

SOLUTIONS

MATH 544: METHODS OF APPLIED MATHEMATICS II

Final Exam:

21 May 2019 Friday 13.40-15.30

SAZ02

QUESTIONS: Solve only three of the following problems.

[35] 1. Let an integral equation of Friedman type be given by

$$y(x) = f(x) + \lambda \int_0^{2\pi} \cos(x+s) y(s) ds$$

where $f(x)$ is a continuous function over $[0, 2\pi]$.

(a) Find the eigenvalues and the corresponding eigenfunctions of the associated homogeneous equation.

(b) Then solve the above integral equation for all values of λ .[35] 2. Find the solution of the following Dirichlet problem. Here the region D is defined as $D = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$

$$\begin{aligned} \nabla^2 u &= 0, \quad \text{in } D, \\ u(x, y, 0) &= f(x, y), \quad \text{for all } (x, y) \in \mathbb{R}^2 \end{aligned}$$

[35] 3. Analyze the following dynamical system. Find the critical points, find the phase portrait near each critical points and find the type of stability and plot them.

$$\begin{aligned} \frac{dx}{dt} &= y^2 - 1, \\ \frac{dy}{dt} &= x^3 - y \end{aligned}$$

[35] 4. (Homework Problem)

(a) Let $\Psi(\vec{r})$ be a solution of the Laplace equation, i.e., $\nabla^2 \Psi = 0$ in a region $D \subset \mathbb{R}^3$, then $\Phi = \frac{a}{r} \Psi(\frac{a^2}{r^2} \vec{r})$ solves also the Laplace equation, i.e., $\nabla^2 \Phi = 0$. Here a is a constant.(b) Let $\vec{R} = \vec{r} - \vec{r}' = (x - x')\vec{i} + (y - y')\vec{j} + (z - z')\vec{k}$ and

$$R = ||\vec{R}|| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

. Show that $\nabla^2 \frac{1}{R} = -4\pi \delta(\vec{r} - \vec{r}')$ in $D \subset \mathbb{R}^3$.

Solution of (1)

(2)

a) Homogeneous equation:

$$y(x) = \lambda \int_0^{2\pi} \cos(x+s) y(s) ds = \lambda A \cos x + \lambda B \sin x$$

$$A = \int_0^{2\pi} \cos s y(s) ds = \int_0^{2\pi} \cos s [\lambda A \cos s + \lambda B \sin s] ds$$

$$= \lambda A \pi$$

$$B = - \int_0^{2\pi} \sin s y(s) ds = - \int_0^{2\pi} \sin s [\lambda A \cos s + \lambda B \sin s] ds$$

$$= - \lambda B \pi$$

$$i) A \neq 0, \lambda_1 = 1/\pi, B = 0 \Rightarrow y_1(x) = \frac{A}{\pi} \cos x$$

$$ii) A = 0, \lambda_2 = -1/\pi, B \neq 0 \Rightarrow y_2(x) = -\frac{B}{\pi} \sin x$$

$x \in [0, 2\pi]$

$$b) y(x) = f(x) + \lambda \int_0^{2\pi} \cos(x+s) y(s) ds$$

$$= f(x) + \alpha \cos x + \beta \sin x$$

where

$$\alpha = \lambda \int_0^{2\pi} \cos s y(s) ds = \lambda \int_0^{2\pi} \cos s [f(s) + \alpha \cos s + \beta \sin s] ds$$

$$= \lambda \beta_1 + \lambda \alpha \pi \Rightarrow \alpha = \frac{\lambda \beta_1}{1 - \lambda \pi}$$

$$\beta = - \lambda \int_0^{2\pi} \sin s y(s) ds = - \lambda \int_0^{2\pi} \sin s [f(s) + \alpha \cos s + \beta \sin s] ds$$

$$= - \lambda \beta_2 - \lambda \beta \pi \Rightarrow \beta = \frac{- \lambda \beta_2}{1 + \lambda \pi}$$

