

MATH 544: METHODS OF APPLIED MATHEMATICS II

Second Midterm Exam:

26 April 2019 Friday 13.40-15.30

SAZ02

QUESTIONS: SOLUTIONS

[35]1. Let $D \in \mathbb{R}^2$ be an infinite rectangular region $0 < x < a$, $0 < y < \infty$. Consider the following Dirichlet's problem

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in D \quad (1)$$

$$u(0, y) = 0, \quad u(a, y) = 0, \quad 0 \leq y < \infty \quad (2)$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq a \quad (3)$$

and $u(x, y) \rightarrow 0$ as $y \rightarrow \infty$ for all $x \in [0, a]$

a. Find the formal solution of the problem.

b. Find restrictions of $f(x)$ so that the boundary value problem is well-posed

[35]2. Assuming Theorem 1 prove Theorem 2.

Theorem 1. Let $u \in C^2(D) \cap C^1(\bar{D})$, where D is a bounded domain.

(i) If $\nabla^2 u \leq 0$ in D and $u \geq 0$ on B , then $u \geq 0$ in D ,

(ii) If $\nabla^2 u \geq 0$ in D and $u \leq 0$ on B , then $u \leq 0$ in D .

Theorem 2. Let $u \in C^2(D) \cap C^1(\bar{D})$, where D is a bounded domain.

(i) If $\nabla^2 u \geq 0$ in D , then $u(x) \leq M = \max_{y \in B} u(y)$ for all $x \in D$.

(ii) If $\nabla^2 u \leq 0$ in D , then $u(x) \geq m = \min_{y \in B} u(y)$ for all $x \in D$.

(iii) If $\nabla^2 u = 0$ in D , then $m \leq u(x) \leq M$ for all $x \in D$.

(iv) If $\nabla^2 u = h$ in D and $u = f$ on B (boundary), then there is at most one solution of this boundary value problem.

[35]3. Consider the following initial and boundary value problem

$$u_t = au_{xx}, \quad (a > 0), \quad t > 0, \quad x \in (0, L), \quad (4)$$

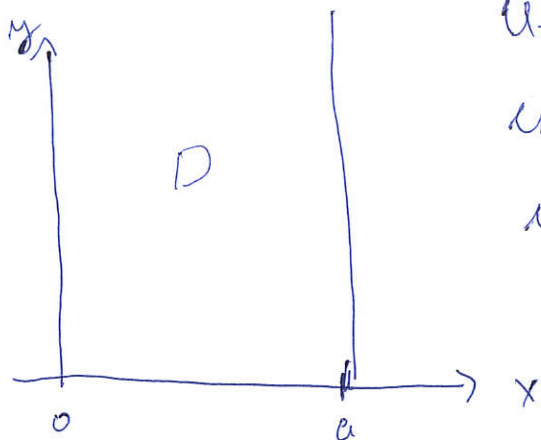
$$u_x(0, t) = g(t), \quad u(L, t) = h(t), \quad t \geq 0, \quad (5)$$

$$u(x, 0) = f(x), \quad x \in [0, L] \quad (6)$$

Assuming the existence, prove the uniqueness of the solutions of this initial and boundary value problem by the use energy functional

Solutions

1)



$$u_{xx} + u_{yy} = 0 \quad \text{in } D$$

$$u(0, y) = u(a, y) = 0, \quad y > 0$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq a$$

$$\lim_{y \rightarrow \infty} u(x, y) = 0$$

a) Formal solution. Since the DE and BCs are linear and homogeneous we can use the method of separation of variables and at the end we find that

$$c) \quad u(x, y) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n y} \sin(\lambda_n x), \quad \lambda_n = \frac{n\pi}{a} \quad (n=1, 2, \dots)$$

and

$$(2) \quad f(x) = \sum_{n=1}^{\infty} c_n \sin(\lambda_n x)$$

with

$$c_n = \frac{\pi}{a} \int_0^a f(x) \sin(\lambda_n x) dx, \quad n=1, 2, \dots$$

b) Justification of the solution. We have two infinite series in (1) and in (2)

(3)

for the first one it is enough to have f bounded in $[0, a]$, i.e.

$|f| < M$ for all $x \in [0, a]$ Then

$$\left| \sum_{n=1}^{\infty} c_n e^{-\lambda_n y} \sin(\lambda_n x) \right| \leq \sum |c_n| e^{-\lambda_n y}$$

but

$$|c_n| \leq \frac{\pi}{a} M \cdot a = \pi M \Rightarrow$$

$$\left| \sum_{n=1}^{\infty} c_n e^{-\lambda_n y} \sin(\lambda_n x) \right| \leq \pi M \sum_{n=1}^{\infty} e^{-\lambda_n y} = \frac{\pi M e^{-\pi y/a}}{1 - e^{-\pi y/a}} < \infty$$

for all $y > 0$

For the second one we are familiar with the form of the equation. We had this series both in parabolic and hyperbolic initial and boundary value problems. Infinite series in (2) uniformly converge if $f(0) = f(a) = 0$ and $f''(x)$ is integrable in $(0, a)$

2) This problem was solved in lecture 5, on Laplace equation

3) This problem was solved in lech
4 on heat equation.