PHYS 405: THEORY OF GENERAL RELATIVITY
First Midterm Exam solutions
November 08, 2016; 10.40-12.30

QUESTIONS: Choose any four of the following six problems

[25] **1**. **a)**. Write down the definition of a geodesic in a manifold. **b)**. Write down the geodesic equation. **c)**.The line element of the cylinder in $\mathbb{R}^3$ is given by $ds^2 = dz^2 + a^2\, d\phi^2$ where a is the radius of the cylinder , ϕ is the polar angle and z is the Cartesian coordinate orthogonal to the xy-plane. Write down the geodesic equations for the cylinder and solve them. Show these geodesics on a cylindrical surface.

Solution: (a) A curve in a manifold is a geodesic whose tangent vector is parallel transported along the curve. (b) Geodesic equation:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\,\frac{dx^\alpha}{ds}\,\frac{dx^\beta}{ds} = 0$$

where s is the affine parameter (arclength parameter of the curve). (c) The metric on the cylinder is constant hence the Christoffel symbol vanishes identically. Then the geodesic equation becomes

$$\frac{d^2x^\mu}{ds^2} = 0, \quad x^\mu = (\phi, z)$$

The solution of this equation yields $z = as + b, \quad \phi = cs + d$ where a, b, c, d are arbitrary constants. Eliminating the variable s from these equation the solution becomes $pz + q\phi = r$ where p, q, r are constant related to the constants a, b, c, d. We have three distinct geodesic curves (i) z = constant curves ($q = 0$). These curves are circles on the cylinder orthogonal to the $z-$ axis. (ii) ϕ = constant curves ($p = 0$). These are straight line on the cylinder parallel to the $z-$ axis. (iii) When p and q are different than zero the corresponding curve is a hellix on the cylinder.

[25] **2**.By transforming from a local inertial coordinate system ξ^μ in which

$$ds^2 = c^2 d\tau^2 = \eta_{\mu\nu}\, d\xi^\mu\, d\xi^\nu$$

to a general coordinate system x^μ, show that freely falling particle,i.e.,

$$\frac{d^2\xi^\mu}{d\tau^2} = 0$$

obey the geodesic equation of motion

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

where

$$\Gamma^\mu_{\alpha\beta} = \frac{\partial x^\mu}{\partial \xi^\rho}\frac{\partial^2 \xi^\rho}{\partial x^\alpha \partial x^\beta}$$

Solution: Since $\xi^\mu = \xi^\mu(x^\alpha)$ we get, by chain rule

$$\frac{d\xi^\mu}{d\tau} = \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}$$

Taking one more τ derivative of both sides we get

$$\frac{d^2\xi^\mu}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial \xi^\mu}{\partial x^\alpha}\right)\frac{dx^\alpha}{d\tau} + \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{d}{d\tau}\left(\frac{dx^\alpha}{d\tau}\right) = 0 \tag{1}$$

Using the chain rule again we get

$$\frac{d}{d\tau}\left(\frac{\partial \xi^\mu}{\partial x^\alpha}\right) = \frac{\partial}{\partial x^\beta}\left(\frac{\partial \xi^\mu}{\partial x^\alpha}\right)\frac{dx^\beta}{d\tau}$$

hence (1) becomes

$$\frac{d^2\xi^\mu}{d\tau^2} = \frac{\partial}{\partial x^\beta}\left(\frac{\partial \xi^\mu}{\partial x^\alpha}\right)\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} + \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{d}{d\tau}\left(\frac{dx^\alpha}{d\tau}\right) = 0 \tag{2}$$

Multiplying both sides by $\frac{\partial x^\nu}{\partial \xi^\mu}$ which is the inverse of $\frac{\partial \xi^\mu}{\partial x^\alpha}$, (i.e., $\frac{\partial x^\nu}{\partial \xi^\mu}\frac{\partial \xi^\mu}{\partial x^\alpha} = \delta^\nu_\alpha$) we get

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

where

$$\Gamma^\mu_{\alpha\beta} = \frac{\partial x^\mu}{\partial \xi^\rho}\frac{\partial^2 \xi^\rho}{\partial x^\alpha \partial x^\beta}$$