Hence

i) if $\lambda \neq 1/\pi, -1/\pi$

$$\Rightarrow y(x) = f(x) + \frac{\lambda}{1-\lambda\pi} \beta_1 \cos x - \frac{\lambda}{1+\lambda\pi} \beta_2 \sin x$$

ii) if $\lambda = 1/\pi \Rightarrow \beta_1 = \int_0^{2\pi} \cos s f(s) ds = 0$ (constraint)

$$\Rightarrow y(x) = f(x) + \alpha \cos x - \beta_2 \sin x, \quad \alpha \text{ is arbitrary}$$

if $\beta_1 = \int_0^{2\pi} \cos s f(s) ds = 0$ there are infinitely many solutions.

if $\beta_1 = \int_0^{2\pi} \cos s f(s) ds \neq 0$ then there exists no solution.

i) if $\lambda = -1/\pi \Rightarrow \beta_2 = \int_0^{2\pi} \sin(s) f(s) ds = 0$ (constraint)

$$\Rightarrow y(x) = f(x) - \beta_1 \cos x + \beta \sin x, \quad \beta \text{ is arbitrary}$$

if $\beta_2 = \int_0^{2\pi} \sin(s) f(s) ds = 0$ then there are infinitely many solutions.

if $\beta_2 = \int_0^{2\pi} \sin s f(s) ds \neq 0$ then there exists no solution.

Solution f(2): This problem is was solved in class and it is also given in Lecture Notes 5 on Laplace Equation

Solution of (3)

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$$\frac{dx}{dt} = y^2 - 1, \quad \frac{dy}{dt} = x^3 - y$$

Critical points: $(1, 1)$ and $(-1, -1)$.

a) linearized system about the critical point $(1, 1)$

$$f(x, y) = y^2 - 1, \quad g(x, y) = x^3 - y.$$

$$f_x(x, y) = 0 \Rightarrow f_x(1, 1) = 0$$

$$f_y(x, y) = 2y \Rightarrow f_y(1, 1) = 2$$

$$g_x(x, y) = 3x^2 \Rightarrow g_x(1, 1) = 3$$

$$g_y(x, y) = -1 \Rightarrow g_y(1, 1) = -1$$

$$\frac{d\bar{x}}{dt} = 2\bar{y}, \quad \frac{d\bar{y}}{dt} = 3\bar{x} - \bar{y}$$

$$u = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

$$\frac{du}{dt} = A u$$

$$A = \begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda(1+\lambda) - 6 = 0$$

Eigen values of A are $\lambda_1 = -3, \lambda_2 = +2$

(Two distinct eigen values of opposite sign)

$$(A - \lambda_1 I) v_1 = \left[\begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix} - \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \right] \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \quad 3a + 2b = 0$$

$$v_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \frac{1}{\sqrt{13}}$$

$$(A - \lambda_2 I) v_2 = \left[\begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right] \begin{bmatrix} a \\ b \end{bmatrix}$$

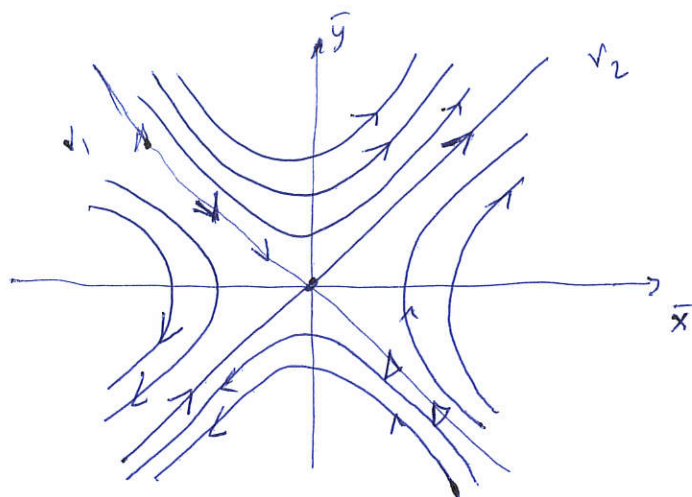
$$= \begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \quad a = b$$

$$v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow u = c_1 v_1 e^{-3t} + c_2 v_2 e^{2t}$$

(5)

where c_1 and c_2 are constant.



