ASSIGNED EXERCISES OF MATH544: PDE set 7 *April, 2001*

THE LAPLACE EQUATION

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In all types of equations we studied the following:

1. **Exact Solutions**
2. **Uniqueness of solutions**
3. **Maximum-Minimum Principles**

Here, in the elliptic case, I will include into our study the subject **Green's Function Technique** for the Laplace operator.

1. **Green's function for the Laplace operator**: Let D be a domain and B its boundary. Let the closure of D be $\bar{D} = D \cap B$. Let $u, v \in C^2(D) \cap C^1(\bar{D})$. Then the following is an identity

$$\int_D [v\nabla^2 u - u\nabla^2 v]\, dV = \int_B [v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}]\, dS \tag{1}$$

where $\frac{\partial u}{\partial n} = < N, grad\, u >$. Here N is the unit normal vector field defined at each point of the surface B. Now consider the following Dirichlet problem

$$\nabla^2 u = h, \quad in \quad D, \tag{2}$$

$$u = f, \quad on \quad B. \tag{3}$$

where h and f some continuous functions in D and on B respectively. Let $v = G(x, y)$ with

$$\nabla_y^2 G(x, y) = \delta(x - y), \quad in \quad D \tag{4}$$

$$G(x, y)|_{y \in B} = 0. \tag{5}$$

Then we find that

$$u(x) = \int_D G(x,y)h(y)\,dv_y + \int_B f(y)\,\frac{\partial G(x,y)}{\partial n_y}\,dS_y \tag{6}$$

Hence the solution of a Dirichlet problem reduces to the determination of Green's function $G(x,y)$ satisfying (4)-(5).

2. Find the Green function when D is a ball and B is the sphere with radius a. Since $\frac{1}{|x-y|}$ satisfies

$$\nabla^2 \frac{1}{|x-y|} = 4\pi\delta(x-y)$$

then the following form for G is more suitable

$$G(x,y) = \frac{1}{4\pi}[\frac{1}{|x-y|} - \frac{q}{|x^* - y|}]$$

where q and x^* are to be determined thorough the boundary condition. Let $y \in B$ then

$$G|_{y\in B} = 0$$

or

$$\frac{1}{|x-y|} = \frac{q}{|x^* - y|}$$

This gives

$$q = \frac{a}{r}, \quad x^* = q^2\,x. \tag{7}$$

where $r^2 = |x|^2$. We find also

$$\frac{\partial G(x,y)}{\partial n_y} =< \frac{y}{a}, grad_y G(x,y) > |_{y\in B} = \frac{r^2 - a^2}{4\pi\,|x-y|^3}$$

Hence the solution of the Dirichlet problem when $h = 0$ is

$$u(x) = \int_B \frac{a^2 - r^2}{4\pi\,|x-y|^3}\,f(y)\,dS_y$$

3. Using the Green's function technique solve the following Dirichlet problem.

$$\nabla^2 u = 0, \qquad -\infty < x, y < \infty, \quad z > 0, \tag{8}$$

$$u(x, y, 0) = f(x, y), \qquad -\infty < x, y < \infty. \tag{9}$$

where $f(x, y)$ is a continuous function in R^2.

4. Solve the above problem by use of separation of variables

5. Prove the following theorem:

Theorem. Let D be a bounded normal domain, and suppose that the Green's function $G(x, y)$ for the Dirichlet problem exists. Then

$$G(x, y) = G(y, x)$$

for $x, y \in D$. Consequently , in addition to $\nabla_y^2 G(x, y) = 0$ for $y \neq x$, we also have $\nabla_x^2 G(x, y) = 0$ for $x \neq y$.

Solution. *First way.* Let $u = G(z, x)$ and $v = G(z, y)$ in (1). Using (4) and (5) and using the property

$$\int_D f(y)\, \delta(y - x)\, d^3 y = f(x)$$

of the delta-distribution one obtains $G(x, y) = G(y, x)$. *Second way.* Let $K_a(c)$ represent a ball with radius a centered at c and $S_a(c)$ is the sphere bounding this ball.

$$K_a(a) = \{x \in R^3 || x - c| < a\}, \quad S_a(c) = \{x \in R^3 || x - c| = a\}$$

Fix $x, x' \in D$. Fix $\varepsilon > 0$ so small that $K_\varepsilon(x), K_\varepsilon(x') \subset D$ and $|x - x'| > 2\varepsilon$. Apply (1) to the domain

$$D_\varepsilon = D - (K_\varepsilon(x) \cup K_\varepsilon(x'))$$

for $u(y) = G(x, y)$, and $v(y) = G(x', y)$

6. In two dimensions let D be a disc with radius a and B is the circle bounding D with radius a. Consider

$$\nabla^2 u = 0, \quad r < a, \tag{10}$$

$$u = f \quad on \quad B. \tag{11}$$

a. Find the formal solution.

b. Find reasonable restrictions on f so that the formal solution is a solution.

