Ch.4. INTEGRAL EQUATIONS AND GREEN'S FUNCTIONS
Ronald B Guenther and John W Lee, Partial Differential Equations of Mathematical Physics and Integral Equations.
Hildebrand, Methods of Applied Mathematics, second edition

In the study of the partial differential equations of hyperbolic and parabolic types we solved several initial and boundary value problems. While solving these equations we used the method separation of variables which reduces the problem to one of the following types of *Sturm-Liouville problems*

Sturm-Liouville Eigen value problem: Let $p(x) > 0, q(x) \geq 0, r(x) \geq 0$ in $I = (a, b)$. Here we assume $b > a$. Let $X \in C^2(I) \cap C^1(\bar{I})$, then the set of equations given below

$$-\frac{d}{dx}\,[p(x)\,\frac{dX}{dx}] + q(x)X = \lambda\, r(x)X, \tag{1}$$
$$X(a) - h_0 X'(0) = 0, \tag{2}$$
$$X(b) + h_1 X'(b) = 0. \tag{3}$$

is called the *The Sturm-Liouville Eigenvalue Problem* (SLEVP). Here $h_0 \geq 0$ and $h_1 \geq 0$ are constants. Let the Sturm-Liouville operator L be defined by

$$L = -\frac{d}{dx}\,p(x)\,\frac{d}{dx} + q(x), \tag{4}$$

then the *Sturm-Liouville Eigenvalue equation becomes*

$$L\,X = \lambda\, r(x)\,X, \quad X(a) - h_0 X'(a) = 0, \quad X(b) + h_1 X'(b) = 0. \tag{5}$$

Note that L is self-adjoint, i.e., $L = L^*$

The Sturm-Liouville Boundary Value Problem: With the same p, q, r defined above we have the Sturm-Liouville Boundary Value Problem (SLBVP)

$$L\,X = f(x), \quad in \quad I, \tag{6}$$
$$X(a) - h_0 X'(a) = 0, \tag{7}$$
$$X(b) + h_1\, X'(b) = 0. \tag{8}$$

Here $f(x)$ is a continuous function in I.

Sturm-Liouville Problem: Combination of SLEVP and SLBVP may be given as

$$L\,X = \lambda r(x) X + f(x), \quad in \quad I, \tag{9}$$
$$X(a) - h_0 x'(a) = 0, \tag{10}$$
$$X(b) + h_1 x'(b) = 0. \tag{11}$$

4.1. The Sturm Liouville Boundary Value Problem and Green's Functions

Let us formulate the SLBVP as

$$L\,u(x) = f(x), \quad x \in I, \tag{12}$$
$$u(a) - h_0 u'(a) = 0, \tag{13}$$
$$u(b) + h_1 u'(b) = 0. \tag{14}$$

We shall solve this problem by the use of Green's function technique. Let $u, v \in C^2(I) \cap C^1(\bar{I})$ then it is easy to prove the *Lagrange identity*

$$vLu - uLv = -\frac{d}{dx}[p(x)(vu' - uv')], \tag{15}$$

Now letting $v = G(x, y)$ and

$$L_x\, G(x,y) = \delta(x-y), \quad x, y \in I, \tag{16}$$

$$G(a,y) - h_0\, \frac{dG(x,y)}{dx}|_{x=a} = 0, \tag{17}$$

$$G(b,y) + h_1\, \frac{dG(x,y)}{dx}|_{x=b} = 0. \tag{18}$$

then we have

$$u(x) = \int_a^b G(x,y)\, f(y) dy, \tag{19}$$

Hence the SLBVP reduces to the construction of the Green's function $G(x,y)$ satisfying (16)-(18). In addition to these properties $G(x,y)$ satisfy also the following properties

1. $G(x,y)$ is continuous in $x, y \in I$ (*prove it*) in particular we have as $\varepsilon \to 0_+$

$$G(x,y)|_{x=y-\varepsilon} = G(x,y)|_{x=y+\varepsilon}. \tag{20}$$

2. $\frac{dG(x,y)}{dx}$ is discontinuous at $x=y$. As $\varepsilon \to 0_+$ we have the jump condition *prove it*

$$\frac{dG(x,y)}{dx}|_{x=y-\varepsilon} - \frac{dG(x,y)}{dx}|_{x=y+\varepsilon} = p(y) \tag{21}$$

