ASSIGNED EXERCISES OF MATH544: PDE set 6 *April, 2001*

THE HEAT EQUATION

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1. Solve the following initial value problem

$$u_t = a u_{xx}, \quad (a > 0), t > 0, \quad x \in [0, L], \tag{1}$$

$$u(0,t) = u(L,t) = 0, \quad t \geq 0, \tag{2}$$

$$u(x,0) = f(x), \quad x \in [0, L] \tag{3}$$

step 1. Formal Solution: Using the separation of variables we first let $u(x,t) = T(t)X(x)$. Then find that

$$T' = akT, \quad X'' = kX$$

where k is any constant. It must be a negative number $k = -\lambda^2$. we find that

$$T = e^{-a\lambda_n^2 t}, X = \sin(\lambda_n x) \quad , n = 1, 2, \cdots$$

where $\lambda_n = \frac{n\pi}{L}$. Hence

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \, e^{-a\lambda_n^2 t} \, \sin(\lambda_n x) \tag{4}$$

Here α_n are constants , will be determined by the initial condition (assuming the t goes to zero exists which requires the uniform convergence of the sine series)

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin(\lambda_n x), \tag{5}$$

with

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin(\lambda_n x) dx, \quad n = 1, 2, \cdots \tag{6}$$

step 2. Is the formal solution a solution of the initial and boundary value problem we started with?. For this the given function must satisfy certain conditions. Term-by term differentiation of the series representing $u(x,t)$ in (4) can be justified at each point $(\tilde{x},\tilde{t})$ with $\tilde{t} > 0$ and $0 \le \tilde{x} \le L$ by showing that the series in (4) and the series representing u_t, u_x and u_{xx} that result from term-by-term differentiation , converge uniformly in a region D which contains the point $(\tilde{x},\tilde{t})$. Such a region is

$$D = \{(x,t) | 0 \le x \le L,\ t \ge \frac{\tilde{t}}{2}\}.$$

We first assume that the function $f(x)$ is bounded in $[0,L]$

$$|f(x)| \le M, \quad for\ all\ x \in [0,L]$$

Then from (6) we have

$$|\alpha_n| \le 2M$$

and series (4) satisfies

$$|\sum_{n=1}^{\infty} \alpha_n\, e^{-a\lambda_n^2\, t} \sin(\lambda_n x)\,| \le 2M \sum_{n=1}^{\infty} \lambda_n^2\, e^{-a\lambda_n^2\, \tilde{t}/2}.$$

The series at the right hand side converges uniformly in D. Similar reasoning applies also for the series representing u_t, u_x, and u_{xx}. Consequently (4) satisfies the equations (1) and (2).

To guarantee that (4) will also satisfy the initial condition (5), we need further assumptions on $f(x)$. For the uniform convergence of the series in (5) we need $\sum_{n=1}^{\infty} |\alpha_n| < \infty$. If $f(0) = f(L) = 0$ then

$$\alpha_n = -\frac{2}{\lambda_n^2 L} \int_0^L f'' \sin(\lambda_n x) dx.$$

Hence , if f'' is continuous on $0 \le x \le$, then it is bounded say by M'', and

$$|\alpha_n| \le \frac{2M'' L^2}{\pi^2\, n^2}$$

This estimate implies that the series (4) for $u(x,t)$ is uniformly convergent for $t \geq 0$ and $0 \leq x \leq L$. Since $u(x,t)$ is the uniform limit of continuous functions , it is continuous. Hence we have the following:

Theorem. *If $f(x)$ is bounded and integrable, the series in (4) yields a solution to (1) and (2) for $t > 0$ and $0 \leq x \leq L$. If in addition f has continuous second derivative and $f(0) = f(L) = 0$, then $u(x,t)$ defined in (4) is uniformly convergent and continuous for $t \geq 0$ and $0 \leq x \leq L$ and also satisfies the initial condition (3).*

2. Solve the following initial and boundary value problem:

$$u_t - au_{xx} = F(x,t), \quad t > 0,\ 0 < x < L, \tag{7}$$

$$u(0,t) = u(L,t) = 0, \quad t \geq 0, \tag{8}$$

$$u(x,0) = f(x), \quad 0 \leq x \leq L \tag{9}$$

Complete the method of solution given in class. The method of solution is similar to previous problem.

