**ASSIGNED EXERCISES OF MATH544:**

**Stability and Bifurcations: set 10**

*May, 2001*

## STABILITY AND BIFURCATIONS: One dimensional case

Classification of points of $f(\mu, u) = 0$

1. *Regular point* $P_0$ is the point where the implicit function theorem works: $f_\mu(P_0) = f_u(P_0) = 0$. Hence there exists a unique curve passing through this point.
2. *A regular turning point* $P_0$ where the $\alpha$ indicator or the slope of the tangent at $P_0$ changes its sign. This means that the stability is different on both sides of this point
3. *Singular point* $P_0$ is a point at which $f_\mu(P_0) = f_u(P_0) = 0$.
4. *A double point* $P_0$ is a singular point through which pass two and only two branches of $f = 0$ possessing distinct tangents. We shall assume that $f_{uu}(P_0)$, $f_{\mu u}(P_0)$, and $f_{\mu\mu}(P_0)$ are not vanishing simultaneously at the double point.
5. *A singular turning (double) point* $P_0$ is a double point at which $\frac{d\mu}{du}$ changes sign on one branch.
6. *A cusp point* $P_0$ is a point of second order contact between two branches of the curve. The two branches have the same tangent at a cusp point.
7. *A conjugate point* $P_0$ is an isolated point.
8. *higher-order singular point* $P_0$ is a singular point at which all second partial derivatives vanish simultaneously.

Summary of what we did so far.

**Theorem 1**. *Let $u_0$ be an equilibrium solution of*

$$\frac{du}{dt} = f(\mu, u), \quad t > 0, \tag{1}$$

*where $\mu$ is a parameter, and assume that*

$$f(\mu, u_0 + \bar{u}) = f_u(\mu, u_0)\bar{u} + R(u_0, \bar{u})$$

*where the remainder satisfies $|R(u_0, \bar{u})| \leq K|\bar{u}|^2$ for $\bar{u}$ sufficiently small where $K$ is a positive constant. Then $u_0$ is asymptotically stable if $\alpha = f_u(\mu, u_0) < 0$ and unstable if $\alpha > 0$.*

Hence the indicator $\alpha$ is important in the study of stability. At the critical points defined above the determination of the $\alpha$ indicator depend on the slopes of the tangents about these points on the branches of equilibrium solutions. An important theorem (the implicit function theorem) for such a purpose is given by

**Theorem 2**. *Let $f(\mu, u)$ be a continuously differentiable function in a region $U$ of the $\mu u$ plane containing the point $P_0 = (\mu_0, u_0)$. If $f(P_0) = 0$ and $f_u(P_0) \neq 0$, then there is a rectangle $R$ , defined by $R = \{(\mu, u)| \ |u - u_0| < a, \ |\mu - \mu_0| < b\}$ contained in $U$ such that*
*(i) The equation $f(\mu, u) = 0$ has a unique solution $u = u(\mu)$ on $R$. (ii) The function $u(\mu)$ is continuously differentiable on the interval $|\mu - \mu_0| < b$ and its derivative is given by*

$$\frac{du}{d\mu} = -\frac{f_\mu(\mu, u(\mu))}{f_u(\mu, u(\mu))}$$

On the regular points of the bifurcating curves we can use above theorems and find the $\alpha$ indicator and investigate the stability of the model. Let us now assume that a point $P_0$ on the branches is a singular point. To study the stability about such points we have the following theorem. First we need the slopes of the curves about these points. For this purpose we have the following results.

**Theorem 3**. *Let $P_0$ be a double point of $f(\mu, u) = 0$. Then either* *(i)* $f_{\mu\mu}(P_0) \neq 0$ *and two tangents are given by*

$$\frac{d\mu}{du} = -\frac{f_{\mu u}}{f_{\mu\mu}} \pm \frac{1}{|f_{\mu\mu}|}\sqrt{\Delta} \tag{2}$$

*where*

$$\Delta = {f_{\mu u}}^2 - f_{\mu\mu}\, f_{uu}$$

**Remark**: If $\Delta > 0$ then the curves intersecting at $P_0$ have different tangents. If $\Delta = 0$ then these curves have the same tangent line at the point $P_0$. If $\Delta < 0$ then there exists no tangent line passing through $P_0$ such points are called *isolated points.*