Since $G(x,y)$ satisfies the homogeneous equation for $x<y$ and $x>y$ we then have

$$G(x,y) = \begin{cases} a_1(y)\, v_0(x) + a_2(y)\, v_1(x), & a \leq x < y \leq b \\ b_1(y)\, v_0(x) + b_2(y)\, v_1(x), & a \leq y < x \leq b \end{cases} \tag{22}$$

where $v_0(x)$ and $v_1(x)$ are the solutions of the homogeneous SL-equation and a_1, a_2, b_1 and b_2 are functions of y to be determined through the conditions (17), (18), (20), and (21). If we choose v_0 as a solution of the homogeneous SL equation satisfying the *left boundary condition*

$$v_0(a) - h_0\, v_0'(a) = 0, \tag{23}$$

and $v_1(x)$ as the solution of the SL equation satisfying the *right boundary condition*

$$v_1(b) + h_1 v_1'(b) = 0, \tag{24}$$

then $a_2 = b_1 = 0$ and due to the symmetry $G(x, y) = G(y, x)$ and remaining conditions we show that

$$G(x, y) = \begin{cases} v_0(x)\, v_1(y), & a \leq x < y \leq b \\ v_0(y)\, v_1(x), & a \leq y < x \leq b \end{cases} \tag{25}$$

The jump condition (21) becomes

$$p(x)\, W(v_0, v_1) = -1 \tag{26}$$

where $W(u, v) = uv' - vu'$ is the Wronskian of u and v. We then have the following theorem:

Theorem 1. *The SLBVP (12) has a Green's function if and only if the corresponding homogeneous SLEVP [with $f(x) = 0$] has only the trivial solution, in which case the Green's function is given in (25) with the boundary conditions in (23) and (24).*

proof: **(a)** If we assume the existence of the Green's function we have the solution (19) for each function $f(x)$ in I. The corresponding homogeneous solution (with $f = 0$ in (19) goes to the trivial solution $u = 0$. **(b)** Let us assume that the only solution of homogeneous SLBVP is only the trivial one. Hence we should be able to find nontrivial solutions so that $pW = C$. Let $C = 0$. This means v_0 and v_1 are proportional hence , for example v_0 satisfies the homogeneous SLBVP which is nontrivial, hence we obtain a contradiction. This means $C =\neq 0$. By a simple scaling $C = -1$ (by redefining v_0 for instance). Hence $pW = -1$. hence we have v_0 and v_0 solving the homogeneous SL equation, v_0 satisfying the left boundary condition (23) and v_1 the right boundary condition (24). Then there exist a unique Green's function given in (25).

To summarize all properties of the Green's function we formulate the following theorem

Theorem 2. *Assume that the homogeneous SLBVP ($f = 0$ problem has only the trivial solution. Let $G(x, y)$ be the function given in (25) and with the conditions (23) and (24). Then*

i *G(x,y) is continuous on the square $a \le x, y \le b$ and has continuous second derivatives on each of the triangles $a \le x < y \le b$ and $a \le y < x \le b$. On each triangle $L_x\, G(x, y) = 0$*
(ii) *G(x,y) satisfies the boundary conditions at $x = a$ and $x = b$,*
(iii) *For each $y \in (a, b)$, G(x,y) satisfies the jump condition (21)*

Conversely , properties (i)-(iii) uniquely determine G such that (19) solves the SLBVP in (19).

We have remarked that , since L is self adjoint, the Green's function is symmetric. If $p(x) > 0, q \ge 0$ and $h_0, h_1 \ge 0$ the we have the following result

Theorem 3. *The Green's function of the SLBVP exists if $p(x) > 0, q(x) \ge 0$, and $h_0, h_1 \ge 0$.*

proof. Th.1 says that the Green's function exists if and only if the homogeneous SLBVP has only the trivial solution. The homogeneous SLBVP is given as follows. Let $z \in C^2(I) \cap C^1(\bar{I})$ satisfy

$$-[p(x)z']' + q(x)z = 0, \quad x \in I, \tag{27}$$

$$z(a) - h_a z'(a) = 0, \quad z(b) + h_1 z'(b) = 0. \tag{28}$$

Multiplying the differential equation by Z and integrating over I and using integration parts we obtain

$$p(b)h_1z'(b)^2 + p(a)h_0z'(a)^2 + \int_a^b [p(x)z'^2 + q(x)z^2]dx = 0 \tag{29}$$

Here, since $p > 0$ in I then $z'(x) = 0$ in I. Hence $z =$ a constant everywhere in I, but this constant should vanish at the boundary points a and b, then $z(x)$ must vanish at all pints in I. This proves that under the given circumstances the only solution of the SLBVP is the trivial one ,$z = 0$.