[25] **3.** If a particle's worldline is described by giving x,y and z as function of t in some inertial frame S then components of its velocity in some other

inertial frame S' (moving in the x-direction with speed v relative to S) are usually obtained by taking differentials of the Lorentz transformations

$$\begin{aligned} ct' &= \gamma(ct - \beta x), \\ x' &= \gamma(x - \beta ct), \\ y' &= y, \\ z' &= z, \end{aligned}$$

where $\beta = \frac{v}{c}$ and $\gamma = (1-\beta^2)^{-1/2}$.If the velocity of the particle in S is given by

$$u_x = \frac{dx}{dt}, \quad u_y = \frac{dy}{dt}, \quad u_z = \frac{dz}{dt},$$

then show that the components of the velocity in frame S' are given by

$$\begin{aligned} u'_x &= \frac{dx'}{dt'} = \frac{u_x - v}{1 - u_x\, v/c^2}, \\ u'_y &= \frac{dy'}{dt'} = \frac{u_y}{\gamma_v(1 - u_x\, v/c^2)}, \\ u'_z &= \frac{dz'}{dt'} = \frac{u_z}{\gamma_v(1 - u_x\, v/c^2)}, \end{aligned}$$

Solution: Taking the differentials of the Lorentz transformations we get

$$\begin{aligned} cdt' &= \gamma(cdt - \beta dx), \\ dx' &= \gamma(dx - \beta cdt), \\ dy' &= dy, \\ dz' &= dz, \end{aligned}$$

Using the first two equations above we get

$$u'_x \equiv \frac{dx'}{dt'} = \frac{\gamma(dx - \beta cdt)}{\gamma(dt - \frac{\beta}{c}dx)} == \frac{dx - \beta cdt}{dt - \frac{\beta}{c}dx} = \frac{u_x - v}{1 - u_x\, v/c^2} \tag{3}$$

Dividing third equation by the first one we get

$$u'_y \equiv \frac{dy'}{dt'} = \frac{dy}{\gamma(dt - \frac{\beta}{c}dx)} = \frac{u_y}{\gamma(1 - u_x\, v/c^2)} \tag{4}$$

Similarly deviding fourth equation by the first one we obtain

$$u'_z \equiv \frac{dz'}{dt'} = \frac{dz}{\gamma(dt - \frac{\beta}{c}dx)} = \frac{u_z}{\gamma(1 - u_x\, v/c^2)} \tag{5}$$

[25] **4.a)**. Show that in an inertial frame S velocity 4-vector of a massive particle can be given as follows $u^\mu = (\gamma_u\, c, \gamma_u\, \vec{u})$ where $\gamma_u = (1 - u^2/c^2)^{-1/2}$, $u^2 = \vec{u} \cdot \vec{u}$. **b).** Assume that the particle is moving along the x-direction. One can find an inertial frame S' moving along the x- direction with speed v relative to S so that $u'^\mu = (c, 0, 0, 0)$. **c)**. Show that photons speed is the same c in all inertial frames (For simplicity assume that photon is also moving along x-direction)

Solution: (a) On the trajectory of the particle

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

where τ is the proper time of the particle. Hence we have

$$c^2 \frac{dt^2}{d\tau^2} - \frac{dx^2}{d\tau^2} - \frac{dy^2}{d\tau^2} - \frac{dz^2}{d\tau^2} = c^2 \tag{6}$$

Remember that the 4-velocity of the particle is

$$u^\mu = \frac{dx^\mu}{d\tau} = (c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}) = \frac{dt}{d\tau}\,(c, \vec{u}) \tag{7}$$

where $\vec{u} = \frac{\vec{x}}{dt}$. From (6)

$$(\frac{dt}{d\tau})^2\,(c^2 - u^2) = c^2,$$

or

$$\frac{dt}{d\tau} = \gamma_u = (1 - u^2/c^2)^{-1/2}$$

using this in (7) we get the required result. (b) Taking the relative speed $v = u_x$ in

$$u'_x = \frac{dx'}{dt'} = \frac{u_x - v}{1 - u_x\, v/c^2},$$

we get

$$u'_x = \frac{u_x - v}{1 - u_x\, v/c^2} = 0,$$

and since $\gamma_{u'} = 1$ we get $u'^{\mu} = (c, 0, 0, 0)$. (c) Using the transformation of velocities in problem 3 and letting $u_x = c$ we get $u'_x = c$. We assume the the direction where the photon propagates is the x-direction.

[20] **5**.If $t_{\mu\nu}$ are the components of a symmetric tensor and v_α the components of a vector show that if

$$v_\mu\, t_{\alpha\beta} + v_\alpha\, t_{\beta\mu} + v_\beta\, t_{\mu\alpha} = 0 \tag{8}$$

then either $t_{\alpha\beta} = 0$ or $v_\alpha = 0$.

