ASSIGNED EXERCISES OF MATH544:
Integral Equations set 9
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THE INTEGRAL EQUATIONS, GREEN'S FUNCTION AND THE NEUMANN SERIES

1. Ronald B Guenther and John W Lee, 2. Hildebrandt
See also Ch.4 of the lecture notes.

1. An integral equation of the form

$$y(x) = f(x) + \lambda \int_a^x g(x,s)\, y(s) ds$$

is called a *Volterra equation.*

a. Show that , if $y(x)$ satisfies the differential equation

$$y'' + xy = 1$$

and the conditions $y(0) = y'(0) = 0$, then y satisfies the Volterra equation

$$y(x) = \int_0^x (s-x)\, s\, y(s) ds + \frac{x^2}{2}$$

b. Prove that the converse of the preceding statement is also true.

2. Transform the problem

$$y'' + xy = 1, \quad y(0) = y(1) = 0$$

to the integral equation

$$y(x) = -\frac{1}{2}s(1-x) + \int_0^1 G(x,s) s\, y(s) ds$$

where $G(x,s) = x(1-s)$ when $x < s$ and $G(x,s) = s(1-x)$ when $x > s$.

3. Consider the integral equation

$$y(x) = f(x) + \lambda \int_a^b e^{x-s}\, y(s)ds.$$

(i) Find the iterated kernels $g_n(x,s)$ defined in the lecture (ii) the resolvent kernel $\gamma(x,s)$ (iii) Find the solution (iv) find the cases (restrictions on λ where there is no solution and (v) the solution is not unique (vi) find the solution when $f(x) = x^2$.

4. **a**. Consider the integral equation

$$y(x) = f(x) + \lambda \int_a^b q(x)\, q(s)\, y(s)$$

where $q(x)$ with $x \in [a,b]$ is a continuous function. Answer all questions (i)-(v) as in Pr.3. Consider also **b**. let $q(x) = x$ discuss the cases (i)-(v) and **c**. let $f(x) = \sin x$ the consider part **b**.

5. Consider the integral equation

$$y(x) = f(x) + \lambda \int_0^1 (1 - k\, x\, s) y(s)ds$$

where $f(x)$ is a continuous function in $[0,1]$, k is a constant and λ is the parameter of the integral equation. Follow the cases (i)-(v) in Problem 3 and answer them. Discuss all possible cases.

6. Solve the following equation

$$y(x) = \lambda \int_0^{2\pi} \sin(x+s)\, y(s)ds.$$

Discuss all possibilities

7. Obtain the most general solution of the integral equation

$$y(x) = f(x) + \lambda \int_0^{2\pi} \sin(x+s)\, y(s)ds.$$

when $f(x) = x$ and $f(x) = 1$. Discuss all cases.

8. Prove that the equation

$$y(x) = f(x) + \frac{1}{\pi} \int_0^{2\pi} \sin(x+s)\, y(s)\, ds$$

has no solution when $f(x) = x$, but that it possesses infinitely many solutions when $f(x) = 1$. Determine all such solutions