Example 1. Solve $u'' = f(x), \quad x \in (a, b)$ and $u(a) = u(b) = 0$.

solution. Let

$$G(x, y) = \begin{cases} a_1x + a_2, & a \leq x < y \leq b \\ b_1x + b_2, & a \leq y < x \leq b \end{cases} \tag{30}$$

where the coefficient are to be determined through the conditions (17), (18), (20) and (21). We find

$$G(x, y) = \begin{cases} \frac{(x-a)(y-b)}{b-a}, & a \leq x < y \leq b \\ \frac{(y-a)(x-b)}{b-a}, & a \leq y < x \leq b \end{cases} \tag{31}$$

Hence the solution is given by

$$u(x) = \int_a^b G(x, y)f(y)dy = \tag{32}$$

$$\int_a^x \frac{(y-a)(x-b)}{b-a} f(y)dy + \int_x^b \frac{(y-b)(x-a)}{b-a} f(y)dy, \tag{33}$$

$$= \frac{x-b}{b-a} \int_a^x (y-a)\, f(y)dy + \frac{x-a}{b-a} \int_x^b (y-b)\, f(y)dy \tag{34}$$

Example 2. Solve the same problem by changing the boundary conditions to $u(a) = 0, \quad u'(b) = 0$.

Example 3. Solve the same problem by changing the boundary conditions to $u(a) = 0, \quad u'(a) = 0$.

Example 4. Solve the same problem by changing the boundary conditions to $u(a) - \alpha^2 \, u'(a) = 0$ and $u(b) + \beta^2 \, u'(b) = 0$. Here α and β are some constants.

4.2 The Neumann Series

Let us now consider the general SL problem

$$L\, u = \lambda\, r(x)\, u + f(x), \quad x \in (a, b), \tag{35}$$

$$u(a) - h_0 u'(a) = 0, \quad u(b) + h_1 u'(b) = 0. \tag{36}$$

Since the Green's function $G(x, y)$ outlined in the first section belongs to the operator L we then have

$$u(x) = \int_a^b G(x, y)\, [f(y) + \lambda\, r(y) u(y)] dy \tag{37}$$

or, equivalently we have

$$u(x) = g(x) + \lambda \int_a^b G(x, y)\, r(y)\, u(y) dy, \tag{38}$$

where

$$g(x) = \int_a^b G(x, y) f(y) dy$$

Eq(38) is called the *The Fredholm integral equation of the second kind* which can be put into a more symmetrical one. Let

$$y(x) = \sqrt{r(x)}\, u(x), \quad g(x, s) = \sqrt{r(x)} \sqrt{r(s)}\, G(x, s), \quad f_1(x) = \sqrt{r(x)}\, g(x). \tag{39}$$

then

$$y(x) = f_1(x) + \lambda \int_a^b g(x,s)\, y(s) ds, \tag{40}$$

In our future analysis we let $f_1 \to f$. Here $g(x,s)$ is called the *kernel* of the integral equation, $f(x)$ is given and λ is in general a complex parameter. In most cases it is real. And also we can assume that $g(x,s)$ is continuous in $a \le x, s \le b$ and $f(x)$ is continuous in $a \le x \le b$. The integral equation given in (40) may be solved by using several methods. Let us assume that y is a power series in λ

$$y(x) = \sum_{n=0}^{\infty} \lambda^n\, y_n(x), \tag{41}$$

where the coefficients $y_n(x)$ will be determined by the use of the integral equation (40) (we let $f_1 = f$). We obtain

$$y_0(x) = f(x), \tag{42}$$

$$y_1(x) = \int_a^b g(x,s)\, y_0(s) ds, \tag{43}$$

$$y_2(x) = \int_a^b g(x,s)\, y_1(s) ds, \tag{44}$$

$$y_{n+1} = \int_a^b g(x,s)\, y_n(s) ds, \tag{45}$$

for $n = 0, 1, \cdots$ Hence all $y_n(x)$ can be calculated recursively. We have shown that the series (41) must be uniformly convergent to justify the term by term integration in (40).

