**ASSIGNED EXERCISES OF MATH544** *February, 2001*

## CALCULUS OF VARIATIONS

**1**. Find the shortest distance berween two points $A$ and $B$ on the plane using the polar coordinates.

**2**. Find the shortest distance between two points $A$ and $B$ on a sphere.

**3**. Find the shortest distance between two pints $A$ and $B$ on a right cylinder.

In the above problems (2) and (3) you may write the expressions for the length on the sphere and on the cylinder then use the calculus variations directly. On the other hand you may use the length formula in $R^3$ with the constarints $x^2 + y^2 = 1$ and $x^2 + y^2 + z^2 = 1$ respectively.

**4**. Let $J(y) = \int_a^b L(x, y, y')dx$ be a functional. Let $y$ be continuous in $[a, b]$. Assume that the Euler Lagrange equation holds on the interval $(a, b)$. Let $y$ be differentiable at all $x\varepsilon[a, b]$ except one point $x_0\varepsilon(a, b)$. Assume that $y'$ tends to $m$ or $n$ as $x$ tends to $x_0$ from the left or from the right.

**4.a**. Porve that $\frac{\partial L}{\partial y'}$ and $L - y' \frac{\partial L}{\partial y'}$ must be continuous functions of x at all points in $[a, b]$ including the corner points like $x_0$.

**4.b**. Let $J(y) = \int_a^b (y'^2 - 1)^2 dx$. Suppose a corner exists at $x_0\varepsilon[a, b]$. Minimize this functional (porve that there exist $m$ and $n$ where the functional $J(y)$ has a minimum value).

**5**. Determine the function extremizing (sometimes called the stationary function associated with) the functional $J(y) = \int_0^1 y'^2 f(x)dx$ when $y(0) = 0$ and $y(1) = 1$, where $f(x) = -1$ for $0 \leq x < \frac{1}{4}$ and $f(x) = +1$ for $\frac{1}{4} < x \leq 1$

**6**. The Euler-Lagrange (EL) equation is in general an ODE of the second order. We have (in the following we use the notation $p = y'$)

$$L_{px} + L_{py}y' + L_{pp}\, y'' = L_y$$

In the derivation of the EL we usualy do not assume (in advance) the existance of the second derivative of $y$. It can easily be shown, however, that $y''$ exists at least at those points where $L_{pp} \neq 0$ (prove this). Consider the functional $J(y) = \int_{-1}^{1} y^2 (p - 2x)^2 \, dx$ with $y(-1) = 0$ and $y(1) = 1$. Find the the function which minimisez the functional under the given boundary conditions. Prove that $y''$ does exist at $x = 0$.

**7.** Let $J(y) = \int_0^{\pi} y''^2 dx$ with the constraint $\int_0^{\pi} y^2 dx = 1$ and with the boundar conditions $y(0) = y''(0) = 0$ , $y(\pi) = y''(\pi) = 0$.

**7.a.** Find the function extremizing the functional $J(y)$

**7.b.** Consider the above problem without the constraint.

**8.** If $l$ is not preassigned, show that the stationar functions corresponding to the problem $\delta \int_0^l y'^2 dx = 0$ with $y(0) = 2$ and $y(l) = \sin l$ are of the form $y = 2 + 2x \cos l$, where $l$ satisfies the transcendental equation $2 + 2l \cos l - \sin l = 0$. Also verify that the smallest positive value of $l$ is between $\pi/2$ and $3\pi/4$.

**9.** If $l$ is not preassigned, show that the stationary functions corresponding to the problem $\delta \int_0^l [y'^2 + 4(y - l)] dx = 0$, with $y(0) = 2$ and $y(l) = l^2$ are of the form $y = x^2 - 2(x/l) + 2$, where $l$ is one of two real roots of the equation $2l^4 - 2l^3 - 1 = 0$.

**10.** Find the shortest distance berween the line $y = x$ and the parabola $y^2 = x - 1$.

**11.** Find the Lagrangian $L(x, y, y')$ such that Euler-Lagrange equation coincides with the (Emden-Fowler) equation $y'' + \frac{2}{x} y' + y^5 = 0$.

**12.** Determine the natural boundary condition at $x = b$ for the variational problem defined by $J(y) = \int_a^b L(x, y, y' \, dx + G(y(b))$, $y \in C^2[a, b]$ , $y(a) = y_0$ , where $G$ is a given differentiable function. As an application of this problem let $L = y'^2$, $y \in [0, 1]$, $y(0) = 1$, $y(1)$ is unspecified and $G = y^2$. Find $y$ extremizing this problem.

**13**. Give a proof that if $\int_a^b h'(x)f(x)dx = 0$ for all $h(x)$ satisfying $h(a) = h(b) = 0$, $h(x)$ is continuous and $h'(x)$ is piecewise continuous in $[a, b]$ where $f(x)$ is piecewise continuous in $[a, b]$.

