ASSIGNED EXERCISES OF MATH544: PDE set 4 *March, 2001*

INITIAL AND BOUNDARY VALUE PROBLEMS OF SOME HYPERBOLIC TYPE OF EQUATIONS

Ronald B Guenther and John W Lee, Partial Differential Equations of Mathematical Physics and Integral Equations

1. Homogeneous wave equations with initial conditions: Let $f(x)$ and $g(x)$ be some arbitrary functions and

$$u_{tt} - c^2\, u_{xx} = 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

Theorem 1. *Let $f(x)$ have a second derivative and $g(x)$ have a first derivative for $-\infty < x < \infty$. Then the initila value problem given above is well-posed. Its solution is given by*

$$u(x,t) = \frac{1}{2}\,[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

2. Homogeneous wave equation with initial and boundary value problem. Let

$$u_{tt} - c^2\, u_{xx} = 0,$$
$$u_{x,0} = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le L$$
$$u(0,t) = u(L,t) = 0, \quad t \ge 0$$

Theorem 2. *Let $f(x) =$ have a continuous fourth derivative and $g(x)$ have a continuous third derivative for $0 \le x \le L$, and $f(0) = f(L) = 0$, $f''(0) = f''(L) = 0$, and $g(0) = g(L) = 0$. Then the initial boundary value problem above has a solution given by*

$$u(x,t) = \sum_{n=1}^{\infty} [A_n\, \cos(\lambda_n ct) + B_n\, \sin(\lambda_n ct)]\, \sin(\lambda_n x),$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx,$$

$$B_n = \frac{2}{\lambda_n cL}\int_0^L g(x)\sin(\frac{n\pi x}{L})dx.$$

where $\lambda_n = \frac{n\pi}{L}$.

Comment . Uniform convergence requires

$$|\lambda_n^2 A_n| \leq \frac{C}{n^2}, \quad |\lambda_n^2 B_n| \leq \frac{C}{n^2}$$

where $C = max\{C_1, C_2\}$, here

$$C_1 = \frac{2l}{\pi^2}\int_0^L |f^4(x)|dx,$$

$$C_2 = \frac{2L}{c\pi^2}[|g''(0)| + |g''(L)| + \int_0^L |g'''(0)|dx]$$

3. **Theorem 3**. *Let the initila data f and g in the previous problem satisfy the hypothesis of Theorem 2 (above). Then this problem is well posed.*

Hint. Let (f_1, g_1) and (f_2, g_2) are the data for the solutions u_1 and u_2 respectively. Then prove that

$$|u_1(x,t) - u_2(x,t)| \leq D\, max\{||f_1 - f_2||_2, ||g_1 - g_2||_1\}$$

where D is a constant and

$$||f_1 - f_2||_2 = max\{||f_1 - f_2||, ||f_1' - f_2'||, ||f_1'' - f_2''||\}$$

$$||g_1 - g_2||_1 = max\{||g_1 - g_2||, ||g_1' - g_2'||\}$$

Here the norm is defined as $||h(x)|| = max|h(x)|$, for all $x \in [0, L]$. The way one proves this theorem: Let $u = u_1 - u_2$, $f = f_1 - f_2, g = g_1 - g_2$. Use the compatibility conditions and integration by parts to find

$$|\lambda_n B_n| \leq \frac{C_1}{n^2}, \quad |A_n| \leq \frac{C_2}{n^2}$$

where A_n and B_n are Fourier coefficients based on $f = f_1 - f_2$ and $g = g_1 - g_2$ and

$$C_1 = \frac{2L}{c\pi^2}\int_0^L |g'(x)|dx, \quad C_2 = \frac{2L}{\pi^2}\int_0^L |f''|\,dx$$

Using the solution given in the problem 2 above one can find the estimate given above at the beginning of this problem.

4. Inhomogeneous Initial and Boundary value problem: (k¿0)

$$u_{tt} + ku_t - c^2 u_{xx} = F(x,t), \quad 0 \le x \le L, \quad t > 0$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le L$$

$$u(0,t) = r(t), \quad u(L,t) = s(t), \quad t \ge 0$$

Theorem 4. There exists at most one solution to the above inhomogeneous initial boundary value problem.

Let the data $f = 0, g = 0, r = 0, s = 0$. Then we have

Theorem 5. Let $F(x,t)$ have continous third order partial deriveatives for $0 \le x \le L$ and $t \ge 0$. Assume that $F(0,t) = F(L,t) = F_{xx}(0,t) = F_{xx}(L,t) = 0$ for all $t \ge 0$. Then the above inhomegenous initial and boundary value problem has a unique solution given by

$$u(x,t) = \sum_0^\infty u_n(t)\, sin(\lambda_n x)$$

$$u_n(t) = \frac{1}{c\lambda_n}\int_0^t F_n(\tau)\, G(t,\tau) d\tau$$

where$G(t,\tau)$ is the Green's function corresponding to the equation $u_n(t)$

$$u_n'' + ku_n' - c^2\lambda_n^2\, u_n = F_n(t)$$

We assumed $F(x,t) = \sum_0^\infty F_n(t)\sin(\lambda_n x)$ with

$$F_n(t) = \frac{2}{L}\int_0^L F(x,t)\sin(\lambda_n x) dx.$$