Proposition 4. *Assume that $g(x,s)$ is continuous in $a \le x, s \le b$ and $f(x)$ is continuous in $a \le x \le b$. The series in (40) is uniformly convergent if* $|\lambda| M(b-a) \le 1$

proof. Since both g and f are continuous we have

$$M = max|g(x,s)|, \quad N = max|f(x)|$$

in $a \le x, s \le b$. Using (42), (43, (44), and (45) we obtain

$$|y_0(x)| \le N, \quad (46)$$

$$|y_1(x)| \le MN(b-a), \quad (47)$$

$$|y_2(x)| \le NM^2(b-a)^2, |y_n(x)| \le NM^n(b-a)^n| \quad (48)$$

which leads to

$$|\sum_{n=0}^{\infty} \lambda^n y^n(x)| \le N \sum_{n=0}^{\infty}[|\lambda|M(b-a)|^n < \infty$$

This result provides also a solution of the integral equation (40)

Proposition 5. *Let*

$$y(x) = f(x) + \lambda \int_a^b g(x,s)\, y(s) ds \quad (49)$$

be the integral equation. Let $g(x,s)$ and $f(x)$ be continuous in $a \le x, s \le b$. If $|\lambda| < \frac{1}{M(b-a)}$ then

$$y(x) = f(x) + \lambda \int_a^b \gamma(x,s)\, f(s) ds \quad (50)$$

where $\gamma(x,s)$ is called the resolvent kernel and given by

$$\gamma(x,s) = \sum_{n=1}^{\infty} \lambda^{n-1}\, k_n(x,s) \quad (51)$$

proof: Using (42)-(45) we obtain

$$\begin{aligned} y_0(x) &= f(x), & (52) \\ y_1(x) &= \int_a^b g(x,s)\, f(s) ds, & (53) \end{aligned}$$

$$
\begin{aligned}
y_2(x) &= \int_a^b g(x,s)\, y_s ds, && (54)\\
&= \int_a^b g_2(x,s)\, f(s) ds, && (55)\\
& && (56)
\end{aligned}
$$

where $k_2(x,s) = \int_a^b g(x,t)\, g(t,s) dt$. Defining, in general

$$
\begin{aligned}
g_1(x,s) &= g(x,s), && (57)\\
g_n(x,s) &= \int_a^b g(x,t) g_{n-1}(t,s) dt, \quad n = 2, 3, \cdots && (58)
\end{aligned}
$$

then it is straightforward to establish the relation

$$
y_n(x) = \int_a^b g_n(x,s)\, f(s) ds, \quad n = 1, 2, \cdots
$$

and hence

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} \lambda^n\, y_n(x) && (59)\\
&= f(x) + \sum_{n=1}^{\infty} \lambda^n\, y_n(x) && (60)\\
&= f(x) + \sum_{n=1}^{\infty} \lambda^n \int_a^b y_n(s) f(s) ds && (61)\\
&= f(s) + \lambda \int_a^b \gamma(x,s) f(s) ds, && (62)
\end{aligned}
$$

where

$$
\gamma(x,s) = \sum_{n=1}^{\infty} \lambda^n g_n(x,s) \tag{63}
$$

Convergence of the series in (62) is guaranteed by Prop.4, and the solution given in Prop.5 is unique.

Theorem 6. *Let g(x,s) and f(x) be continuous on* $a \leq x, s \leq b$. *Let* $M = max|g(x,s)|$ *for all* $(x,s) \in [a,b]$. *If* $|\lambda| < \frac{1}{M(b-a)}$ *the the solution*

$$y(x) = f(x) + \lambda \int_a^b \gamma(x,s)\, f(s) ds$$

is unique and continuous.

proof: Let us assume that there are two different solutions $z_1(x)$ and $z_2(x)$ of the integral equation (49). Then their difference $w(x) = z_2(x) - z_1(x)$ satisfies

$$w(x) = \lambda \int_a^b g(x,s) w(s) ds$$

Hence we have for all $x \in [a,b]$ we have

$$|w(x)| \leq |\lambda| MW(b-a)$$

where $W = max|w(x)|$. Then

$$[1 - |\lambda| M(b-a)]W \leq 0$$

This implies that $W = 0$. Hence $z_1 = z_2$ everywhere in $[a,b]$.

Example 5. Find the solution of

$$y(x) = f(x) + \lambda \int_a^b e^{x-s}\, y(s) ds$$

Solution: Try to find the following solution. Given a continuous $f(x)$ for $x \in [a,b]$ we have

$$y(x) = f(x) + \frac{\lambda\, e^x}{1 - \lambda\,(b-a)} \int_a^b e^{-s}\, f(s) ds$$

where $\lambda \neq \frac{1}{b-a}$. (i) Discuss the case $\lambda = \frac{1}{b-a}$ (ii) and as an example let $f(x) = x^2$. Discuss also the application of (iii) Prop.4, (iv) Prop.5 and (v) Thm.6 to this example. (vi) find also the Neumann series corresponding to this example.