3. Solve the problem:

$$u_t - au_{xx} = F(x,t), \quad 0 < x < L,\ t > 0, \tag{10}$$

$$u_x(0,t) = u_x(L,t) = 0, \quad t \geq 0, \tag{11}$$

$$u(x,0) = f(x), \quad 0 \leq x \leq L. \tag{12}$$

Explain also how you can prove the uniqueness of the solution of this problem.

4. Let $u(x,t)$ be a solution to the problem

$$u_t - au_{xx} = 0, \quad t > 0,\ 0 \leq x \leq L, \tag{13}$$

$$u(0,t) = u(L,t) = 0, \quad t \geq 0 \tag{14}$$

$$u(x,0) = f(x), \quad 0 \leq x \leq L \tag{15}$$

let $\tilde{u}$ be a solution of the above problem with f replaced by $\tilde{f}$. Denote the respective Fourier coefficients by α_n nad $\tilde{\alpha_n}$. Show $|\alpha_n - \tilde{\alpha_n}| \leq 2||f'' - \tilde{f}''||/\lambda_n^2$

and hence

$$||u - \tilde{u}|| \le 2||f'' - \tilde{f}''|| \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \right)$$

where $||h|| = max|h(x)|$ for $0 \le x \le L$. Deduce continuous dependence and hence the well-posedness of the problem.

5. For $T > 0$ let

$$Q_T = \{(x,t)|0 < x < L,\ 0 < t \le T\}$$

(this is exactly the region D defined in the lecture), be a rectangle with its bottom and two vertical sides removed. Let

$$\bar{Q}_T = \{(x,t)|0 \le x \le L,\ 0 \le t \le T\}.$$

Then $B_T = \bar{Q}_T - Q_T$ consists of the three missing sides of the original rectangle Q_T. Suppose that $w(x,t)$ satisfies the differential inequality

$$w_t - a(x,t)w_{xx} - b(x,t)w_x - c(x,t)w < 0 \quad (x,t) \in Q_T$$

where $a(x,t) \ge 0$ and $c(x,t) \le 0$ in Q_T. Then $w(x,t)$ can not achieve a positive local maximum in Q_T

6. Let a, b and c be as in the previous problem. Suppose that $u(x,t)$ is continuous on $\bar{Q}_T$ and satisfies

$$u_t - a(x,t)u_{xx} - b(x,t)u_x - c(x,t)u \le 0, \quad (x,t) \in Q_T, \tag{16}$$

$$u(x,t) \le 0, \quad (x,t) \in B_T \tag{17}$$

Then $u(x,t) \le 0$ on $\bar{Q}_T$.

7. Let a, c and c be as in Prob.4. Suppose that $u(x,t)$ is continuous on $\bar{Q}_T$ and satisfies

$$u_t - a(x,t)u_{xx} - b(x,t)u_x - c(x,t)u = 0$$

in Q_T. If u has a positive maximum , it is assumed on the boundary B_T and if u has a negative minimum , it is assumed on B_T.

8. Let a, b and c be as in Prob. 4. Suppose that $F(x,t)$ is defined in Q_T and bounded by the constant N there. Suppose that $u(x,t)$ is continuous on $\bar{Q}_T$, $M = max_{B_T}|u(x,t)|$, and $u(x,t)$ satisfies

$$u_t - a(x,t)u_{xx} - b(x,t)u_x - c(x,t)u = F(x,t)$$

in Q_T. The $|u(x,t)| \leq M + TN$ in Q_T.

9. Let $a(x,t) > 0$ and $c(x,t) \leq 0$ for all $0 < x < L$ and $t >$). Show that the initial boundary value problem

$$u_t - a(x,t)u_{xx} - b(x,t)u_x - c(x,t)u = F(x,t), \quad 0 < x, L, \quad t > 0, \qquad (18)$$

$$u(0,t) = g(t), \quad u(L,t) = h(t), \quad t \leq 0, \qquad (19)$$

$$u(x,0) = f(x), \quad 0 \leq x \leq L \qquad (20)$$

has at most one solution which is continuous on $0 \leq x \leq L$ and $t \geq 0$ [Notice that if there is , in fact a continuous solution, then the boundary and initial data must be compatible. That is ,$g(0) = f(0), \quad h(0) = f(L)$].