$(1,1)$ is a saddle point

if $c_2 = 0$ stable

if $c_2 \neq 0$ unstable

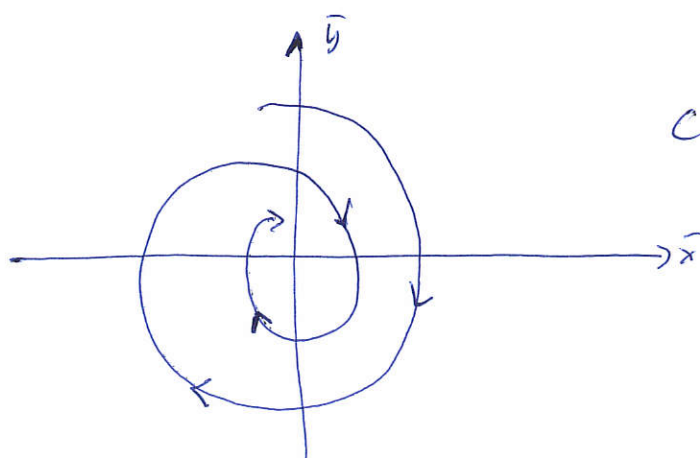
b) linearized system about the critical point $(-1, -1)$

$$\begin{array}{l|l} f_x(-1, -1) = 0 & \frac{du}{dt} = A u, \quad A = \begin{pmatrix} 0 & -2 \\ 3 & -1 \end{pmatrix} \\ f_y(-1, -1) = -2 & \\ g_x(-1, -1) = 3 & \\ g_y(-1, -1) = -1 & \end{array} \quad \left| \begin{array}{l} -\lambda - 2 \\ 3 - 1 - \lambda \end{array} \right| = \lambda(1 + \lambda) + 6 = 0$$

$$\lambda_{1,2} = \frac{-1}{2} \pm \alpha i, \quad \alpha = \frac{\sqrt{23}}{2}$$

$$\frac{d\bar{x}}{dt} = -2\bar{y}, \quad \frac{d\bar{y}}{dt} = 3\bar{x} - \bar{y}$$

$$u = e^{-t/2} (v_1 \cos \alpha t + v_2 \sin \alpha t)$$

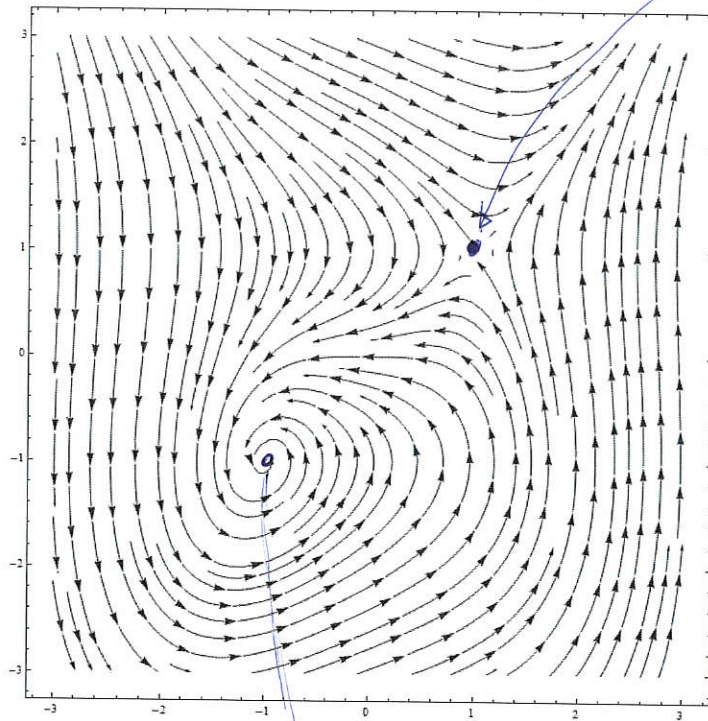


critical point $(-1, -1)$

is a spiral source
and stable.