*(ii) If* $f_{\mu\mu} = 0$ *and the two tangents are given by*

$$\frac{du}{d\mu}(P_0) = 0, \quad \textit{and} \quad \frac{d\mu}{du} = -\frac{f_{uu}(P_0)}{2\, f_{\mu u}(P_0)} \tag{3}$$

For stability analysis we need the $\alpha$ indicator about the point $P_0$. For this purpose, in the neighborhood of a regular turning point we have

**Theorem 4**. *Let* $P_0$ *be a regular turning point of* $f = 0$. *Then equilibrium solutions on one side are stable and on the other side are not stable.*

**Remark**: The $\alpha$ indicator, $\alpha = f_u(\mu(u), u) = -\frac{d\mu}{du}(u)\, f_\mu(\mu(u), u)$. Since the slope changes its sign then $\alpha$ changes its sign on different sides of of the point $P_0$.

For double points to find the $\alpha$ indicator we consider two distinct cases.

**case 1**. The two curves $\mu = \mu^+(u)$ and $\mu = \mu^-(u)$ passing through (intersecting at) $P_0$ have tangents given by (2). Let the corresponding stability indicators are $\alpha^+$ and $\alpha^-$. They are given by

**Theorem 5**. *Let* $P_0$ *be a double point with* $f_{\mu\mu}(P_0) \neq 0$. *Then*

$$\alpha^+ = -\frac{d\mu^+}{du}(u)\,[\frac{f_{\mu\mu}(P_0)}{|f_{\mu\mu}(P_0)|}\sqrt{\Delta}\,(u-u_0)+o(|u-u_0|)], \quad (4)$$

$$\alpha^- = \frac{d\mu^-}{du}(u)\,[\frac{f_{\mu\mu}(P_0)}{|f_{\mu\mu}(P_0)|}\sqrt{\Delta}\,(u-u_0)+o(|u-u_0|)] \quad (5)$$

**case 2**. One of the curves is $u = u_1(\mu)$ with $\frac{du_1}{d\mu}(P_0) = 0$ and other is $\mu = \mu_2(u)$ with $\frac{d\mu_2}{du} = -\frac{f_{uu}(P_0)}{2\,f_{\mu u}(P_0)}$. Their corresponding $\alpha$ indicators are given by

**Theorem 6**. *Let $P_0$ be a double point with $f_{\mu\mu}(P_0) = 0$. Then*

$$\alpha^1(\mu) = \frac{f_{\mu u}(P_0)}{|f_{\mu u}(P_0)|}\sqrt{\Delta}\,(u-u_0)+o(|u-u_0|), \quad (6)$$

$$\alpha^2(\mu) = -\frac{f_{\mu u}(P_0)}{|f_{\mu u}(P_0)|}\sqrt{\Delta}\,(u-u_0)+o(|u-u_0|). \quad (7)$$

*where $\alpha^1(\mu) = f_u(\mu, u_1(\mu)$ and $\alpha^2(u) = f_u(\mu_2(u), u)$ are the stability indicators of the curves $u_1(\mu)$ and $\mu_2(u)$*

In addition to the **solved problems and exercises given in Logan page 434-436** solve also the following problems

**QUESTIONS**

**1**. Check the stability of all equilibrium solutions of the equation (2) with

$$f = u(9-\mu u)(\mu + 2u - u^2)[(\mu-10)^2 + (u-3)^2 - 1]$$

and draw a branching (bifurcation) diagram using the dashed lines for unstable equilibria.

**2**. Consider the integral equation

$$\lambda y(t) = \frac{2}{\pi}\int_0^{\pi}(3\sin\tau\,\sin t + 2\sin 2\tau\sin 2t)\,(y(\tau)+y^3(\tau))d\tau$$

Solutions $y(t,\lambda)$ of this integral equation depend on the independent variable $t$ and on the parameter $\lambda$.

