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New nonlinear evolution equations from surface theory

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We point out that the connection between surfaces in three-dimensional flat space and the inverse scattering problem provides a systematic way for constructing new nonlinear evolution equations. In particular we study the imbedding for Guichard surfaces which gives rise to the Calapso–Guichard equations generalizing the sine-Gordon (SG) equation. Further, we investigate the geometry of surfaces and their imbedding which results in the Korteweg–deVries (KdV) equation. Then by constructing a family of applicable surfaces we obtain a generalization of the KdV equation to a compressible fluid.

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I. INTRODUCTION

The Korteweg–deVries and sine-Gordon equations are classic examples of nonlinear evolution equations which form completely integrable systems. They belong to a class of partial differential equations which can be solved by the inverse scattering method of Gardner, Greene, Kruskal, and Miura[1] and Zakharov and Shabat.[2] At present we do not have a set of partial differential equations encompassing all such systems. Thus, given a set of equations we must first check whether or not they can be formulated as an inverse scattering problem according to the general framework provided by Lax[3] and Ablowitz, Kaup, Newell, and Segur,[4] (LAKNS). It is therefore of interest to construct new equations generalizing the KdV and SG equations and in this paper we shall present such new systems. We shall verify that these generalizations are viable by showing that the new equations form LAKNS systems. For this purpose we shall in each case obtain the linear equations of the scattering problem for which the new nonlinear equations act as integrability conditions. Finding the explicit form of the potentials for the appropriate inverse scattering problem is also the first step toward the solution of these equations. But a discussion of solutions will be postponed to a future paper because here we shall be concerned solely with the problem of constructing new nonlinear evolution equations.

We have obtained the new equations by exploiting the one-to-one correspondence between LAKNS systems and the classical theory of surfaces in three-dimensional space. For the SG equation where the underlying surface is pseudospherical this is very familar territory,[5] while for the general case this correspondence has been discussed by Crampin Pirani and Robinson[6] at the level of connection. The "soliton connection" is a flat linear connection in a principal fiber bundle with structure group SL $(2,R)$. Its relationship to the problem of imbedding surfaces in three-dimensional Euclidean space arises from the fact that the Gauss–Codazzi equations are in this case equivalent to Cartan's equations of structure for SO (3).[7] This correspondence suggests that the soliton connection can be given a richer structure at the level of Riemannian metrics. We shall discuss this problem in Sec. II where we shall formulate the LAKNS equations in terms of the first and second fundamental forms of the surface. Then, quite generally, the linear equations to be solved by the inverse scattering method are the Weingarten equations and their integrability conditions which result in the nonlinear evolution equations consist of the Gauss–Codazzi equations.

There are advantages to be derived from the recognition of the metric level appropriate to the soliton connection. First of all it will enable us to correct misleading statements in the literature to the effect that all LAKNS systems correspond to pseudospherical surfaces.[8] The validity of such a statement requires the use of the equations of motion in the definition of the connection, but this is an identity which does not yield any new information about metric structure. On the other hand, if we can properly identify the surface at the metric level we can consider its generalizations in classical differential geometry and such surfaces will provide new examples of completely integrable systems. As an illustration of this point, in Sec. III we shall consider the problem of imbedding surfaces which are applicable to quadrics. These are known as surfaces of Guichard[9] and they are generalizations of pseudospherical surfaces. The Gauss–Codazzi equations for Guichard surfaces, first obtained by Calapso,[10] provide an attractive generalization of the SG equation.[11] We shall give an updated derivation of the Calapso–Guichard equations (CG) and cast them into the form of an inverse scattering problem.

The equivalence between LAKNS systems and surface theory at the metric level can be exploited in a systematic way in order to construct new nonlinear evolution equations. We shall discuss this process in Sec. II and in Sec. IV apply it to the KdV equation to obtain a generalization of this equation.