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Phase portrait of the non-linear system:



$(1, 1)$

Critical point: $(1, 1)$

From the linearized system: saddle

From the phase portrait: saddle

Critical point: $(-1, -1)$

From the linearized system: spiral source

From the phase portrait: spiral source

$(-1, -1)$

(7)

4 Solution of (4)

$$(a) \quad \tilde{x} = \frac{a^2}{r^2} x, \quad \tilde{y} = \frac{a^2}{r^2} y, \quad \tilde{z} = \frac{a^2}{r^2} z$$

$$\partial_x = \frac{\partial \tilde{x}}{\partial x} \partial_{\tilde{x}} + \frac{\partial \tilde{y}}{\partial x} \partial_{\tilde{y}} + \frac{\partial \tilde{z}}{\partial x} \partial_{\tilde{z}}$$

$$\begin{aligned} \partial_x^2 &= \frac{\partial^2 \tilde{x}}{\partial x^2} \partial_{\tilde{x}} + \frac{\partial \tilde{x}}{\partial x} \left(\frac{\partial \tilde{x}}{\partial x} \partial_{\tilde{x}} + \frac{\partial \tilde{y}}{\partial x} \partial_{\tilde{y}} + \frac{\partial \tilde{z}}{\partial x} \partial_{\tilde{z}} \right) \partial_{\tilde{x}} \\ &\quad + \frac{\partial^2 \tilde{y}}{\partial x^2} \partial_{\tilde{y}} + \frac{\partial \tilde{y}}{\partial x} \left(\frac{\partial \tilde{x}}{\partial x} \partial_{\tilde{x}} + \frac{\partial \tilde{y}}{\partial x} \partial_{\tilde{y}} + \frac{\partial \tilde{z}}{\partial x} \partial_{\tilde{z}} \right) \partial_{\tilde{y}} \\ &\quad + \frac{\partial^2 \tilde{z}}{\partial x^2} \partial_{\tilde{z}} + \frac{\partial \tilde{z}}{\partial x} \left(\frac{\partial \tilde{x}}{\partial x} \partial_{\tilde{x}} + \frac{\partial \tilde{y}}{\partial x} \partial_{\tilde{y}} + \frac{\partial \tilde{z}}{\partial x} \partial_{\tilde{z}} \right) \partial_{\tilde{z}} \\ &= \frac{\partial^2 \tilde{x}}{\partial x^2} \partial_{\tilde{x}} + \frac{\partial^2 \tilde{y}}{\partial x^2} \partial_{\tilde{y}} + \frac{\partial^2 \tilde{z}}{\partial x^2} \partial_{\tilde{z}} \\ &\quad + \left(\frac{\partial \tilde{x}}{\partial x} \right)^2 \partial_{\tilde{x}}^2 + \left(\frac{\partial \tilde{y}}{\partial x} \right)^2 \partial_{\tilde{y}}^2 + \left(\frac{\partial \tilde{z}}{\partial x} \right)^2 \partial_{\tilde{z}}^2 \\ &\quad + \left(2 \frac{\partial \tilde{x}}{\partial x} \frac{\partial \tilde{y}}{\partial x} \partial_{\tilde{x}} \partial_{\tilde{y}} + 2 \frac{\partial \tilde{x}}{\partial x} \frac{\partial \tilde{z}}{\partial x} \partial_{\tilde{x}} \partial_{\tilde{z}} \right. \\ &\quad \left. + 2 \frac{\partial \tilde{y}}{\partial x} \frac{\partial \tilde{z}}{\partial x} \partial_{\tilde{y}} \partial_{\tilde{z}} \right) \end{aligned}$$