(a) Show that every solution is of the form

$$y = A(\lambda)\sin t + B(\lambda)\sin 2t$$

(b) Are there nontrivial solutions for $\lambda = 0$?
(c) Calculate all solutions for $\lambda > 0$ (for $\lambda = 10$ there are nine different solutions).
(d) For each solution, calculate

$$[y] = \int_0^{\pi} y^2(t,\lambda)dt$$

and sketch a branching diagram

**3**. *Cusp point Bifurcation* At the cusp points it is not possible in general to obtain $u$ as a function of $\mu$ or $\mu$ as a function of $u$. We then introduce a new parameter $\eta$ in order to obtain a parametric representations $u(\eta)$ and $\mu(\eta)$. Let $P_0 = (0,0)$ be a singular point

$$f(P_0) = 0, \quad f_\mu(P_0) = f_u(P_0) = 0 \quad and \quad \Delta = 0$$

We assume that the second partial derivatives of $f$ are not vanishing simultaneously at the point $P_0$. Here we assume that $f_{uu}(P_0) \neq 0$. Show that in the neighborhood of the point $P_0$ we have

$$\mu = \eta^2 \quad or \quad \mu = -\eta^2, \tag{8}$$
$$u = \frac{1}{2}u_{\eta\eta}\,\eta^2 + \frac{1}{6}\,u_{\eta\eta\eta}\,\eta^3 O(\eta^4), \tag{9}$$

**Hint**: The above representations can be obtained by differentiating $f(\mu(\eta), u(\eta)) = 0$ with respect $\eta$ in the neighborhood of the point $P_0$ and letting $u(0) = \eta(0) = 0$. For instance differentiating twice we obtain

$$f_{uu}\,(u_\eta)^2 + 2f_{\mu u}\,u_\eta\,\mu_\eta + f_{\mu\mu}\,(\mu_\eta)^2 = 0$$

and so on.

**STABILITY AND BIFURCATIONS: Two dimensional case**

Let $x$ and $y$ be functions of $t \in I$ satisfying the system of ODEs

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \quad t \in I \tag{10}$$

where $P$ and $Q$ are functions of $x, y$ having continuous partial derivatives of all orders in some domain of the $xy$ plane which is called the *phase plane*. Let us consider the following initial value problem

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad t \in I \tag{11}$$

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad t_0 \in I \tag{12}$$

Form the theory of ODEs we have

**Lemma 6**. *If $P$ and $Q$ have continuous partial derivatives in some domain $D$ the the initial value problem stated above has unique solutions*

Some definitions

1. *Phase Plane*. The xy plane.
2. *Phase variables or state variables* : x and y
3. *Trajectory or path*. A Curve $C$ defined by the solution $x = x(t), y = y(t)$ where $t \in I$ of (10)
4. *Phase portrait*. The totality of all paths and critical points of (10) graphed in the phase plane.
5. *Critical Points*. All solution of $P(x_0, y_0) = Q(x_0, y_0) = 0$. Critical points are the equilibrium or steady state solutions of (10).

**A corollary of Lemma 6**. *There exists only one path passing through each point of the phase plane.*

This means that the paths corresponding to different initial values do not intersect each other. Critical points are also solutions of (10) which are called the equilibrium solutions. According to the above Corollary of Lemma 6 none of the trajectories pass through such critical points. Critical points are called isolated it there exists a neighborhood which contains only this critical point. Without proof we state the following results

**(a)**. If a path approaches a critical point then $t \to \pm\infty$
**(b)**. As $t \to \pm\infty$ a path approaches *a critical point*, *moves on a closed path*, *approaches a closed path*, or *leaves every bounded set*

**Stability Analysis**. In two dimensional case there stability definitions are given as follows

**Definition**( of stability). Let $u = (x(t), y(t))$ define the state vector for $t \in I$. Let $P_0 = (x_0, y_0)$ be a critical point and $u(t_0) = (x(t_0), y(t_0), \quad t_0 \in I$ be the initial value of the state vector. Then for given positive number $\varepsilon > 0$ there exists a positive number $\delta_\varepsilon$ such that $||u(t) - u_0|| \leq 0$ for all $t \in I$ whenever $||u(0) - u_0|| \leq \delta_\varepsilon$

This implies that any path starting sufficiently closer to the critical point remains closer to the point.

