MATH 443: PARTIAL DIFFERENTIAL EQUATIONS
**Final Exam:** 29 December 2005 Thursday 09.00-12.00(SAZ19)

**QUESTIONS: Solve five problems out of the following seven problems**

[25]**1**. Find the equation of the integral surface of the equation

$$2y(z-3)p + (2x-z)q = y(2x-3)$$

passing through the circle $z = 0, \quad x^2 + y^2 = 2x$.

**Solution**

$$x^2 + y^2 - z^2 - 2x + 4z = 0$$

[25]**2**. Find the surface which is orthogonal to the one-parameter system $z = cxy(x^2 + y^2)$ which passes through the hyperbola $z = 0, \quad x^2 - y^2 = a^2$.

**Solution**

$$(x^2 - y^2)^2 (x^2 + y^2) = a^4 (x^2 + y^2 - 4z^2)$$

[20]**3**. Show that the equations $xp - yq = x$ and $x^2 p + q = xz$ are compatible and find their solution.

**Solution**

$$p = \frac{1+yz}{1+yx}, \quad q = \frac{xz - x^2}{1+xy}$$

Then

$$dz = pdx + qdy$$

gives

$$(1+yz)dx + (xz - x^2)dy - (1+xy)dz = 0$$

Hence $X = (1 + yz, xz^2 - 1 - xy)$. Then $X \cdot (\nabla \times X) = 0$. This means that given equations are compatible and the solution of the Pfaffian differential equation is

$$z = x + c\,(1 + yx)$$

where $c$ is an arbitrary constant.

[20]**4.** Let $u(x, y, z)$ satisfy the Laplace equation, $\nabla^2 u = 0$, in the region $z > 0$ of $\mathbb{R}^3$ and $u(x, y, 0) = f(x, y)$. Prove that

$$u(x, y, z) = \frac{z}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x', y')dx'\,dy'}{[z^2 + (x - x')^2 + (y - y')^2]^{3/2}}$$

**Solution** Solved in the class.

[20] **5**. Solve the following differential equation

$$u_{xxx} - 6u_{xxy} + 12u_{xyy} - 8u_{yyy} = 12\,e^{y+2x}$$

**Solution**

$$u = x^2\,f_1(2x + y) + x\,f_2(2x + y) + f_3(2x + y) + 2x^3\,e^{2x+y}$$

[20]**6**. Solve the following initial value problem

$$u_{tt} - \frac{1}{c^2}\,u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$
$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad x \in \mathbb{R}.$$

where $f(x)$ and $g(x)$ are some differentiable functions.

**Solution** Solved in class (D'Alembert's solution)

[20]**7**. An initial and boundary value problem is given as follows:

$$
\begin{aligned}
u_t - 2t\, u_{xx} = 0, \quad & t \in \mathbb{R}, \quad 0 < x < a, \\
u(x,0) = f(x), \quad & 0 < x < a, \\
u(0,t) = u(a,t) = 0, \quad & t \in \mathbb{R}
\end{aligned}
$$

(a) Find the solution of this problem.
(b) Justify your solution (find conditions on the given function $f(x)$ for the existence of the solutions).
(c) Discuss the uniqueness of the solution.

**Solution** By letting $t \to t^2$ in the heat equation one obtains the above initial and boundary value problem. Following the approach done in the class for the heat equation one solves the above problem.