MATH 584; TOPICS IN MATHEMATICAL PHYSICS II

Homework Set I

March 18, 2005

QUESTIONS:

Let $f \in T^{(0,0)}M$, $K \in T^{(1,0)}M$, $\gamma \in T^{(0,1)}M$, $\theta \in T^{(2,0)}M$, $J \in T^{(0,2)}M$, $\Phi \in T^{(1,1)}M$ where M is a smooth manifold.

1. Let $\phi : M \to \bar{M}$ be a C^∞ map. Let $u \in M$ and $\bar{u} \in \bar{M}$. Prove that

(i). $\bar{f}(\bar{u}) = f(u)$.

(ii). $\bar{K}(\bar{u}) = \phi'(u)\, K(u)$.

(iii). $\gamma(u) = \phi'^{\dagger}(u)\, \bar{\gamma}(\bar{u})$.

(iv). $\bar{\theta}(\bar{u}) = \phi'(u)\, \theta(u)\, \phi'^{\dagger}(u)$.

(v). $J(u) = \phi'^{\dagger}(u)\, \bar{J}(\bar{u})\, \phi'(u)$.

(vi). $\phi'(u)\, \Phi(u) = \bar{\Phi}(\bar{u})\, \phi'(u)$.

2. Prove that the Lie derivatives of these tensors with respect to the vector filed v are as follows:

(i). $L_v\, f = f' \circ v =< \nabla f, v >$.

(ii). $L_v\, K = K' \circ v - v' \circ K = [v, K]$.

iii. $L_v\, \gamma = \gamma' \circ v + v'^{\dagger} \circ \gamma$.

(iv). $L_v\, \Phi = \Phi' \circ v + \Phi \circ v' - v' \circ \Phi$.

(v). $L_v\, \theta = \theta' \circ v - \theta \circ v'^{\dagger} - v' \circ \theta$.

(vi). $L_v\, J = J' \circ v + J \circ v' + v'^{\dagger} \circ J$.

(vii). $L_v\, \Psi = \Psi' \circ v - \Psi \circ v'^{\dagger} + v'^{\dagger} \circ \Psi$. Here $\Psi = \Phi^{\dagger}$

3. Consider an evolution equation

$$u_t = K(u) \tag{1}$$

where $K \in T^{(1,0)}M$. We assume that u and its derivatives go to zero as $|x| \to \infty$

(i). A vector field $\sigma \in TM$ is a *strong symmetry* if and only if

$$\frac{\partial \sigma}{\partial t} + \sigma'[K] - K'[\sigma] = 0. \tag{2}$$

(ii). A co-vector $\gamma \in T^*M$ is *conserved gradient* (gradient of a functional I) if and only if

$$\frac{\partial \gamma}{\partial t} + \gamma'[K] + K'[\gamma] = 0. \tag{3}$$

and $\gamma' = \gamma'^{\dagger}$. Co-vectors satisfying only (3) are called *conserved covariants.*
(iii). An operator valued function $\Phi \in T^{(1,1)}M$ is *recursion operator of (1)* (or it is a *strong symmetry* of K) if and only if the following operator equation is valid.

$$\Phi'[K] - K'\,\Phi + \Phi\,K' = \tag{4}$$

(iv). An operator valued function $\Phi \in T^{(1,1)}M$ is called *hereditary* (or *Nijenhuis* or *regular*) if and only if

$$\Phi'\,[\Phi\,v]w - \Phi\,\Phi'[v]w = \Phi'\,[\Phi\,w]v - \Phi\,\Phi'[w]v \tag{5}$$

for all vector fields $v, w \in TM$.
(v). An operator valued function $\theta \in T^{(2,0)}M$ is called a *Hamiltonian operator* if and only if it is skew symmetric, $\theta = -\theta^{\dagger}$, and satisfies

$$< a, \theta'[\theta b]c > + \text{cyclic permutations} = 0 \tag{6}$$

for all $a, b, c \in T^*M$.
(vi). An equation (1) is a *Hamiltonian system* if and only if it can be written in the form

$$u_t = \theta\,\gamma \tag{7}$$

where θ is a Hamiltonian operator and γ is a closed co-vector ($\gamma = grad\ I$ which implies $\gamma' = \gamma'^{\dagger}$).
(vii). Associated with (7) we define the following Poisson bracket:

$$\{I, H\} = < grad\ I, \theta\, grad\ H > \tag{8}$$

Try to write the above equations in the form of Lie derivatives and give explanations.

4. Let $K(u) = u_{xxx} + 6uu_x$.
(i). Find K' and $K'^{\dagger}$.
(ii). Let $I = \int_R (u^3 - \frac{1}{2}\, u_x^2)dx$ Find the co-vector $\gamma = grad\ I$. Find γ' and show that $\gamma' = \gamma'^{\dagger}$. Show that γ is a *conserved gradient*.
(iii). Let $\sigma = 1 + 6tu_x$. Show that σ is a *symmetry* of the KdV equation (1).
(iv). Show that the KdV equation is a Hamiltonian system by writing (1) in the form (7) where

$$\theta = D, \quad \gamma = u_{xx} + 3u^2 \tag{9}$$

Show that θ given above is a Hamiltonian operator and γ given above is conserved gradient.
(v). KdV equation has also a second Hamilton operator.

$$\theta = D^3 + 4uD + 2u_x, \quad \gamma = u \tag{10}$$

where $u_t = (D^3 + 4uD + 2u_x)\, u$. Call the previous Hamilton operator as $\theta_0 = D$ and conserved gradient as γ_1. For the new pair call θ_1 and γ_0 respectively. Prove that θ_1 is a Hamilton operator and γ_0 is a conserved covariant.
(vi). Prove that the operator $\Phi = D^2 + 4u + 2u_x\, D^{-1}$ is a recursion operator of the KdV equation. Here $D^{-1}\, f = \int_{-\infty}^{x} f(\xi)d\xi$
(vii). Prove that Φ is hereditary.
(viii). Show that $\Phi = \theta_1\, {\theta_0}^{-1}$.
(ix). Using the recursion operator $\Phi \in T(1,1)M$ find the first four symmetries of the KdV equation

$$\sigma_{n+1} = \Phi\, \sigma_n, \quad n = 0, 1, 2, 3, 4 \quad (\sigma_0 = u_x)$$

(x). The adjoint $\Phi^{\dagger}$ of Φ maps conserved gradients to conserved gradients $(\Phi^{\dagger} : T^*M \to T^*M)$

$$\gamma_{n+1} = \Phi^{\dagger}\, \gamma_n, \quad n = 0, 1, 2, \cdots$$

Find first four conserved gradients of the KdV by taking $\gamma_0 = u$.
(xi). Prove the following: Hamilton operators map conserved gradients γ_n to symmetries σ_n ($\theta : T^*M \to TM$)

$$\sigma_n = \theta\, \gamma_n, \quad n = 0, 1, 2, \cdots$$