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$$\lambda_t + 2u\lambda_x + 2\kappa u_x = 0,$$
$$u_t + 6uu_x + u_{xxx} - \lambda_x u_{xx} = 0, \tag{1.1}$$

where we have introduced a new field λ and κ is constant. This pair of coupled partial differential equations can be interpreted as the continuity and Euler equations for a compressible fluid. In deriving these equations the first step was to investigate the family of surfaces for which the imbedding problem results in the KdV equation. It appears that this question has not been asked before even though it is of basic interest. We shall obtain the expression for the first and second fundamental forms characterizing the geometry of surfaces underlying the KdV equation. Then we shall consider the problem of constructing applicable surfaces, that is, surfaces with the same intrinsic geometry as that appropriate to the KdV equation but with a different imbedding into three-dimensional flat space. The resulting Gauss–Codazzi equations (1.1) are new nonlinear evolution equations. We shall formulate them as an inverse scattering problem and conclude by pointing out an alternative derivation of these equations. The connection 1-form appropriate to the KdV equation can in particular be subjected to gauge transformations belonging to $R^{\cdot}$, an abelian subgroup of SL $(2,R)$, which is also familiar as the group of scale transformations leaving the KdV equation invariant. If we let the "scale parameter" become a function of position and time while preserving the connection 1-form appropriate to this abelian subgroup, we obtain a new realization of the solition connection. Equations (1.1) are the conditions for this new connection to have zero curvature.

II. SURFACES

We shall briefly review the LAKNS equations of the inverse scattering method and the Weingarten and Gauss–Codazzi equations of the imbedding problem in order to point out the one-to-one correspondence between them. The rest of this section is devoted to the construction of applicable surfaces, which provides a systematic way for generalizing familar examples of LAKNS systems.

We consider a moving frame in a three-dimensional flat space M with Euclidean or Lorentzian signature. At a point P in M we have

$$dP = \omega^i \mathbf{e}_i, \tag{2.1}$$

$\mathbf{e}_i$, $i = 1,2,3$, are the basis vectors and ω^i the basis 1-forms. A surface S is defined by

$$\omega^3 = 0, \tag{2.2}$$

accordingly $\mathbf{e}_3$ is the normal vector to the surface. We shall let the metric of M be of the form

$$ds^2 = \mathrm{diag}(1,\epsilon,\eta), \quad \epsilon = \pm 1, \quad \eta = \pm 1 \tag{2.3}$$

so that there is no loss of generality in always defining the surface by Eq. (2.2). Cartan's equations of structure are

$$d\mathbf{e}_i = \omega_i^{\ k}\mathbf{e}_k, \tag{2.4}$$

where $\omega_i^{\ k}$ are connection 1-forms and their integrability conditions become

$$\Theta^i_{\ k} = 0, \tag{2.5}$$

where

$$\Theta^i_{\ k} = d\omega^i_{\ k} + \omega^i_{\ j}\wedge\omega^j_{\ k} \tag{2.6}$$

are the curvature 2-forms. We can write these equations in a more familar form if we introduce the first and second fundamental forms of the surface. That is, we consider a surface with the intrinsic metric

$$ds_1^{\ 2} = (\omega^1)^2 + \epsilon(\omega^2)^2 \tag{2.7}$$

and the Riemannian connection on S is given by the 1-form $\omega^1_{\ 2}$ where

$$d\omega^\alpha + \omega^\alpha_{\ \beta}\wedge\omega^\beta = 0 \tag{2.8}$$

with the Greek indices ranging over two values only. Finally we have

$$d\omega^\alpha_{\ \beta} = K\omega^\alpha\wedge\omega^\beta, \tag{2.9}$$

where K is the Gaussian curvature of the surface. We shall now consider the imbedding problem and to this end we let

$$-ds_2^{\ 2} = \omega^1\otimes\pi^1 + \epsilon\omega^2\otimes\pi^2 \tag{2.10}$$

denote the extrinsic curvature, or the second fundamental form of S. The Gauss–Codazzi equations for imbedding S in M are the same as Eqs. (2.5)–(2.6) with the identification

$$\omega^1_{\ 3} = \pi^1, \qquad \omega^2_{\ 3} = \pi^2, \tag{2.11}$$

and therefore we have

$$d\omega^1_{\ 2} - \epsilon\eta\pi^1\wedge\pi^2 = 0,$$
$$d\pi^1 + \omega^1_{\ 2}\wedge\pi^2 = 0, \tag{2.12}$$
$$d\pi^2 - \epsilon\omega^1_{\ 2}\wedge\pi^1 = 0.$$

Finally, from Eq. (2.2) which defines the surface, we have the condition

$$\omega^1\wedge\pi^1 + \epsilon\omega^2\wedge\pi^2 = 0. \tag{2.13}$$

In the imbedding problem the equations of structure (2.4) are also known as Weingarten equations and together with their integrability conditions which are the Gauss–Codazzi equations (2.12) they constitute the fundamental equations of the subject.