**Definition**(of asymptotical stability). A critical point is asymptotically stable if it is stable and if all paths starting sufficiently closer to this point approach this point asymptotically (as $t \to \infty$). It means $\lim_{t\to\infty} ||u(t) - u_0|| = 0$.

This means that a critical point is *asymptotically stable* if all paths starting sufficiently closer to it asymptotically reaches that point. There are four types of Critical point, *center*, *node*, *saddle*, and *spiral*.

**Remark**. The critical points of center type are stable but not asymptotically stable

**Linearized Stability**. Let $P_0 = (x_0, y_0)$ be a critical point. Stability of this critical point is investigated by using linear perturbations about this equilibrium solution, $x = x_0 + \bar{x}, y = y_0 + \bar{y}$. Let us choose $P_0 = (0, 0)$ for simplicity then by using the differentiability properties of $P$ and $Q$ at all orders we get

$$\dot{x} = ax + by, \quad \dot{y} = cx + dy, \tag{13}$$

where $a, b, c, d$ are arbitrary constants which are given by

$$a = P_x(0,0), \quad b = P_y(0,0), \quad c = Q_x(0,0), \quad d = Q_y(0,0)$$

This system of equations (13) may also be written by

$$\dot{u} = Au, \tag{14}$$

where $A$ is the coefficient matrix in (13) and $u$ is the column state vector. Let $\lambda_1$ and $\lambda_2$ be eigenvectors of the matrix $A$ and $v_1$ and $v_2$ be the corresponding eigenvectors. For the solution of (14) we two distinct cases

**A**. Two eigenvalues are different ($\lambda_1 \neq \lambda_2$). then the solution is given by

$$u(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} \tag{15}$$

where $c_1$ and $c_2$ are arbitrary constants to be determined by the initial conditions.
**B**. Eigenvalues are the same ($\lambda_1 = \lambda_2$). We have two sub cases. ($B_1$): $A = \lambda_1 I$, then

$$u(t) = [c_1 v_1 + c_2 v_2] e^{\lambda_1 t}. \tag{16}$$

Here $v_1$ and $v_2$ correspond to two different eigenvectors of corresponding to the same eigenvalue $\lambda_1$. ($B_2$): $(A - \lambda_1 I)^2 = 0$, then

$$u(t) = [vt + w] e^{\lambda_1 t}, \tag{17}$$

where $v = Aw - \lambda_1 w$.

The classification of the critical point $(0,0)$ is achieved by the eigenvalues of the matrix $A$

(1). Eigenvalues have the same sign. Critical point is a **node**. Node is stable if the eigenvalues are negative and unstable if the eigenvalues are positive. Solution is given (15).
(2). Eigenvalues have opposite sign. Critical point is a **saddle point**. Solution is given (15).
(3). Eigenvalues are equal. It is a **stable node** if the eigenvalue is negative. It is an **unstable node** otherwise. The solutions is given in (16) and (17).
(4). Eigenvalues are complex. Let $\lambda_1 = \alpha + i\beta$ then $\lambda_2 = \alpha - i\beta$. Let the eigenvector $v_1$ corresponding to eigenvalue be $v_1 = w + iv$. We have two types. Path about the critical point is a **spiral** if $\alpha \neq 0$ and **center** type of critical points if $\alpha = 0$ where the path is a closed curve (periodic solutions). The solutions are found from (15) by replacing the eigenvalues and taking the real part. We obtain $u(t) = c_1 u_1 + c_2 u_2$ where

$$u_1 = e^{\alpha t}(w \cos \beta t - v \sin \beta t)$$

$$u_2 = e^{\alpha t}(w \cos \beta t + v \sin \beta t)$$

As a summary we have the following theorem

**Theorem 7**. *The critical point of the system (13) is stable if and only if , the eigenvalues of $A$ have non positive real parts, it is asymptotically stable if and only if the eigenvalues have negative real parts.*

So far we investigated the stability of the linear systems (13). For the non-linear systems we let

$$\begin{aligned} \dot{x} &= ax + by + f(x,y), & (18) \\ \dot{y} &= cx + dy + g(x,y) & (19) \end{aligned}$$

where $a, b, c, d$ are constant with (i). $ad - bc \neq 0$ and (ii).