7. Let $D = \{(x, y, z) \in R^3 | z > 0\}$ and $B = \{(x, y, z) | z = 0\}$. Solve the following Dirichlet problem

$$\nabla^2 u = 0, \quad in \quad D, u = f(x, y) \quad on \quad B. \tag{12}$$

Remark 1. The last two problems were solved in class. They respectively represent discrete and continuous spectrum in the method of separation of variables.

8. In two dimensions consider

$$\nabla^2 u = 0, \quad 0 < x < a, \quad 0 < y < b, \tag{13}$$

$$u(x, 0) = f(x), \quad u(x, b) = 0, \quad 0 \leq x \leq a, \tag{14}$$

$$u(0, y) = u(a, y) = 0, \quad 0 \leq y \leq b. \tag{15}$$

a. Find the formal solution,

b. Find reasonable restrictions on $f(x)$ so that the formal solution is a solution.

9. Consider

$$\nabla^2 u(r,\theta) = 0, \quad 0 \leq r < a, \tag{16}$$
$$\frac{\partial u(a,\theta)}{\partial r} + \alpha\, u(a,\theta) = f(\theta), \tag{17}$$

with f is 2π periodic continuous function and $\alpha > 0$ a constant.

a. Solve formally by separation of variables.

b. Find reasonable conditions on f so that the formal solution is a solution.

10. Suppose that $k > 0$ is a constant and $\nabla^2 u - k^2 u = 0$. Formulate Dirichlet, Neumann , and Robin problems of this operator $L = \nabla^2 - k^2$ and prove that each problem has at most one solution.

11. Prove the following theorems.

Theorem 1. *Let $u \in C^2(D) \cap C^1(\bar{D})$, where D is abounded domain.*

(i) If $\nabla^2 u \leq 0$ in D and $u \geq 0$ on B. then $u \geq 0$ in D,
ii) If $\nabla^2 u \geq 0$ in D and $u \leq 0$ on , B then $u \leq 0$ in D.

Theorem 2. *Let $u \in C^2(D) \cap C^1(\bar{D})$, where D is a bounded domain.*

(i) If $\nabla^2 u \geq 0$ in D, then $u(x) \leq M = max_{y\in B}\, u(y)$ for all $x \in D$.
(ii) If $\nabla^2 u \leq 0$ in D, then $u(x) \geq m = \min_{y\in B}\, u(y)$ for all $x \in D$.
(iii) If$\nabla^2 u = 0$ in D, then $m \leq u(x) \leq M$ for all $x \in D$.

Theorem 3. *Let D be a bounded domain and B its boundary. Let $u \in C^2(D) \cap C^1(\bar{D})$. Then there is at most one solution of the problem*

$$\nabla^2 u = h, \quad in \quad D, \tag{18}$$
$$u = f, \quad on \quad B. \tag{19}$$

Theorem 4. *Suppose that $u \in C^2(D) \cap C^1(\bar{D})$ and satisfies the Dirichlet problem as in the previous theorem (Thm. 3). Suppose that $h(x)$ is bounded* *by a constant H in D and f is bounded by F on B. Then there is a constant C depending only on the domain D such that $|u(x)| \leq F + CH$*

Proof. *Find a function w such that*

$$\nabla^2 w \leq -H \quad in \quad D \ and \quad w \geq \pm u \quad on \ B. \tag{20}$$

Given such a function we have
$\nabla^2 (w \pm u) \leq -H \pm h \leq 0$ *in D and $w \pm u \leq 0$ on B*
Using Thm. 1 above we must have

$$w \pm u \geq 0$$

or $w \geq |u|$ in D. Hence we have to find an w satisfying (20). Let

$$w(x) = F + H[e^d - ex_1 - c_c], \quad x \in \bar{D} \tag{21}$$

where D lies in between the planes $x_1 = c_1$ and $x_1 = c_1 + d$ and $C = e^d$. Prove that this w satisfies all requirements of (20).

Theorem 5. *Suppose that the functions $u_i \in C^2(D) \cap C^1(\bar{D})$, $i = 1, 2$, satisfies $\nabla^2 u_i = h_i$ in D and $u_i = f_i$ on B, where h_i and f_i are given. If*

$$|h_1(x) - h_2(x)| \leq \varepsilon$$

for all $x \in D$ and

$$|f_1(x) - f_2(x)| \leq \varepsilon$$

for all $x \in B$. then

$$|u_1(x) - u_2(x)| \leq (1 + C)\varepsilon$$

for all $x \in D$, where C is a constant determined by the domain D.

Remark 2. First three theorems were proved in the class.

12. *Prove the following theorem.*

Theorem 6. *Let $u \in C^2(D) \cap C^1(\bar{D})$ for some bounded domain D.*

(1) *If $\nabla^2 u < 0$ in D and $u \geq 0$ on B , then $u \geq 0$ in D.*

(ii) *If $\nabla^2 u > 0$ in D and $u \leq 0$ on B, then $u \leq 0$ in D.*