The Gauss–Codazzi equations for imbedding S in M can be written as Cartan's equations for SL $(2,R)$

$$d\theta^i + \tfrac{1}{2}c_j^{\ i}{}_k\theta^j\wedge\theta^k = 0, \tag{2.14}$$

where $c_j^{\ i}{}_k$ are the structure constants of SL $(2,R)$. Hereafter Latin indices will stand for SL $(2,R)$ values and range over $i = 0,1,2$. Equations (2.14) will be identical to Eqs. (2.12) provided we let

$$\omega^1_{\ 2} = -(2i/\epsilon^{\frac{1}{2}})\theta^0,$$
$$\pi^1 = i\eta^{\frac{1}{2}}(\theta^1 + \theta^2), \tag{2.15}$$
$$\pi^2 = \eta^{\frac{1}{2}}\epsilon^{\frac{1}{2}}(-\theta^1 + \theta^2),$$

since

$$c_1^{\ 0}{}_2 = 1,$$
$$c_0^{\ 1}{}_1 = -c_0^{\ 2}{}_2 = 2, \tag{2.16}$$

are the nonvanishing structure constants of SL $(2,R)$. In the form which will be most useful for our purposes we can express this result as follows:

Given the first and second fundamental forms of the surface as in Eqs. (2.7) and (2.10) we can construct an SL $(2,R)$ valued connection 1-form

$$\Gamma = \begin{pmatrix} \theta^0 & \theta^1 \\ \theta^2 & -\theta^0 \end{pmatrix}, \qquad (2.17)$$

where θ^i are given by Eqs. (2.15), which has a vanishing curvature 2-form

$$\Theta = d\Gamma + \Gamma \wedge \Gamma = 0 \qquad (2.18)$$

by virtue of the Gauss–Codazzi equations (2.12).

When we turn to the problem of solitons, we find that the soliton connection is an SL $(2,R)$ valued 1-form as in Eqs. (2.17), where[6]

$$\begin{aligned} \theta^0 &= -(Adt + i\zeta dx), \\ \theta^1 &= -(Bdt + qdx), \\ \theta^2 &= -(Cdt + rdx), \end{aligned} \qquad (2.19)$$

and A,B,C,q,r are functions of t and x while ζ is a constant. The condition that its curvature 2-form should vanish, [cf. Eqs. (2.18)], reduces to the LAKNS equations

$$\begin{aligned} A_x &= qC - rB, \\ B_x - 2i\zeta B &= q_t - 2Aq, \\ C_x + 2i\zeta C &= r_t + 2Ar, \end{aligned} \qquad (2.20)$$

where here and in the following subscripts such as x, t denote partial differentiation with respect to the coordinates. In both problems we have SL $(2,R)$-valued flat connection forms but there is one further condition which must be satisfied before we can establish their equivalence. In Eqs. (2.19) ζ, which will be the eigenvalue in the inverse scattering method, must be a constant. Starting with the connection given by Eqs. (2.17) we can always perform an SL $(2,R)$ gauge transformation

$$\Gamma' = \Sigma\Gamma\Sigma^{-1} + \Sigma d\Sigma^{-1}, \qquad (2.21)$$

where

$$\Sigma = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1, \qquad (2.22)$$

to cast it into the form given in Eqs. (2.19) and vice versa. The condition (2.18) is gauge-invariant and therefore we have a one-to-one correspondence between LAKNS systems and the classical theory of surfaces imbedded in a three-dimensional flat space M.

