ASSIGNED EXERCISES OF MATH544: PDE set 5 *March, 2001*

THE METHOD OF RIEMANN

Ronald B Guenther and John W Lee, Partial Differential Equations of Mathematical Physics and Integral Equations

1. Solve the following initial value problem

$$u_{xy} - yu_x = 1, \quad y > -x, -\infty \leq x \leq \infty, \tag{1}$$

$$u(x,-x) = x, \quad u_x(x,-x) = x^2, \quad u_y(x,-x) = x^2 - 1, \tag{2}$$

$$\Gamma = \{x, y \in R^2 | y = -x\} \tag{3}$$

step 1. Define the Riemann function $v(x,y)$ which satisfies the homogeneous adjoint equation

$$v_{xy} + (yv)_x = 0$$

Using the equations for u and v we obtain

$$A_y + B_x = v$$

where

$$A = \frac{1}{2}(vu_x - uv_x), \quad B = \frac{1}{2}(vu_y - uv_y) - yuv$$

step 2. Integrate the above divergence form in a triangular region with the vertices $M(x_0, y_0)$, $P(-y_0, y_0)$, and $Q(x_0, -x_0)$ and using Green's theorem we get

$$\int_C [-Adx + Bdy] = \int\int_{T(x_0,y_0)} v(x,y)dxdy$$

where $C = L_{QM} u L_{MP} u L_{PQ}$. Evaluating each integral

$$\int_{QM} = \int_{QM} Bdy = \frac{1}{2}\int_{QM} (vu_y - uv_y)|_{x=x_0} dy - \int_{QM} yuv|_{x=x_0} dy$$

$$= \frac{1}{2}(uv)|_Q^M - \int_{QM} u(x_0, y)\,(v_y + yv)dy$$

Since the last term contains an unknown value $u(x_0, y)$ of the function $u(x, y)$ we then let its coefficient to vanish.

$$v_y(x_0, y) + yv(x_0, y) = 0, \quad y \leq y_0 \tag{4}$$

Hence

$$\int_{QM} = \frac{1}{2}u(M)v(M) - \frac{1}{2}u(Q)v(Q)$$

Similarly we find that

$$\int_{MP} = -\int_{MP} A|_{y=y_0}dx = \frac{1}{2}u(M)v(M) - \frac{1}{2}u(P)v(P)$$

with

$$v_x(x, y_0) = 0, \quad x \leq x_0 \tag{5}$$

We then find that

$$u(M)v(M) = \frac{1}{2}u(P)v(P) + \frac{1}{2}u(Q)v(Q) - \int_\Gamma [-Adx + Bdy] + \int_{T(x_0,y_0} v(x, y)dxdy \tag{6}$$

Here

$$u(M) = u(x_0, y_0), \quad v(M) = v(x_0, y_0),$$

$$u(P) = u(-y_0, y_0) = y_0, \quad , v(P) = v(-y_0, y_0)$$

$$u(Q) = u(x_0, -x_0) = x_0, \quad v(Q) = v(x_0, -x_0),$$

step 3. Determination of the Riemann (Green) function: Let us collect all the conditions on the (Riemann) function $v(x, y)$. We usually use the notation $v(x, y) = R(x_0, y_0, x, y)$

$$v_{xy}(x, y) + yv(x, y) = 0, \quad y > -x, x \in R, \tag{7}$$

$$v_y(x_0, y) + yv(x_0, y) = 0, \quad y \leq y_0, \tag{8}$$

$$v_x(x, y_0) = 0, \quad x \leq x_0, \tag{9}$$

$$v(x_0, y_0) = 1. \tag{10}$$

The complete solution of the above problem (known as the Goursat-Problem) is easy and given by

$$v(x, y) = e^{\frac{1}{2}(-y^2+{y_0}^2)}$$

step 4. Solution: Inserting the above result in the equation (6) we obtain that

$$u(x_0, y_0) = -\frac{1}{2}y_0 + \frac{1}{2}x_0 e^{({y_0}^2-{x_0}^2)/2} + \frac{1}{2}\int_{-y_0}^{x_0} x^2 e^{({y_0}^2-x^2)/2}dx \tag{11}$$

$$-\frac{1}{2}\int_{-x_0}^{y_0} e^{({y_0}^2-y^2)/2}dy + \int_{T(x_0,y_0)} e^{({y_0}^2-y^2)/2}dxdy \tag{12}$$

2. Solve the above problem without using Riemann's method.

3. Solve the initial value problem by the use of Riemann's method:

$$u_{xy} = F(x, y), \quad u(x, x) = f(x), \quad u_x(x, x) = g(x) \tag{13}$$

4. Solve the following initial value problem:

$$u_{xy} + xu_x = 1, \quad y > x, \quad x \in R, \tag{14}$$

$$u(x, -x) = u_x(x, -x) = 0. \tag{15}$$

5. Solve

$$u_{xy} - u_x \;=\; xy, \tag{16}$$

$$u(x, x^3) \;=\; \sin x, \quad u_x(x, x^3) = \cos x, \quad u_y(x, x^3) = 0, \tag{17}$$

in the region above the curve $\Gamma: \ y = x^3$.