Solution; Some information: Contraction of indices: Contraction of u^a with t_{ab} gives a vector,call it v_b

$$u^a\, t_{ab} = v_b$$

Contraction of u^a with t_{ba} gives the same vector v_b, because $t_{ab} = t_{ba}$ is a symmetric tensor.

(i) If $u^a = 0$ for all $a = 1, 2, \cdots, n$, then equation (8) is satisfied identically.
(ii) If $u^a \neq 0$ (at least one component is different than zero. Contracting equation (8) by u^a we get

$$\lambda t_{bc} + u_c\, v_b + u_b\, v_c = 0 \tag{9}$$

or

$$t_{bc} = -\frac{1}{\lambda}[u_c\, v_b + u_b\, v_c] \tag{10}$$

where $\lambda = u^a\, u_a \neq 0$. Contracting equation (10) by u^b we get

$$v_c = -\frac{1}{\lambda}[u_c\, (u^b\, v_b) + \lambda\, v_c] \tag{11}$$

or

$$v_c = -\frac{1}{2\,\lambda}\, (u^b\, v_b)\, u_c \tag{12}$$

Lastly contraction the las equation by u^b we obtain

$$\lambda\, u^b\, v_b = 0 \tag{13}$$

Since $\lambda \neq 0$ then $u^b\, v_b = 0$. This implies that, from equation (12) $v_c = 0$. This finally implies, from equation (10) that $t_{bc} = 0$.

[25] **6**. Let $\{\mathbf{e}_\alpha\}$ be a basis of a manifold M and let $\{\mathbf{e}^\alpha\}$ be the dual basis so that $\mathbf{e}_\alpha \cdot \mathbf{e}^\beta = \delta^\beta_\alpha$. We know that

$$\begin{aligned} \partial_\alpha\, \mathbf{e}_\beta &= \Gamma^\rho_{\alpha\beta}\, \mathbf{e}_\rho, \\ \partial_\alpha\, \mathbf{e}^\beta &= -\Gamma^\beta_{\alpha\rho}\, \mathbf{e}^\rho, \end{aligned}$$

a). Any four vector can be expressed as $\mathbf{v} = e^\alpha\, v_\alpha$ or $\mathbf{v} = e_\alpha\, v^\alpha$. Show that $\partial_\mu\, \mathbf{v} = (\nabla_\mu\, v^\alpha)\, \mathbf{e}_\alpha$ where $\nabla_\mu\, v^\alpha$ is the covariant derivative of the contravariant component v^α given as

$$\nabla_\alpha\, v^\beta = \partial_\alpha\, v^\beta + \Gamma^\beta_{\alpha\rho}\, v^\rho$$

or show that $\partial_\mu\, \mathbf{v} = (\nabla_\mu\, v_\alpha)\, \mathbf{e}^\alpha$ where $\nabla_\mu\, v_\alpha$ is the covariant derivative of the covariant component v_α given as

$$\nabla_\alpha\, v_\beta = \partial_\alpha\, v_\beta - \Gamma^\rho_{\alpha\beta}\, v^\rho$$

b). Let $\mathbf{t}$ be a tensor of rank-2,i,e., $\mathbf{t} = t_{\alpha\beta}\, \mathbf{e}^\alpha\, \mathbf{e}^\beta$. Show that the covariant derivative, defined through $\partial_\mu\, \mathbf{t} = (\nabla_\mu\, t_{\alpha\beta})\, \mathbf{e}^\alpha\, \mathbf{e}^\beta$, is given by

$$\nabla_\mu\, t_{\alpha\beta} = \partial_\mu\, t_{\alpha\beta} - \Gamma^\rho_{\mu\alpha}\, t_{\rho\beta} - \Gamma^\rho_{\mu\beta}\, t_{\alpha\rho}$$

c). Show that covariant derivative of the metric tensor $g_{\alpha\beta}$ is zero,i.e., $\nabla_\mu\, g_{\alpha\beta} = 0$

This problem was solved in class.

Appendix:

1. Christoffel symbol

$$\Gamma^a_{bc} = \frac{1}{2}\, g^{ad}\, (\partial_b\, g_{cd} + \partial_c\, g_{bd} - \partial_d\, g_{cb}) \tag{14}$$

2. Lorentz Transformations in matrix form

$$[\Lambda^\mu_\nu] = \begin{pmatrix} \gamma & -\beta\,\gamma & 0 & 0 \\ -\beta\,\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{15}$$