We shall now consider the problem of constructing applicable surfaces. These surfaces will carry the same intrinsic metric as in Eqs. (2.7) but a different imbedding. The choice of the new expression for the second fundamental form (which via the Gauss–Codazzi equations leads to new equations of motion) is best understood by turning to Eqs. (2.21). In these equations we have the transformation rule for the connection under a change of gauge. We have already remarked that the soliton connection is a flat connection and this property is invariant under gauge transformations. It will be useful for our purposes to specialize to transformations belonging to $R^{\cdot}$ which is an abelian subgroup of SL $(2,R)$, where

$$\Sigma = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix} \qquad (2.23)$$

is a typical element. In this case Eqs. (2.21) reduce to

$$\begin{aligned} \theta'^0 &= \theta^0 + d\lambda, \\ \theta'^1 &= e^{2\lambda}\theta^1, \\ \theta'^2 &= e^{-2\lambda}\theta^2, \end{aligned} \qquad (2.24)$$

and through the identifications (2.15) we find that this change of gauge corresponds to the simultaneous rotations

$$\begin{aligned} \omega'^\alpha &= (\Lambda^{-1})^\alpha{}_\beta \omega^\beta, \\ \pi'^\alpha &= \Lambda^\alpha{}_\beta \pi^\beta, \end{aligned} \qquad (2.25)$$

where

$$\Lambda = \begin{pmatrix} \cosh 2\lambda & -(i/\epsilon^1)\sinh 2\lambda \\ i\epsilon^1 \sinh 2\lambda & \cosh 2\lambda \end{pmatrix}. \qquad (2.26)$$

is a position-dependent rotation. The transformation law for the connection 1-form on S is given by

$$\omega^1{}_2 \rightarrow \omega^1{}_2 + 2id\lambda, \qquad (2.27)$$

which follows from Eqs. (2.15) or directly from Eqs. (2.25). Since the equations of motion are obtained from the gauge-invariant condition that the curvature should vanish they are not affected by gauge transformations such as the one implicitly defined by Eqs. (2.25). If, however, we undo the rotation in Eqs. (2.25) for $\{\omega^\alpha\}$ only, we shall find an applicable surface where

$$\begin{aligned} \omega^\alpha &\rightarrow \omega^\alpha, \\ \pi^\alpha &\rightarrow \Lambda^\alpha{}_\beta \pi^\beta, \end{aligned} \qquad (2.28)$$

and λ will now enter as a new field into the equation of motion. In this case $\omega^1{}_2$ remains invariant and we have

$$\begin{aligned} \theta^0 &\rightarrow \theta^0, \\ \theta^1 &\rightarrow e^{2\lambda}\theta^1, \\ \theta^2 &\rightarrow e^{-2\lambda}\theta^2, \end{aligned} \qquad (2.29)$$

in place of Eqs. (2.24). The requirement that the curvature of the new connection be zero results in

$$\begin{aligned} \Theta^0 &= 0, \\ \Theta^1 + 2d\lambda \wedge \theta^1 &= 0, \\ \Theta^2 - 2d\lambda \wedge \theta^2 &= 0, \end{aligned} \qquad (2.30)$$

which are new equations of motion. We note that the Θ^0 component of curvature remains unchanged when we consider applicable surfaces. Therefore if we want to obtain new field equations by using this method, we must start with a connection where Θ^0 vanishes identically. In Sec. IV we shall apply this construction to the KdV equation.

III. CALAPSO–GUICHARD EQUATIONS

The prototype of a completely integrable system where surface theory plays a paramount role is the SG equation.

The Gauss–Codazzi equations for imbedding pseudospherical surfaces in E^3 reduce to the SG equation and there are generalizations of this equation which utilize the same intrinsic geometry.[5,12] In contrast, the 2-surface behind the Calapso–Guichard (CG) equations is a quadric. So we consider an intrinsic metric on the surface with the basis 1-forms

$$\omega^1 = e^{\xi}\sin\vartheta dt,$$
$$\omega^2 = e^{\xi}\cos\vartheta dx, \tag{3.1}$$

which differs from the metric of a pseudospherical surface by a conformal factor. The Riemannian connection on the surface is given by the 1-form

$$\omega^1{}_2 = (\vartheta_x + \tan\vartheta\, \xi_x)dt + (\vartheta_t - \cot\vartheta\, \xi_t)dx \tag{3.2}$$

which follows from Eqs. (2.8) and (3.1). The imbedding of this surface in E^3 is given by the second fundamental form [cf. Eq. (2.10)], where