By using the symmetry one express ∂_y^2 and ∂_z^2 in a similar way in terms of $\partial_{\tilde{x}}, \partial_{\tilde{y}}, \partial_{\tilde{z}}$

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Then we find that

$$\begin{aligned}
 \nabla^2 &= \left(\frac{\partial^2 \tilde{x}}{\partial x^2} + \frac{\partial^2 \tilde{x}}{\partial y^2} + \frac{\partial^2 \tilde{x}}{\partial z^2} \right) \partial_{\tilde{x}} + \left(\frac{\partial^2 \tilde{y}}{\partial x^2} + \frac{\partial^2 \tilde{y}}{\partial y^2} + \frac{\partial^2 \tilde{y}}{\partial z^2} \right) \partial_{\tilde{y}} \\
 &+ \left(\frac{\partial^2 \tilde{z}}{\partial x^2} + \frac{\partial^2 \tilde{z}}{\partial y^2} + \frac{\partial^2 \tilde{z}}{\partial z^2} \right) \partial_{\tilde{z}} \\
 &+ \left[\left(\frac{\partial \tilde{x}}{\partial x} \right)^2 + \left(\frac{\partial \tilde{x}}{\partial y} \right)^2 + \left(\frac{\partial \tilde{x}}{\partial z} \right)^2 \right] (\partial_{\tilde{x}}^2 + \partial_{\tilde{y}}^2 + \partial_{\tilde{z}}^2) \\
 &+ \left[2 \frac{\partial \tilde{x}}{\partial x} \frac{\partial \tilde{y}}{\partial y} \partial_{\tilde{x}} \partial_{\tilde{y}} + 2 \frac{\partial \tilde{x}}{\partial x} \frac{\partial \tilde{z}}{\partial z} \partial_{\tilde{x}} \partial_{\tilde{z}} + 2 \frac{\partial \tilde{y}}{\partial y} \frac{\partial \tilde{z}}{\partial z} \partial_{\tilde{y}} \partial_{\tilde{z}} \right] \\
 &= \nabla^2 \tilde{x} \partial_{\tilde{x}} + \nabla^2 \tilde{y} \partial_{\tilde{y}} + \nabla^2 \tilde{z} \partial_{\tilde{z}} \\
 &+ \vec{\nabla} \tilde{x} \cdot \vec{\nabla} \tilde{x} (\vec{\nabla}^2) + 2 (\vec{\nabla} \tilde{x} \cdot \vec{\nabla} \tilde{y}) \partial_{\tilde{x}} \partial_{\tilde{y}} \\
 &+ 2 (\vec{\nabla} \tilde{x} \cdot \vec{\nabla} \tilde{z}) \partial_{\tilde{x}} \partial_{\tilde{z}} + 2 \vec{\nabla} \tilde{y} \cdot \vec{\nabla} \tilde{z} \partial_{\tilde{y}} \partial_{\tilde{z}}
 \end{aligned}$$

$$\vec{\nabla} \tilde{x} \cdot \vec{\nabla} \tilde{x} = \vec{\nabla} \tilde{y} \cdot \vec{\nabla} \tilde{y} = \vec{\nabla} \tilde{z} \cdot \vec{\nabla} \tilde{z}$$

$$\left. \begin{aligned}
 \vec{\nabla} \tilde{x} &= \frac{a^2}{r^2} \hat{e} - \frac{2a^2}{r^4} x \vec{r} \\
 \vec{\nabla} \tilde{y} &= \frac{a^2}{r^2} \hat{j} - \frac{2a^2}{r^4} y \vec{r} \\
 \vec{\nabla} \tilde{z} &= \frac{a^2}{r^2} \hat{k} - \frac{2a^2}{r^4} z \vec{r}
 \end{aligned} \right\} \begin{aligned}
 \nabla^2 \tilde{x} &= -\frac{2}{r^2} \tilde{x} \\
 \nabla^2 \tilde{y} &= -\frac{2}{r^2} \tilde{y} \\
 \nabla^2 \tilde{z} &= -\frac{2}{r^2} \tilde{z} \\
 \vec{\nabla} \tilde{x} \cdot \vec{\nabla} \tilde{y} &= \vec{\nabla} \tilde{x} \cdot \vec{\nabla} \tilde{z} = \vec{\nabla} \tilde{y} \cdot \vec{\nabla} \tilde{z} = 0
 \end{aligned}$$



