PHYS 405: THEORY OF GENERAL RELATIVITY
Second Midterm Exam
December 16, 2016; 08.40-10.30

QUESTIONS: Choose any four of the following six problems

[25] **1**. Show that the Einstein tensor $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}$ satisfies divergence free condition $\nabla_\mu G^{\mu\nu} = 0$.

[25] **2**. Consider two neighboring geodesics, C given by $x^\mu(u)$ and $\bar{C}$ given by $\bar{x}^\mu(u)$ where u is an affine parameter, and let $\xi^\mu(u)$ be the small vector connecting points on the two geodesics with the same parameter value $\bar{x}^\mu(u) = x^\mu(u) + \xi^\mu(u)$. Derive the Geodesic deviation formula

$$\frac{D\xi^\mu}{du} + R^a{}_{cbd}\, \xi^b\, \dot{x}^c\, \dot{x}^d = 0$$

[25] **3**. Show that the geodesic equation can be written as

$$\dot{p}_\mu = \frac{1}{2} (\partial_\mu\, g_{\alpha\beta})\, p^\alpha\, p^\beta$$

where p^μ is the 4-momentum of the particle with unit mass and $p_\mu = g_{\mu\alpha}\, p^\alpha$.

[25] **4**. **(a)**. A photon moves on circular orbit at $r = 3\mu$ in the Schwarzschild geometry. Show that the period of the orbit as measured by a stationary observer at the this radius is $\tau = 6\pi\mu/c$. **(b)**. What is the period of the orbit as measured by a stationary observer at infinity. **(c)**. For a particle in a circular orbit of radius r in the Schwarzschild geometry show that

$$\frac{d\phi}{dt} = \left(\frac{GM}{r^3}\right)^{1/2}.$$

[25] **5**. Find the stable innermost circular orbit of massive particles in the Schwarzschild geometry.

[25] **6**. Show that the vector $\xi^\mu = \delta^\mu_0$ is a Killing vector field in the Schwarzschild geometry.

Appendix:

1. Christoffel symbol

$$\Gamma^a_{bc} = \frac{1}{2}\, g^{ad}\ (\partial_b\, g_{cd} + \partial_c\, g_{bd} - \partial_d\, g_{cb}) \tag{1}$$

2. Lorentz Transformations in matrix form

$$[\Lambda^\mu_\nu] = \begin{pmatrix} \gamma & -\beta\,\gamma & 0 & 0 \\ -\beta\,\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

3. The curvature tensor

$$R^d{}_{abc} = \partial_b\, \Gamma^d{}_{ac} - \partial_c\, \Gamma^d{}_{ab} + \Gamma^e{}_{ac}\, \Gamma^d{}_{eb} - \Gamma^e{}_{ab}\, \Gamma^d_{ec}$$

4. The Ricci tensor

$$R_{ab} \equiv R^d{}_{abd} = \partial_b\, \Gamma^d{}_{ad} - \partial_d\, \Gamma^d{}_{ab} + \Gamma^e{}_{ad}\, \Gamma^d{}_{eb} - \Gamma^e{}_{ab}\, \Gamma^d_{ed}$$

5. The Ricci Scalar

$$R \equiv g^{ab}\, R_{ab}$$

6. The Scwarschild Metric

$$ds^2 = c^2\,(1 - \frac{2\mu}{r})dt^2 - \frac{1}{1 - \frac{2\mu}{r}}\, dr^2 - r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

where $\mu = \frac{MG}{c^2}$

7. Geodesic Equations for a massive particle at the equatorial plane $\theta = \frac{\pi}{2}$.

$$(1 - \frac{2\mu}{r})\,\dot{t} = k, \tag{3}$$

$$c^2(1 - \frac{2\mu}{r})\dot{t}^2 - (1 - \frac{2\mu}{r})^{-1}\dot{r}^2 - r^2\,\dot{\phi}^2 = c^2, \tag{4}$$

$$r^2\,\dot{\phi} = h \tag{5}$$

where $k = \frac{E}{m_0 c^2}$ and h is the angular velocity of the test particle
8. Geodesic Equations for a massless particle at the equatorial plane $\theta = \frac{\pi}{2}$.

$$(1 - \frac{2\mu}{r})\,\dot{t} = k, \tag{6}$$

$$c^2(1 - \frac{2\mu}{r})\dot{t}^2 - (1 - \frac{2\mu}{r})^{-1}\dot{r}^2 - r^2\,\dot{\phi}^2 = 0, \tag{7}$$

$$r^2\,\dot{\phi} = h \tag{8}$$

where $k = \frac{E}{m_0 c^2}$ and h is the angular velocity of the test particle
9. Symmetries of the Curvature tensor

$$R_{dabc} = -R_{adbc} = -R_{dacb} \tag{9}$$

$$R_{adbc} = R_{bcad}, \quad R_{dabc} + R_{dcab} + R_{dcab} = 0, \tag{10}$$

$$\nabla_e\,R_{abcd} + \nabla_c\,R_{abde} + \nabla_d\,R_{abec} = 0 \tag{11}$$

10. A vector ξ^μ is called a Killing vector field if it satisfies the Killing equation $\nabla_\mu\,\xi_\nu + \nabla_\nu\,\xi_\mu = 0$