QUESTIONS: Choose any four of the following six problems

[25] 1. a). Write down the definition of a geodesic in a manifold. b). Write down the geodesic equation. c). The line element of the cylinder in $\mathbb{R}^3$ is given by $ds^2 = dz^2 + a^2 d\phi^2$ where $a$ is the radius of the cylinder, $\phi$ is the polar angle and $z$ is the Cartesian coordinate orthogonal to the $xy$-plane. Write down the geodesic equations for the cylinder and solve them. Show these geodesics on a cylindrical surface.

[25] 2. By transforming from a local inertial coordinate system $\xi^\mu$ in which $ds^2 = c^2 d\tau^2 = \eta_{\mu\nu} d\xi^\mu d\xi^\nu$ to a general coordinate system $x^\mu$, show that freely falling particle, i.e.,

$$\frac{d^2 \xi^\mu}{d\tau^2} = 0$$

obey the geodesic equation of motion

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

where

$$\Gamma^\mu_{\alpha\beta} = \frac{\partial x^\mu}{\partial x^\rho} \frac{x^\rho}{\partial x^{\alpha} \partial x^{\beta}}$$

[25] 3. If a particle’s worldline is described by giving $x, y$ and $z$ as function of $t$ in some inertial frame $S$ then components of its velocity in some other inertial frame $S'$ (moving in the $x$-direction with speed $v$ relative to $S$) are usually obtained by taking differentials of the Lorentz transformations

$$ct' = \gamma(ct - \beta x),$$
$$x' = \gamma(x - \beta ct),$$
$$y' = y,$$
$$z' = z,$$
where $\beta = \frac{v}{c}$ and $\gamma = (1 - \beta^2)^{-1/2}$. If the velocity of the particle in $S$ is given by (Show that transformation of velocities)

$$u_x = \frac{dx}{dt}, \quad u_y = \frac{dy}{dt}, \quad u_z = \frac{dz}{dt},$$

then show that the components of the velocity in frame $S'$ are given by

$$u'_x = \frac{dx'}{dt'} = \frac{u_x - v}{1 - u_x v/c^2},$$

$$u'_y = \frac{dy'}{dt'} = \frac{u_y}{\gamma v (1 - u_x v/c^2)},$$

$$u'_z = \frac{dz'}{dt'} = \frac{u_z}{\gamma v (1 - u_x v/c^2)}.$$ 

[25] 4.a). In an inertial frame $S$ velocity 4-vector of a massive particle can be given as follows $u^\mu = (\gamma_v c, \gamma_v \vec{u})$ where $\gamma_v = (1 - u^2/c^2)^{-1/2}$, $u^2 = \vec{u} \cdot \vec{u}$. b). One can find an inertial frame $S'$ moving along the $x$-direction with speed $v$ relative to $S$ so that $u'^\mu = (c, 0, 0, 0)$. c). Show that photons speed is the same $c$ in all inertial frames (For simplicity assume that photon is also moving along $x$-direction)

[20] 5. If $t_{\mu\nu}$ are the components of a symmetric tensor and $v_\alpha$ the components of a vector show that if

$$v_\mu t_{\alpha\beta} + v_\alpha t_{\beta\mu} + v_\beta t_{\mu\alpha} = 0$$

then either $t_{\alpha\beta} = 0$ or $v_\alpha = 0$.

[25] 6. Let $\{e_\alpha\}$ be a basis of a manifold $M$ and let $\{e^\alpha\}$ be the dual basis so that $e_\alpha \cdot e^\beta = \delta^\beta_\alpha$. We know that

$$\partial_\alpha e^\beta = \Gamma^\rho_{\alpha\beta} e_\rho,$$

$$\partial_\alpha e^\beta = -\Gamma^\beta_{\alpha\rho} e^\rho,$$

a). Any four vector can be expressed as $v = e^\alpha v_\alpha$ or $v = e_\alpha v^\alpha$. Show that $\partial_\mu v = (\nabla_\mu v^\alpha) e_\alpha$ where $\nabla_\mu v^\alpha$ is the covariant derivative of the contravariant component $v^\alpha$ given as

$$\nabla_\alpha v^\beta = \partial_\alpha v^\beta + \Gamma^\beta_{\alpha\rho} v^\rho$$

2
or show that $\partial_\mu v = (\nabla_\mu v_\alpha) e^\alpha$ where $\nabla_\mu v_\alpha$ is the covariant derivative of the covariant component $v_\alpha$ given as

$$\nabla_\alpha v_\beta = \partial_\alpha v_\beta - \Gamma^\rho_{\alpha\beta} v^\rho$$

b). Let $t$ be a tensor of rank-2, i.e., $t = t_{\alpha\beta} e^\alpha e^\beta$. Show that the covariant derivative, defined through $\partial_\mu t = (\nabla_\mu t_{\alpha\beta}) e^\alpha e^\beta$, is given by

$$\nabla_\mu t_{\alpha\beta} = \partial_\mu t_{\alpha\beta} - \Gamma^\rho_{\mu\alpha} t_{\rho\beta} - \Gamma^\rho_{\mu\beta} t_{\alpha\rho}$$

c). Show that covariant derivative of the metric tensor $g_{\alpha\beta}$ is zero, i.e., $\nabla_\mu g_{\alpha\beta} = 0$

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Appendix:

1. Christoffel symbol

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left( \partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{cb} \right)$$

2. Lorentz Transformations in matrix form

$$[\Lambda^\nu_\mu] = \begin{pmatrix} \gamma & -\beta & \gamma & 0 & 0 \\ -\beta & \gamma & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$