**14**. *Legendre Condition.* In the theory of extrema of functions of single variable, a necessary condition for a minimum, besides $f'(x) = 0$, is that $f'' \geq 0$ (if it exists). A condition somewhat analogous to this holds for functionals. Let us suppose that there is an admissible function $y$. for which $J(y) = \int_a^b L)x, y, y')dx$ is a minimum. Then $f(\epsilon) = J(y+\epsilon h)$ has a minimum at $\epsilon = 0$; accordingly $f'(0) = 0$ (from which follows the Euler-Lagrange equation $E_y(L) = 0$ and $f''(0) \geq 0$, assuming its existence. Hence for all all $h$,

$$f''(0) = \int_a^b [L_{yy}\, h^2 + 2L_{yy'}\, h\, h' + L_{y'y'}\, h'^2]dx \geq 0$$

.

Choose a special variation

$$h = \begin{cases} 0 & a \leq x \leq \xi - \epsilon \\ 1 + (x - \xi)/\epsilon & \xi - \epsilon \leq x \leq \xi \\ 1 - (x - \xi)/\epsilon & \xi \leq x \leq \xi + \epsilon \\ 0 & \xi + \epsilon \leq x \leq b \end{cases} \tag{1}$$

If we substitute this function in the above expression an let $\epsilon \to 0$, then the term

$$\frac{1}{\epsilon^2} \int_{\xi-\epsilon}^{\xi+\epsilon} L_{y'y'} dx$$

will dominate the left hand side of the above inequality and determine its sign. Thus the the sign of $L_{y'y'}$ determines the sign of $f''(0)$, and for minimum, the Legendre condition $L_{y'y'} \geq 0$ must hold. We have the Legendre test:

**Theorem 1**. *The Legendre test. If*
*(i) Euler-Lagrange equation is satisfied,*

*(i) the range of intgeration is sufficiently small,*
*(iii) the sign of $L_{y'y'}$ is constant throughout this range,*
*then J(y) is a minimum or a maximum value of J according as the sign of $L_{y'y'}$ is positive or negative.*
In case $L$ contains more dependent functions, the Legendre condition is that the matrix $L_{y'_i y'_j}$ be positive or negative definite, that is

$$\Sigma_{(i,j)} \lambda_i \lambda_j L_{y'_i y'_j} \geq 0 \text{ or } \leq 0$$

for all $\lambda$.

**The following part is prepared by using the book** *An Intoruction to the Calculus of Variations* **by Charles Fox, Oxford Press (1950). In particular the second chapter on the second variations.**

**15**. *The second variation*: Let $y = y(x)$ be the path of integration for which the integral $J(y) = \int_a^b L(x, y, y')dx$ is minimium or maximum (sometimes called the integral is stationary). Consider all admissible functions $y + \varepsilon h$ where both $y$ and $h$ belong to $C^2[a, b]$. Then

$$J(y + \varepsilon h) = J(y) + \frac{\varepsilon^2}{2} J_2 + O(\varepsilon^3)$$

where

$$J_2 = \int_a^b [h^2 \frac{\partial^2 L}{\partial y^2} + 2\,h\,h' \frac{\partial^2 L}{\partial y\,\partial y'} + h'^2 \frac{\partial^2 L}{\partial y'^2}]\,dx$$

If $J(y)$ is minimum (or maximum) the sign of $J_2$ must be positive (or negative) without depending upon the choice of $h(x)$ , for all sufficiently small $\varepsilon$. For simplicity we use the following notation: $L_0 = \frac{\partial L}{\partial y}$, $L_1 = \frac{\partial L}{\partial y'}$. $L_{00} = \frac{\partial^2 L}{\partial y^2}$, $L_{01} = \frac{\partial L}{\partial y\,\partial y'}$, $L_{11} = \frac{\partial^2 L}{\partial y'^2}$. Then $J_2$ takes the form

$$J_2 = \int_a^b [L_{00}\,h^2 + 2L_{01}\,hh' + L_{11}\,h'^2]dx$$

.

We have the following Lemma. Proof is straightforward (done in the class)

**Lemma 2**. *Let $h(a) = h(b) = 0$, then*

$$J_2 = \int_a^b \{h^2\,(L_{00} - \frac{d}{dx}\,L_{01}) - h\,\frac{d}{dx}\,(h'\,L_{11})\}dx$$

.

**Definition**. *(Jacobi equation and Jacobi field). Let $u \in C^2[a,b]$. Then the folowing equation is called the Jacobi equation.*

$$\{L_{00} - \frac{d}{dx}(L_{01})\}\,u - \frac{d}{dx}(L_{11}\frac{du}{dx}) = 0$$

*the function $u$ satisfying this second order ODE is called the Jacobi field*

**Lemma 3**. *If $h(a) = h(b) = 0$ and is a Jacobi field then*

$$J_2 = \int_a^b L_{11}\,[h' - h\,\frac{u'}{u}]^2\,dx,$$

**Proof**. Taking the term $L_{00} - \frac{d}{dx}(L_{01})$ from the Jacobi equation and inserting into the expression for $J_2$ given in the first Lemma one obtains the required result.

