MATH 443: PARTIAL DIFFERENTIAL EQUATIONS

Second Midterm Exam: 24 November 2005 Thursday 10.00-12.30

(SAZ19)

QUESTIONS: Solve four problems out of the following five problems

[25]**1**. Reduce the equation

$$y^2\, z_{xx} - 2xy\, z_{xy} + x^2\, z_{yy} = \frac{y^2}{x}\, z_x + \frac{x^2}{y}\, z_y \tag{1}$$

to its canonical form and solve it.

Solution. Since $R = y^2$, $S = -2xy$ and $T = y^2$ then there exists one solution (*hence the equation is of parabolic type*) of

$$R\lambda^2 + S\lambda + T = 0$$

which is given by $\lambda = \frac{x}{y}$. This leads to

$$y\xi_x - x\xi_y = 0$$

or $\xi = x^2 + y^2$. Choose $\eta = x^2 - y^2$ the we have

$$z_x = 2xz_\xi + 2yz_\eta, \quad z_y = 2yz_\xi - 2yz_\eta, \tag{2}$$

$$z_{xx} = 4x^2\,[z_{\xi\xi} + 2z_{\xi\eta} + z_{\eta\eta}] + 2z_\xi + 2z_\eta, \tag{3}$$

$$z_{xy} = 4xy\,[z_{\xi\xi} - z_{\eta\eta}], \tag{4}$$

$$z_{yy} = 4y^2\,[z_{\xi\xi} - 2z_{\xi\eta} + z_{\eta\eta}] + 2z_\xi - 2z_\eta \tag{5}$$

then (1) reduces to its *canonical form* $z_{\eta\eta} = 0$. Hence the solution is

$$z(x,y) = f(x^2+y^2)\,(x^2 - y^2 + g(x^2+y^2)$$

where f and g are arbitrary functions.

[25]**2**. Solve the following initial value problem by using the Riemann-Green function method

$$\begin{aligned} u_{xx} - u_{yy} = F(x,y), \quad (x,y) \in U \subset \mathbb{R}^2, \\ u(x,0) = f(x), \\ u_y(x,0) = g(x), \end{aligned} \tag{6}$$

where $F(x,y)$ is a continuous function over U, $f(x)$ and $g(x)$ are differentiable functions. (**Hint**: The initial curve is the $x-$ axis and the region Σ is given below and take $G(x,y,\xi,\eta) = 1$.

Solution. Integrating (6) over Σ we get

$$\int\int_{\Sigma} F(x,y)\,dx\,dy = \int_{\partial\Sigma} [u_x dy + u_y dx]$$

where $\partial\Sigma$ is the boundary of Σ. Then (6) reduces to

$$\int\int_{\Sigma} F(x,y)\,dx\,dy = \int_A^B u_y dx + \int_B^P [u_x dy + u_y dx] + \int_P^A [u_x dy + u_y dx]$$

On AB we have $dy = 0$, BP we have $dx = -dy$ and on PA we have $dx = dy$ then

$$\int\int_{\Sigma} F(x,y)\,dx\,dy = \int_{\xi-\eta}^{\xi+\eta} g(x)dx - u|_B^P + u|_P^A$$

or

$$\int\int_{\Sigma} F(x,y)\,dx\,dy = \int_{\xi-\eta}^{\xi+\eta} g(x)dx - 2u(P) + u(\xi-\eta,0) + u(\xi+\eta,0)$$

this leads to the solution of the problem

$$u(\xi,\eta) = -\frac{1}{2}\int_0^{\eta}\int_{y+\xi-\eta}^{-y+\xi+\eta} F(x,y)\,dx\,dy + \frac{1}{2}\left[f(\xi-\eta)+f(\xi+\eta)\right] + \frac{1}{2}\int_{\xi-\eta}^{\xi+\eta} g(x)dx$$

[25]**3**. Solve the equation

$$z_{xx} + 2\alpha\, z_{xy} + \beta^2\, z_{yy} = x^2, \tag{7}$$

where α and β are constants. Consider all the cases (i) $\alpha > \beta$, (ii) $\alpha = \beta$, and $\alpha < \beta$.

Solution. DE (7) can be written in the form

$$(D + k_1\, D')\,(D + k_2\, D') = x^2$$

where

$$k_{1,2} = \alpha \pm \sqrt{\alpha^2 - \beta^2}$$

Hence we have the following solutions
(i) when $\alpha > \beta$ then both k_1 and k_2 are real the the solution is simply given by

$$z(x, y) = f(y - k_1\, x) + g(y - k_2\, x) + \frac{1}{12}\, x^4$$

(ii) when $\alpha = \beta$ $k_1 = k_2$ then the DE reduces to

$$(D = \alpha D')^2\, z = x^2$$

This equation has the solution

$$z(x, y) = x\, f(y - \alpha\, x) + g(y - \alpha x) + \frac{1}{4}\, x^4$$

where f and g are arbitrary functions.
(iii) when $\alpha < \beta$ DE is irreducible. This means that k_1 and k_2 become imaginary. The way we solve the DE equation is similar to part (i), provided that we take real part of the solution.

[25]**4**. Show that the surfaces $x^2 + y^2 + z^2 = cx^{2/3}$, where c is a constant, can form equipotential surfaces, and find the general form of the potential function ψ satisfying the Laplace equation, $\nabla^2\, \psi = 0$.

Solution. *Solved in class.*

[25]**5**. Show that if the two-dimensional harmonic equation (Laplace equation) $\nabla^2\, V = 0$ is transformed to plane polar coordinates r and θ, defined by

$x = r\cos\theta$, $y = r\sin\theta$ it takes the form

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

and show that, by use of separation of variables, potential function is given by

$$V(r,\theta) = \sum_0^\infty [A_n\, r^n + B_n\, r^{-n}][C_n\, \cos n\theta + D_n\, \sin n\theta] + K \ln r, \qquad (8)$$

where A_n, B_n, C_n, D_n and K are constants. Determine these constants if
(i) V is finite as $r \to 0$,
(ii) $V = \sum_{n=0}^{\infty} c_n \cos n\theta$ on $r = a$ where c_n are given constants.

Solution. Using the transformation $x = r\cos\theta, \quad y = r\sin\theta$ we obtain

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2\, d\theta^2$$

Hence $h_1 = 1$, $h_2 = r$ then

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

Assuming $V = R(r)\,\Theta(\theta)$ we get

$$\frac{r}{R}\,[rR']' + \frac{1}{\Theta}\,\Theta'' = 0$$

Hence assuming the single valuedness of the solution

$$\frac{1}{\Theta}\,\Theta'' = -n^2, \quad \frac{r}{R}\,[rR']' = n^2$$

Solutions of these equations give

$$R(r) = A_n r^n + B_n\, r^{-n}, \qquad (9)$$

$$\Theta = C_n \cos n\theta + D_n \sin n\theta \qquad (10)$$

In addition to these we have $V = K \ln r$ solution. Adding all these we get the general solution (8). Since the solution is regular at $r = 0$ then B_n and K must be zero. Hence we have

$$V(r,\theta) = \sum_0^\infty r^n [\tilde{A}_n \cos n\theta + \tilde{B}_n \sin n\theta], \tag{11}$$

Since solution is given as $V = \sum_0^\infty c_n \cos n\theta$ at $r = a$ then $\tilde{A}_n a^n = c_n$ and $\tilde{B}_n = 0$. We have finally

$$V(r,\theta) = \sum_0^\infty c_n \, (r/a)^n \cos n\theta. \tag{12}$$