PHYS 405: THEORY OF GENERAL RELATIVITY
First Midterm Exam
November 08, 2016; 10.40-12.30

QUESTIONS: Choose any four of the following six problems

[25] **1**. **a)**. Write down the definition of a geodesic in a manifold. **b)**. Write down the geodesic equation. **c)**.The line element of the cylinder in $\mathbb{R}^3$ is given by $ds^2 = dz^2 + a^2\, d\phi^2$ where a is the radius of the cylinder , ϕ is the polar angle and z is the Cartesian coordinate orthogonal to the xy-plane. Write down the geodesic equations for the cylinder and solve them. Show these geodesics on a cylindrical surface.

[25] **2**.By transforming from a local inertial coordinate system ξ^μ in which

$$ds^2 = c^2 d\tau^2 = \eta_{\mu\nu}\, d\xi^\mu\, d\xi^\nu$$

to a general coordinate system x^μ, show that freely falling particle,i.e.,

$$\frac{d^2\xi^\mu}{d\tau^2} = 0$$

obey the geodesic equation of motion

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\,\frac{dx^\alpha}{d\tau}\,\frac{dx^\beta}{d\tau} = 0$$

where

$$\Gamma^\mu_{\alpha\beta} = \frac{\partial x^\mu}{\partial x^\rho}\,\frac{x^\rho}{\partial x^\alpha\,\partial x^\beta}$$

[25] **3**. If a particle's worldline is described by giving x,y and z as function of t in some inertial frame S then components of its velocity in some other inertial frame S' (moving in the x-direction with speed v relative to S) are usually obtained by taking differentials of the Lorentz transformations

$$\begin{aligned} ct' &= \gamma(ct - \beta x), \\ x' &= \gamma(x - \beta ct), \\ y' &= y, \\ z' &= z, \end{aligned}$$

where $\beta = \frac{v}{c}$ and $\gamma = (1-\beta^2)^{-1/2}$.If the velocity of the particle in S is given by (Show that transformation of velocities)

$$u_x = \frac{dx}{dt}, \quad u_y = \frac{dy}{dt}, \quad u_z = \frac{dz}{dt},$$

then show that the components of the velocity in frame S' are given by

$$\begin{aligned} u'_x &= \frac{dx'}{dt'} = \frac{u_x - v}{1 - u_x\, v/c^2}, \\ u'_y &= \frac{dy'}{dt'} = \frac{u_y}{\gamma_v(1 - u_x\, v/c^2)}, \\ u'_z &= \frac{dz'}{dt'} = \frac{u_z}{\gamma_v(1 - u_x\, v/c^2)}, \end{aligned}$$

[25] **4.a)**. In an inertial frame S velocity 4-vector of a massive particle can be given as follows $u^\mu = (\gamma_u\, c, \gamma_u\, \vec{u})$ where $\gamma_u = (1 - u^2/c^2)^{-1/2}$, $u^2 = \vec{u} \cdot \vec{u}$. **b).** One can find an inertial frame S' moving along the x- direction with speed v relative to S so that $u'^\mu = (c, 0, 0, 0)$. **c)**. Show that photons speed is the same c in all inertial frames (For simplicity assume that photon is also moving along x-direction)

[20] **5**.If $t_{\mu\nu}$ are the components of a symmetric tensor and v_α the components of a vector show that if

$$v_\mu\, t_{\alpha\beta} + v_\alpha\, t_{\beta\mu} + v_\beta\, t_{\mu\alpha} = 0$$

then either $t_{\alpha\beta} = 0$ or $v_\alpha = 0$.

[25] **6**. Let $\{\mathbf{e}_\alpha\}$ be a basis of a manifold M and let $\{\mathbf{e}^\alpha\}$ be the dual basis so that $\mathbf{e}_\alpha \cdot \mathbf{e}^\beta = \delta^\beta_\alpha$. We know that

$$\begin{aligned} \partial_\alpha\, \mathbf{e}_\beta &= \Gamma^\rho_{\alpha\beta}\, \mathbf{e}_\rho, \\ \partial_\alpha\, \mathbf{e}^\beta &= -\Gamma^\beta_{\alpha\rho}\, \mathbf{e}^\rho, \end{aligned}$$

a). Any four vector can be expressed as $\mathbf{v} = e^\alpha\, v_\alpha$ or $\mathbf{v} = e_\alpha\, v^\alpha$. Show that $\partial_\mu\, \mathbf{v} = (\nabla_\mu\, v^\alpha)\, \mathbf{e}_\alpha$ where $\nabla_\mu\, v^\alpha$ is the covariant derivative of the contravariant component v^α given as

$$\nabla_\alpha\, v^\beta = \partial_\alpha\, v^\beta + \Gamma^\beta_{\alpha\rho}\, v^\rho$$

or show that $\partial_\mu \mathbf{v} = (\nabla_\mu v_\alpha)\, \mathbf{e}^\alpha$ where $\nabla_\mu v_\alpha$ is the covariant derivative of the covariant component v_α given as

$$\nabla_\alpha v_\beta = \partial_\alpha v_\beta - \Gamma^\rho_{\alpha\beta} v^\rho$$

b). Let $\mathbf{t}$ be a tensor of rank-2,i,e., $\mathbf{t} = t_{\alpha\beta}\, \mathbf{e}^\alpha\, \mathbf{e}^\beta$. Show that the covariant derivative, defined through $\partial_\mu \mathbf{t} = (\nabla_\mu t_{\alpha\beta})\, \mathbf{e}^\alpha\, \mathbf{e}^\beta$, is given by

$$\nabla_\mu t_{\alpha\beta} = \partial_\mu t_{\alpha\beta} - \Gamma^\rho_{\mu\alpha} t_{\rho\beta} - \Gamma^\rho_{\mu\beta} t_{\alpha\rho}$$

c). Show that covariant derivative of the metric tensor $g_{\alpha\beta}$ is zero,i.e., $\nabla_\mu g_{\alpha\beta} = 0$

Appendix:

1. Christoffel symbol

$$\Gamma^a_{bc} = \frac{1}{2}\, g^{ad}\, (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{cb}) \tag{1}$$

2. Lorentz Transformations in matrix form

$$[\Lambda^\mu_\nu] = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$