MATH544: First Midterm Exam*March 18, 2010*

1. Prove the following. (a) Theorem. A function $L(x, y, y', \cdot, y^n)$ is defined on $X \times \mathbb{R}^n$ where $X = I \times \mathbb{R}^m$ $(x \in I, y \in \mathbb{R}^m)$ is a null Lagrangian, meaning that Euler Lagrange equations, $E(L) = 0$, vanish identically for all x and y, if and only if L is a total derivative, i.e., $L = \frac{dP}{dx}$ for some function P of x, y and the derivatives of y. **(b). A corollary of the above theorem**: *Two Lagrange functions L and $\tilde{L}$ have the same EL equations if and only if they differ by a total derivative, $L = \tilde{L} + \frac{dP}{dx}$.*

2. Let

$$u_{tt} + a^2\, u_{xxxx} = 0, \quad 0 < x < L, \quad t > 0,$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad 0 \le x \le L, \quad t \ge 0$$

$$u(0, t) = u_x(0, t) = 0, \; u(L, t) = u_{xx}(L, t) = 0, \quad t \ge 0$$

(a) Find the formal solution of the above IVP.
(b) Find reasonable restrictions on $f(x)$ so that the above formal solution in part (a) is a solution of the IVP.
(c) Prove the uniqueness of this IVP. [*Hint*: Consider the energy functional as $E(t) = \frac{1}{2} \int_0^L [(u_t)^2 + a^2\, (u_{xx})^2]\, dx\,]$
(d) Prove that the solution in part (b) continuously depends on the initial data.

3. Consider the IVP defined on the whole real line

$$u_{tt} + a^2\, u_{xxxx} = 0, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad -\infty < x < \infty$$

Use Fourier transform

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iwx}\, \tilde{u}(w, t)\, dw$$

to solve this problem. Find the restrictions on the data (f, g) so that the above IVP has a unique solution