$$\pi^1 = (\cos\vartheta + h\sin\vartheta)dt,$$
$$\pi^2 = (-\sin\vartheta + h\cos\vartheta)dx \tag{3.3}$$

are the basis 1-forms. We can now verify the fundamental property of Guichard surfaces which is responsible for the parametrization used in Eqs. (3.3): If we consider two Guichard surfaces S,S' which have the same extrinsic curvature, then their principal radii of curvature ρ_i,ρ_i' where

$$\pi^1 = \rho_1\omega^1 = \rho_1'(\omega^1)',$$
$$\pi^2 = \rho_2\omega^2 = \rho_2'(\omega^2)'$$

satisfy the relation

$$\tfrac{1}{2}(\rho_1\rho_1' + \rho_2\rho_2') = -1. \tag{3.4}$$

The surface S' is said to be the associate of S and they are related by the transformations

$$e^{i\vartheta'} = [(1+ih)/(1-ih)]e^{-i\vartheta},$$
$$e^{\xi'} = e^{-\xi}(1+h^2), \tag{3.5}$$
$$h' = h.$$

Historically Eq. (3.4) has been the starting point of the investigations on the surfaces of Guichard. Using Eqs. (3.2) and (3.3) in Eqs. (2.12) we obtain the Gauss–Codazzi equations for imbedding S in E^3

$$(\vartheta_x + \tan\vartheta\, \xi_x)_x - (\vartheta_t - \cot\vartheta\, \xi_t)_t$$
$$+ (\cos\vartheta + h\sin\vartheta)(-\sin\vartheta + h\cos\vartheta) = 0,$$
$$h_x = (h - \tan\vartheta)\xi_x,$$
$$h_t = (h + \cot\vartheta)\xi_t. \tag{3.6}$$

These are the Calapso–Guichard equations. We may also call them sine-Guichard equations because they are obtained for Guichard surfaces of the second kind. Needless to say, there is also a sinh-Guichard system which generalizes the sinh-Gordon equation

$$(\vartheta_x + \coth\vartheta\xi_x)_x + (\vartheta_y + \tanh\vartheta\, \xi_y)_y$$
$$+ (\cosh\vartheta + h\sinh\vartheta)(\sinh\vartheta + h\cosh\vartheta) = 0,$$
$$h_x = (h + \coth\vartheta)\xi_x,$$
$$h_y = (h + \tanh\vartheta)\xi_y. \tag{3.7}$$

These equations are Gauss–Codazzi equations for the surfaces of Guichard of the first kind. We shall not consider them any further in this paper except to note that every statement concerning the sine-Guichard equations can be translated into an analoguous one about the sinh-Guichard equations.

We shall now consider the formulation of CG equations as an inverse scattering problem. For this purpose we need to transform these equations into a form whereby the correspondence limit of our results with those of Ablowitz, *et al.*[4] will become manifest. Hence we shall first introduce null coordinates.

$$u = t - x, \quad v = t + x, \tag{3.8}$$

and rewrite Eqs. (3.6) in this new coordinate system. We find

$$(\vartheta_u - \cot2\vartheta\, \xi_u - \csc2\vartheta\, \xi_v)_v$$
$$+ (\vartheta_v - \cot2\vartheta\, \xi_v - \csc2\vartheta\, \xi_u)_u$$
$$+ (\cos\vartheta + h\sin\vartheta)(-\sin\vartheta + h\cos\vartheta) = 0,$$
$$h_v = (h + \cot2\vartheta)\xi_v + \csc2\vartheta\, \xi_u,$$
$$h_u = (h + \cot2\vartheta)\xi_u + \csc2\vartheta\, \xi_v, \tag{3.9}$$

which has a symmetric dependence on the null coordinates but the lack of Lorentz covariance of the CG equations has resulted in long expressions. We note that either in Eqs. (3.6) or in Eqs. (3.9) we can completely eliminate ξ to obtain two coupled partial differential equations for ϑ and h. This form of the equations is useful in the formulation of the initial value problem but we shall keep ξ for ease of handling the equations. Finally, we shall remark that in CG equations h behaves in a manner similar to a stereographic variable. In particular the introduction of stereographic angle where

$$h = \tan(\phi/2) \tag{3.10}$$

simplifies the calculations. This relation is suggested, for example, by Eqs. (3.5) which now reduce to the transformations

$$\vartheta' = \phi - \vartheta,$$
$$\phi' = \phi, \tag{3.11}$$

relating a Guichard surface S to its associate S'.