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0, \quad \lim_{(x,y)\to(0,0)} \frac{g(x,y)}{\sqrt{x^2+y^2}} = 0$$

It is obvious that $(0,0)$ is the critical point. Then we have the following theorem for the general case with the above conditions

**Theorem 8**. (Poincare' and Liapunov) *Let $(0,0)$ be a critical point of the linear system (13) and the nonlinear system (19) with the conditions (i) and (ii) stated above. The critical point $(0,0)$ of the nonlinear system is the same type as of the linear case if*

*a) The eigenvalues of $A$ are real, distinct , and have the same sign (node).*
*b) The eigenvalues of $A$ are real and have opposite signs (saddle).*
*c) The eigenvalues of $A$ are complex, but not purely imaginary (spiral).*

*In addition if $(0,0)$ is asymptotically stable for the linear system (13), then it is asymptotically stable for the nonlinear system (19)*

To understand whether the path is a closed curve we have the theorem of Bendixon and Dulac

**Theorem 9**. (Bendixon & Dulac) *If $P_x + Q_y$ has a fixed sign in a region of phase plane then (19) can not have a closed path in that region*

Assuming there exists a closed a path $C$ and consider the form $Pdy - Qdy$ and integrating over $C$ one obtains a contardiction. We have another useful theorems due to Poincare' and Bendixon.

**Theorem 10**. (Poincare') *A closed path of the system (19) surrounds at least one critical point of the system.*

**Theorem 11**. (Poncare' and Bendixon) *Let $R$ be a closed and bounded region in the phase plane containing no critical points of (19). If $C$ is a path of (19) that lies in $R$ for some time $t_0$ and remains in $R$ for all $t > t_0$, then $C$ is either a closed path or spirals toward a closed path as $t \to \infty$*

**Exercises. In addition to the assigned and solved exercises in Logan solve also the following problems.**

**1**. (Duffing equation). Let $u$ be a function of $t \in I$ satisfying the second order equation

$$\ddot{u} + \dot{u} - u + u^3 = 0$$

which is called the Duffing equation. The stability analysis of this equation is easily studied by letting $x = u, \quad y = \dot{u}$. We then obtain the following nonlinear system

$$\begin{aligned} \dot{x} &= P(x, y) = y, & (20) \\ \dot{y} &= Q(x, y) = x - x^3 - y & (21) \end{aligned}$$

We have the following critical points $(0, 0)$ , $(1, 0)$ and $(-1, 0)$. For the first critical point we have $a = 0, b = 1, c = 1, d = -1$ then the eigenvalues are $\lambda_{1,2} = \frac{1}{2}(-1 \pm \sqrt{5})$ (calculate the eigenvectors). This critical point is a **saddle** point. For the other critical points $(\pm 1, 0)$ we have $a = 0, b = 1, c = -2, d = -1$. Hence $\lambda_{1,2} = \frac{1}{2}(-1 \pm i\sqrt{7})$. Hence $\alpha < 0$, then these critical points are *stable foci* (**spiraling paths**)

**2**. Discuss the stability of the equilibrium solutions of the Van der Pol equation

$$\ddot{u} - \lambda(1 - u^2) + \dot{u} + u = 0$$

for all possible values of $\lambda$.

**3**. Consider the system of ODEs

$$\begin{aligned} \dot{x} &= -y + x\frac{1 - x^2 - y^2}{\sqrt{x^2 + y^2}}, & (22) \\ \dot{y} &= x + y\frac{1 - x^2 - y^2}{\sqrt{x^2 + y^2}} & (23) \end{aligned}$$

Show that the unit circle in the pats plane is a stable cycle. **Hint**. Use polar coordinates.

**4**. Consider the ordinary differential system

$$\dot{x} = -y + x(x^2 + y^2 - 1), \tag{24}$$
$$\dot{y} = x + y(x^2 + y^2 - 1). \tag{25}$$

Show that the periodic path is unstable.