MATH 443: PARTIAL DIFFERENTIAL EQUATIONS

**First Midterm Exam:** 20 October 2005 Thursday 10.00-12.30 (SAZ18)

**QUESTIONS: You will solve four problems. The last three problems are compulsory. Choose any one out of the first three problems**

[25]**1**. Find the integral curves of the vector filed $V = (x(y-z), y(z-x), z(x-y))$.

**Answer**:

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

solving these equations we get the integral curves of $V$ as

$$\{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = c_1,\ xyz = c_2\}$$

where $c_1$ and $c_2$ are arbitrary constants.

[25]**2**. Find the general solution of

$$y\,\frac{\partial u}{\partial x} + x\,\frac{\partial u}{\partial y} = 1$$

in the domain $0 < y < x$. Find the solution which equals $x$ when $y = \frac{1}{2}x$.

**Answer**:

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{1}$$

Solving these equations we get

$$u = x^2 - y^2, \qquad v = e^{-z}\,(x+y)$$

Using the initial conditions we get

$$e^{-z} = \frac{3\varepsilon}{2}\,\frac{\sqrt{x^2-y^2}}{x+y}\,e^{2\varepsilon\frac{\sqrt{x^2-y^2}}{\sqrt{3}}}$$

where $\varepsilon = \pm 1$.

[25]**3**. Find the equation of the integral surface of the equation

$$(y + xz)p + (x + yz)q = z^2 - 1$$

which passes through the parabola $y = 1, x^2 = z$.

**Answer**:

$$\frac{dx}{y + xz} = \frac{dy}{x + yz} = \frac{dz}{z^2 - 1}$$

solving these equations we get

$$u = \frac{z - 1}{x + y} = c_1, \quad v = \frac{z + 1}{x - y} = c_2$$

Using the initial conditions we obtain

$$(c_1 + 1)^2 + 1 = c_1\, c_2$$

or

$$\left(\frac{z - 1 + x + y}{x + y}\right)^2 + 1 = \frac{z^2 - 1}{x^2 - y^2}$$

[25]**4**. Find the integral surface of the equation

$$z = 3p^2 + q^2$$

which passes through the line $z = 1, y = -x$.

**Answer**: $F = 3p^2 + q^2 - z = 0$. Usint the method of characteristics we get

$$x_\tau = 6p,\ y_\tau = 2q,\ z_\tau = 2z,\ p_\tau = p,\ q_\tau = q$$

Initial conditions are

$$x_0 = s,\ y_0 = -s,\ z_0 = 1,\ p_0 = \varepsilon,\ q_0 = \varepsilon$$

where $\varepsilon = \pm 1/2$. Solving these equations we get

$$x = 6\varepsilon\, e^{\tau}, \quad y = 2\varepsilon e^{\tau}, \quad z = e^{2\tau}, \quad p = q = \varepsilon e^{\tau}$$

Using $x, y, z$ we get

$$z = \left(1 + \frac{x+y}{8\varepsilon}\right)^2$$

[25]**5**. Is the following Pfaffian differential equation integrable? Prove your statement. If it is, find its solution.

$$2xzdx + zdy - dz = 0$$

**Answer**: $X = (2xz, z, -1)$ is an integrable vector filed ($X \cdot curl\, X = 0$). Solution of the pfaffian differential equation is

$$x^2 + y - \ln|z| = c$$

or

$$z = c_1 e^{y+x^2}$$

where $c$ and $c_1$ are arbitrary constants.

[25]**6**. Find a complete integral of the equation

$$\left(\frac{\partial z}{\partial x}\right)^4 + 4\,\frac{\partial z}{\partial x}\,\frac{\partial z}{\partial y} + 1 = 0$$

**Answer**:

$$F(p,q) = p^4 + 4pq + 1 = 0$$

since $F$ depends only on $p$ and $q$ we have

$$\cdots = \cdots = \frac{dp}{0} = \frac{dq}{0}$$

hence $p = a =$ constant and $q = b =$ constant, where

$$a^4 + 4ab + 1 = 0$$

which is an equation for $b$. Then using

$$dz = adx + bdy$$

Since $a$ and $b$ are constant the above pfaffian differential equation is easily integrated. We obtain

$$z = ax + by$$