The formulation of CG equations as Cartan's equations for SL $(2,R)$, [cf. Eq. (2.14)] follows from Eqs. (3.2), (3.3) and the identification in Eqs. (2.15). In the null coordinate system the 1-forms are given by

$$\theta^0 = -\tfrac{1}{2}i(\vartheta_u - \cot2\vartheta\, \xi_u - \csc2\vartheta\, \xi_v)du$$
$$+ \tfrac{1}{2}i(\vartheta_v - \cot2\vartheta\, \xi_v - \csc2\vartheta\, \xi_u)dv,$$
$$\theta^1 = \tfrac{1}{2}i(1+ih)e^{-i\vartheta}du + \tfrac{1}{2}i(1-ih)e^{i\vartheta}dv,$$
$$\theta^2 = -\overline{\theta^1}, \tag{3.12}$$

where bar denotes complex conjugation. However, as we mentioned in Sec. II, the connection 1-form Γ obtained from Eqs. (3.12) is not directly in the canonical form of a soliton connection. In order to achieve the canonical form required by the inverse scattering method we must perform a gauge transformation, as in Eqs. (2.21). We find that

$$\Sigma = \begin{pmatrix} \alpha & \alpha \\ -1/2\alpha & 1/2\alpha \end{pmatrix}, \tag{3.13}$$

where

$$\alpha^2 = [(\zeta + f)/(1 + ih)]e^{i\vartheta}, \tag{3.14}$$

with

$$f^2 = \zeta^2 - \tfrac{1}{4}(1 + h^2) \tag{3.15}$$

the desired transformation matrix. ζ is so far an arbitrary constant which will eventually be interpreted as the eigenvalue in the inverse scattering problem. From Eqs. (2.21) and (2.19) we can now determine the potentials which appear in the inverse scattering problem. The results are

$$\begin{aligned}
q &= -\tfrac{1}{2}i(2\vartheta_v - \cot 2\vartheta\, \xi_v - \csc 2\vartheta\, \xi_u) + Hh_v + if, \\
r &= -\tfrac{1}{2}i(2\vartheta_v - \cot 2\vartheta\, \xi_v - \csc 2\vartheta\, \xi_u) + Hh_v - if, \\
A &= [i/(1 + h^2)][\zeta(1 - h^2) - 2ihf]\cos 2\vartheta \\
&\quad + [i/(1 + h^2)][2\zeta h + i(1 - h^2)f]\sin 2\vartheta, \\
B &= -\tfrac{1}{2}i(\csc 2\vartheta\, \xi_v + \cot 2\vartheta\, \xi_u) + Hh_u \\
&\quad + [1/(1 + h^2)][2\zeta h + i(1 - h^2)f]\cos 2\vartheta \\
&\quad - [1/(1 + h^2)][\zeta(1 - h^2) - 2ihf]\sin 2\vartheta, \\
C &= -\tfrac{1}{2}i(\csc 2\vartheta\, \xi_v + \cot 2\vartheta\, \xi_u) + Hh_u \\
&\quad - [1/(1 + h^2)][2\zeta h + i(1 - h^2)f]\cos 2\vartheta \\
&\quad + [1/(1 + h^2)][\zeta(1 - h^2) - 2ihf]\sin 2\vartheta,
\end{aligned} \tag{3.16}$$

where

$$H = \tfrac{1}{2}(\zeta h + if)/[f(1 + h^2)]. \tag{3.17}$$

We have thus cast the Calapso–Guichard equations into a form whereby solutions can be obtained by a direct application of the inverse scattering method.

IV. SURFACE THEORY AND KdV EQUATION

The theory of surfaces has not played a significant role in the case of KdV equation. In this respect historically the approach to the SG and KdV equations has led through very different routes. But in this paper we have emphasized the one-to-one correspondence between surface theory and LAKNS systems, a distinguished member of which is the KdV equation. It is therefore natural to ask about the nature of the surface, i.e., its intrinsic geometry and imbedding in M, which gives rise to the KdV equation.