$$\Rightarrow \nabla^2 = -\frac{2}{r^2} (\tilde{x} \partial_{\tilde{x}} + \tilde{y} \partial_{\tilde{y}} + \tilde{z} \partial_{\tilde{z}}) + \frac{a^2}{r^4} \tilde{\nabla}^2$$

on the other hand

$$\vec{\nabla} = \frac{a^2}{r^2} \tilde{\nabla} - \frac{2}{r^2} \vec{r} (\tilde{r} \cdot \tilde{\nabla})$$

$$\Rightarrow \vec{r} \cdot \vec{\nabla} = -\tilde{r} \cdot \tilde{\nabla}$$

$$\Rightarrow \nabla^2 = \frac{2}{r^2} \vec{r} \cdot \vec{\nabla} + \frac{a^2}{r^2} \tilde{\nabla}^2$$

$$\Phi = \frac{a}{r} \Psi(\tilde{r})$$

$$\nabla^2 \Phi = -\frac{2a\vec{r}}{r^3} \cdot \vec{\nabla} \Psi + \frac{a}{r} \nabla^2 \Psi$$

$$= -\frac{2a\vec{r}}{r^3} \cdot \vec{\nabla} \Psi + \frac{a}{r} \left[\frac{2}{r^2} \vec{r} \cdot \vec{\nabla} \Psi + \frac{a^2}{r^2} \tilde{\nabla}^2 \Psi \right]$$

$$= \frac{a^3}{r^3} \tilde{\nabla}^2 \Psi$$

$$\text{if } \tilde{\nabla}^2 \Psi = 0 \Rightarrow \nabla^2 \Phi = 0 \quad \text{this}$$

completes the proof

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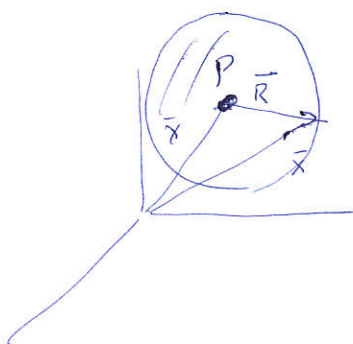
$$(b) \quad \vec{\nabla} \frac{1}{R} = - \frac{1}{R^2} \vec{\nabla} R = - \frac{(x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}}{R^3}$$

$$\vec{\nabla} R = \frac{\vec{R}}{R}, \quad \vec{R} = (x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}$$

$$\vec{\nabla} \frac{1}{R} = - \frac{\vec{R}}{R^3}, \quad \nabla^2 \frac{1}{R} = - \frac{\vec{\nabla} \cdot \vec{R}}{R^3} + 3 \frac{\vec{R} \cdot \vec{R}}{R^5}$$

$$= - \frac{3}{R^3} + \frac{3}{R^3} = 0 \quad R \neq 0$$

Integrating $\nabla^2 \frac{1}{R}$ over a sphere about the point P



$$\iiint_V \nabla^2 \frac{1}{R} dV = \iiint_V \vec{\nabla} \cdot \left(\vec{\nabla} \frac{1}{R} \right) dV$$

$$= \iint_S \vec{\nabla} \frac{1}{R} \cdot \vec{N} dS, \quad \vec{N} = \frac{\vec{R}}{R}, \quad dS = R^2 d\Omega$$

$$= \iint_S - \frac{\vec{R}}{R^3} \cdot \frac{\vec{R}}{R} \cdot R^2 d\Omega$$

$$= - \iint_S d\Omega = - 4\pi$$

$$\Rightarrow \quad \nabla^2 \frac{1}{R} = - 4\pi \delta(\vec{r} - \vec{r}')$$