frac{M}{r-3M} \right) \left(1 - \frac{2M}{r}\right) = c^2 \frac{r}{r-3M} \left(1 - \frac{2M}{r}\right)^2$$

$$k^2 = \frac{1}{\sqrt{1-3M/r}} \left(1 - \frac{2M}{r}\right) \Rightarrow$$

$$\frac{d\phi}{dt} = \left(\frac{GM}{r^3} \right)^{1/2} \frac{1 - \frac{2M}{r}}{(1-\frac{3M}{r})^{1/2}} \cdot \frac{\sqrt{1-3M/r}}{1-2M/r} = \left(\frac{GM}{r^3} \right)^{1/2}.$$

(5). Text book pages (1213-1215)

(6) $\xi^\mu = \delta^\mu_0$, $\xi_\mu = g_{\mu 0}$

$$\begin{aligned}\nabla_\nu \xi_\mu &= \partial_\nu g_{\mu 0} - \Gamma_{\mu\nu}^\alpha g_{\alpha 0} \\ &= \partial_\nu g_{\mu 0} - \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}) g_{\alpha 0} \\ &= \partial_\nu g_{\mu 0} - \frac{1}{2} \delta^\beta_0 (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}) \\ &= \partial_\nu g_{\mu 0} - \frac{1}{2} (\partial_\mu g_{0\nu} + \partial_\nu g_{0\mu}) + \frac{1}{2} \partial_0 g_{\mu\nu}\end{aligned}$$

For the Schwarzschild geometry $\partial_0 g_{\mu\nu} = 0$

$$\begin{aligned}\Rightarrow \nabla_\nu \xi_\mu &= \partial_\nu g_{\mu 0} - \frac{1}{2} \partial_\mu g_{0\nu} - \frac{1}{2} \partial_\nu g_{0\mu} \\ &= \frac{1}{2} (\partial_\nu g_{\mu 0} - \partial_\mu g_{\nu 0})\end{aligned}$$

RHS is antisymmetric with respect to the indices μ and ν

$$\begin{aligned}\nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu &= \frac{1}{2} (\cancel{\partial_\nu g_{\mu 0}} - \cancel{\partial_\mu g_{\nu 0}}) \\ &\quad + \frac{1}{2} (\cancel{\partial_\mu g_{\nu 0}} - \cancel{\partial_\nu g_{\mu 0}}) = 0\end{aligned}$$

$\Rightarrow$ Hence ξ_μ is a Killing vector field of the Schwarzschild geometry