In order to investigate the geometry underlying a given LAKNS system it is necessary to proceed in a direction opposite to that familiar from surface theory. Thus we shall start with the soliton connection of AKNS and finally obtain the first and second fundamental forms of the surface. The AKNS connection for KdV equation is given by the 1-forms.

$$\begin{aligned}
\theta^0 &= (-4i\zeta^3 + 2i\zeta u - u_x)dt + i\zeta dx, \\
\theta^1 &= (4\zeta^2 u + 2i\zeta u_x - 2u^2 - u_{xx})dt + udx, \\
\theta^2 &= (-4\zeta^2 + 2u)dt - dx,
\end{aligned} \tag{4.1}$$

and the requirement that its curvature 2-form should vanish results in

$$u_t + 6uu_x + u_{xxx} = 0, \tag{4.2}$$

which is the KdV equation. The connection 1-form in Eqs. (4.1) is not particularly useful for our purposes but we are free to perform SL $(2,R)$ transformations to write it in any gauge we desire. There is in fact a well-known gauge where simple expressions for the connection 1-form are obtained:

$$\begin{aligned}
\theta^0 &= -u_x dt, \\
\theta^1 &= dx - 2udt, \\
\theta^2 &= -udx + (2u^2 + u_{xx})dt.
\end{aligned} \tag{4.3}$$

and this gauge is interesting because the new θ^0 is manifestly the connection of a Riemannian metric. Hence we can identify the first fundamental form of the surface

$$ds_1^2 = 32udt^2 + 16dtdx, \tag{4.4}$$

where

$$\begin{aligned}
\omega^1 &= dx + 2(u + 2)dt, \\
\omega^2 &= dx + 2(u - 2)dt
\end{aligned} \tag{4.5}$$

are the orthonormal basis 1-forms. We can verify that with

$$\epsilon = -1 \tag{4.6}$$

the Riemannian connection 1-form $\omega^1{}_2$ is consistent with θ^0 given by Eq. (4.3) and the definition in Eq. (2.15); and furthermore the conditions (2.13) are satisfied identically. Further comparison of Eqs. (2.15) and (4.3) leads to

$$\eta = -1 \tag{4.7}$$

and equations (4.6) and (4.7) determine the nature of the imbedding problem for the KdV equation. From Eq. (2.9) we find that

$$K = -\tfrac{1}{4}u_{xx} \tag{4.8}$$

is the Gaussian curvature of the surface. It remains to identify the second fundamental form of this surface and from Eqs. (2.15) and (4.3) we find

$$ds_2^2 = 2dx^2 - 8udtdx + 8(u^2 - 4K)dt^2. \tag{4.9}$$

The imbedding problem of a surface with the first and second fundamental forms given by Eqs. (4.4) and (4.9) is equivalent to the KdV equation. We shall now consider a new family of surfaces applicable to this surface and obtain a generalization of KdV equation.

The KdV equation has well-known scale invariance properties. That is, it remains invariant under the transformations

$$\begin{aligned}
u &\to e^{-4\lambda}u, \\
x &\to e^{2\lambda}x, \\
t &\to e^{6\lambda}t,
\end{aligned} \tag{4.10}$$

where λ is an arbitrary constant parameter. A familiar consequence of this invariance is the similarity solution of the KdV equation which is known as cnoidal waves. The origin of this invariance can be traced to the gauge transformations the KdV soliton connection may be subjected to. In particular we can check that under the transformations (4.10) the soliton connection defined by the 1-forms in Eqs. (4.3) transforms according to Eqs. (2.24). Hence scale transformations leaving the KdV equation invariant can alternatively be regarded as gauge transformations of the KdV connection 1-form for $R^{\cdot}$. In order to construct applicable surfaces to the

KdV-surface we shall proceed as in Sec. II and promote the "scale-parameter" λ to become a function of x and t. We shall further let the KdV connection 1-form (4.3) suffer the transformations (2.28) so that

$$\begin{aligned} \theta^0 &= -u_x dt, \\ \theta^1 &= e^\lambda (dx - 2u dt), \\ \theta^2 &= e^{-\lambda}[-u dx + (2u^2 + u_{xx})dt], \end{aligned} \tag{4.11}$$

and from Eqs. (2.30) we find

$$\begin{aligned} &\lambda_t + 2u\lambda_x = 0, \\ &u_t + 6uu_x + u_{xxx} - \lambda_x u_{xx} = 0, \end{aligned} \tag{4.12}$$

which are new equations of motion. These equations are also scale-invariant under the transformations (4.10).