If the term $\{h' - h\,(\frac{u'}{u})\} \neq 0$ and $L_{11}$ has constant sign for all points of the extremal arc $AB$, where $A = (a, y(a))$ and $B = (b, y(b))$, then $J_2$ must have a sign which is independent of the choice of $h$. Now in the extremal case

$$J(y + \varepsilon h) - J(y) = \frac{\varepsilon^2}{2}\,J_2 + O(\varepsilon^3)$$

Hence $J$ takes its minimum value (or maximum value) if $L_{11}$ is positive (or negative) at all points on the extremal curve $y(x)$ with $x \in [a,b]$. This is essentially the Legendre test stated above.

If , however, $L_{11}$, doesnot keep its sign constant at all $x \in [a,b]$ on the curve $y(x)$, then the value $J(y)$ is neither a minimum nor a maximum.

If the term $\{h' - h\,(\frac{u'}{u})\}$ vanishes at all points of the extremal curve $y(x)$. Then it is clear that $h(x) = \alpha\,u(x)$, where $\alpha$ is an arbitrary constant. Along the extremal curve $y(x)$ the first variation vanishes. If in addition $h(x) = \alpha u(x)$ is choosen then the second variation, $J_2$ vanishes as well. The sign of $\delta J = J(y+\varepsilon h) - J(y)$ will depend on the third variation $\delta J = \frac{\varepsilon^3}{6} J_3$, where

$$J_3 = \int_a^b [h^3 L_{000} + 3h^2 h' L_{001} + 3hh'^2 L_{011} + h'^3 L_{111}]dx$$

Since the sign of $\delta J$ depends on that of $\varepsilon$ there can be no maximum or minimum value of $J$ unless $J_3$ vanishes, in which case the sign of $\delta J$ will depend on that of $J_4$ , the fourth variation. In order to avoid this difficulty Jacobi proposed a test. This test provides whether the $h(x) = \alpha u(x)$ at all points of the extremal curve $y(x)$.

**Definition**: Let $u(x)$ be a solution of the Jacobi equation. Let $u(a) = 0$. This means that the Jacobi field vanishes at the point $A = (a, y(a))$. The all other points on the extremal curve $y(x)$ at which $u(x)$ vanishes are called the *conjugate points* to the point $A$.

The jacobi field ( a function satisfying the Jacobi equation with $u(a) = 0$) may be given as the linear sum of the fundamental solutions $u_1$ and $u_2$ of the Jacobi equation.

$$u(x) = a_1 u_1(x) + a_2 u_2(x)$$

where $a_1$ and $a_2$ are constants. Since $u(a) = 0$, then

$$\frac{u_1(a)}{u_2(a)} = -\frac{a_1}{a_2}$$

Hence if $x$ is the abscissa of the conjugate point to $A$ $(u(x) = 0)$ then

$$\frac{u_1(x)}{u_2(x)} = -\frac{a_1}{a_2}$$

It shows that this ratio is the same for all conjugate points. This is the way obtaining the conjugate points. We now give a nice way of determining the fundamental solutions $u_1$ and $u_2$. Since the function $y(x)$ , the solution of the Euler-Lagrange equation is second order ODE then it contains two independent constants of intgetation $y(x, c_1, c_2)$. It is easy to prove the following Lemma.

**Lemma 4.** $u_i = \frac{\partial y}{\partial c_i}$, $i = 1, 2$.

Hence the conjugate points are easily determined from the folowing formula. Let $Y = (\frac{\partial y}{\partial c_1})/(\frac{\partial y}{\partial c_2})$ then

$$Y(x) = Y(a)$$

Let $A'$ be the first conjuagte point to $A$ when moving on the extremal curve $y(x)$ in the direction from $A = (a, y(a))$ to $B = (b, y(b))$. There are three possibilities

(i) $B$ lies between $A$ and $A'$,
(ii) $B$ coincides with $A'$,
(iii) $B$ lies beyond $A'$.

In all three cases $u(x)$ vanishes at $A$, but in case $(i)$ it cannont vanish again at the point $B$, in case $(ii)$ it vanishes again at $B$, and in case $(iii)$ it vanishes again at some point of the arc $AB$ lying betweein $A$ and $B$. In case $(i)$ we have $u(a) = h(a) = 0$ but $u(b) \neq 0$, $h(b) = 0$. Hence $u(x)$ can not be proportional to $h(x)$ at all points of the extermal arc $AB$. Therefore we must have $(h' - h\frac{u'}{u})^2 > 0$ at all points of $AB$ , except, possibly, at finite number of points where $h(x)$ and $h'(x)$ vanish simultaneously. Then we have the following theorem *Jacobi test.*

**Theorem 5.** *Let $y = y(x)$ be the curve connecting the points $A$ and $B$ for which the functional $J(y) = \int_a^b L(x, y, y')dx$ takes its exremum value. Let $A$ and $A'$ be the conjugate points on this curve. If* **(i)** *$B$ lies between $A$ and $A'$ and* **(ii)** *$L_{11}$ has constant sign for all points on the arc $AB$, then $J(y)$ is a minimum (maximum) when $L_{11} > 0$ $(L_{11} < 0)$*

For all the other cases we need higher variations for extremum tests.