Equations (4.12) can be further extended to Eqs. (1.1) by scaling the connection 1-forms (4.11) according to

$$\begin{aligned} \tilde\theta^0 &= (1-\kappa)\theta^0, \\ \tilde\theta^1 &= (1-\kappa)\theta^1, \\ \tilde\theta^2 &= \theta^2, \end{aligned} \tag{4.13}$$

where $\kappa \neq 1$ is an arbitrary constant. Such a transformation does not belong to SL $(2,R)$ and therefore it does not describe an invariance property of the field equations. Rather, it leads to a new result which consists of the introduction of an arbitray constant into the field equations. This process is consistent because $\Theta^0 = 0$ is once again identically satisfied. The remaining conditions on the curvature result in Eqs. (1.1) and we note that the KdV limit of these equations is given by

$$\lambda \to 0, \quad \kappa \to 0 \tag{4.14}$$

which are both necessary. Finally we shall remark that for $\kappa \neq 0$ the first of Eqs. (1.1) can be written as

$$\rho_t + (2u\rho)_x = 0, \tag{4.15}$$

with

$$\rho = \kappa e^{(1/\kappa)\lambda} \tag{4.16}$$

which is a continuity equation without source terms.

We shall now formulate Eqs. (1.1) as an inverse scattering problem. Once again we must subject the connection 1-forms (4.11) to a gauge transformation in order to cast them into the form of Eqs. (2.19). In this case the required SL $(2,R)$ transformation is given by

$$\Sigma = \begin{pmatrix} 1 & \beta \\ i\zeta & 1 + i\zeta\beta \end{pmatrix}, \tag{4.17}$$

where

$$\beta = i\zeta(e^\lambda - 1)/(ue^{-\lambda} - \zeta^2(1-\kappa)e^\lambda). \tag{4.18}$$

From Eqs. (2.21) and (2.19) we can identify the LAKNS potentials which turn out as follows:

$$\begin{aligned} A &= -(1 + 2i\zeta\beta)(1-\kappa)u_x - 2i\zeta(1-\kappa)u(1 + i\zeta\beta)e^\lambda \\ &\quad - \beta(2u^2 + u_{xx})e^{-\lambda}, \\ B &= 2i\zeta(1-\kappa)u_x - 2\zeta^2(1-\kappa)ue^\lambda + (2u^2 + u_{xx})e^{-\lambda}, \\ C &= -2(1-\kappa)\beta(1 + i\zeta\beta)u_x + \beta_t \\ &\quad - 2(1-\kappa)u(1 + i\zeta\beta)^2 e^\lambda - \beta^2(2u^2 + u_{xx})e^{-\lambda}, \\ q &= \zeta^2(1-\kappa)e^\lambda - ue^{-\lambda}, \\ r &= (1-\kappa)(1 + i\zeta\beta)^2 e^\lambda + u\beta^2 e^{-\lambda} + \beta_x, \end{aligned} \tag{4.19}$$

and solutions can be obtained by application of the inverse scattering method.

There is another aspect of Eqs. (1.1) which is useful from the stand point of constructing solutions. Namely, the scale invariance of these equations allows them to be brought into the form of a single nonlinear ordinary differential equation. We shall introduce

$$\begin{aligned} v &= x^2 u, \\ z &= x^3/t, \end{aligned} \tag{4.20}$$

which together with λ from a set of scale-invariant variables. In terms of these quantities Eqs. (1.1) reduce to

$$\begin{aligned} &[27z^3v''' + z(18v + 24 - z)v' - 12v(2+v)](6v - z) \\ &\quad - 3\kappa(6v' - 4v)(27z^3v'' - 18z^2v' + 18zv) = 0, \end{aligned} \tag{4.21}$$

where prime denotes derivative with respect to z. For $\kappa = 0$ the content of this equation is not different from the corresponding result for the KdV equation.

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