4.3. Fredholm Equation with Separable Kernels.

A separable kernel is given as

$$g(x,s) = \sum_{n=1}^{N} p_n(x)\, q_n(s) \tag{64}$$

Here we assume that the N functions p_n are linearly independent. Then the integral equation (49) takes the form

$$\begin{aligned} y(x) &= f(x) + \lambda \int_a^n g(x,s)\, y(s) ds, && (65)\\ &= f(x) + \lambda \int_a^b \left[\sum_{n=1}^{N} p_n(x)\, q_n(s) \right] y(s) ds, && (66)\\ &= f(x) + \lambda \sum_{n=1}^{N} p_n(x) \left[\int_a^b q_n(s) y(s) ds \right], && (67)\\ &= f(x) + \lambda \sum_{n=1}^{N} c_n\, p_n(x), && (68) \end{aligned}$$

where

$$c_n = \int_a^b q_n(s) y(s) ds, \qquad n = 1, 2, \cdots \tag{69}$$

Using (68) in (69) we obtain the following algebraic linear equations for the constants c_n's

$$c_n = \beta_n + \lambda \sum_{m=1}^{N} \alpha_{nm}\, c_m, \tag{70}$$

where for all $n, m = 1, 2, \cdots$

$$\beta_n = \int_a^b q_n(s)\, f(s) ds, \tag{71}$$

$$\alpha_{mn} = \int_a^b p_n(s)\, q_m(s) ds \tag{72}$$

let A denote the $N \times N$ matrix corresponding to α_{mn} B be the column N-vector corresponding to β_n, and C be the column unknown N-vector to be determined then (70) simply becomes

$$[I - \lambda A]C = B,) \tag{73}$$

where I is the $N \times N$ unit matrix. The above linear equation for C is easily solved , but we have to consider all possible cases. In the above equation (70) we need to determine the unknown coefficients c_n's in terms of the known coefficients β_n and α_{mn} for all $m, n = 1, 2, \cdots$

case (a). $f(x) = 0$ or $B = 0$ the equation (73) becomes homogeneous. For nontrivial solutions $det[I - \lambda A]$ must vanish. Otherwise there is only the trivial solution $c_n = 0$, for all $n = 1, 2, \cdots$. If $det[I - \lambda A] = 0$ at least (depending upon the rank of matrix A one of the c_n's is left arbitrary. In such a case there are infinitely many solutions. To remind you the terminology: Those values of λ where $det[I - \lambda A] = 0$ are called characteristic or eigenvalues and any nontrivial solution of the homogeneous integral equation is called the corresponding characteristic or eigenfunction. If there are k number of constants c_n's $n = 1, 2, \cdots, k$ for a given eigenvalue λ, then k number of linearly independent eigenfunctions are obtained.

case (b). $f(x) \neq 0$ but $\beta_n = 0, \quad n = 1, 2, \cdots$ this means that $f(x)$ is orthogonal to all functions $q_n(x)$, $n = 1, 2, \cdots$. Hence $B = 0$. The case **(a)** applies also here except for the fact that here the solution (68) contains the function $f(x)$. Hence the trivial solution $C = 0$ correspond to the solution

$y(x) = f(x)$. Solutions corresponding to eigenvalues of λ should be expresses as the sum of $f(x)$ and linear sum of the corresponding eigenfunctions.

case (c). $B \neq 0$. We assume that at least for some n , $\beta_n \neq 0$.**if** $det[I - \lambda A] \neq 0$ a unique nontrivial solution of (73) exists, leading to a unique nontrivial solution $y(x)$ of the integral equation (68). **if**$det[I - \lambda A] = 0$ either there is no solution or the solution is not unique meaning that there are infinitely many solutions.

Example 6. Let $g(x, s) = 1 - 3xs$. Solve the corresponding integral equation by considering all three cases above

4.4. Hilbert-Schmidt Theory

When the kernel $g(x, s)$ is not of type (64) there are , in general, infinitely many eigenvalues and eigenfunctions of the homogeneous Fredholm equation. In addition, there may also be more than one eigenfunctions corresponding to one eigenvalue. This is called the degenaracy and the number of eigenfunctions corresponding to a single eigenvalue is called the multiplicity. In this section we assume that the kernel $g(x, s)$ is symmetric with respect to the variables x and s. We also assume that multiplicity is one.

Remark: The homogeneous Fredholm equation

$$y(x) = \lambda \int_a^b g(x, s)\, y(s)\, ds \tag{74}$$

can not have zero eigenvalue , because the corresponding eigenfunction is also zero. We the have the following result.

Proposition 7. *Eigenfunctions of the homogeneous Fredholm integral equation with a symmetric kernel corresponding to different eigenvales are orthogonal.*

proof: By its definition we have that

$$y_m = \lambda_m \int_a^b g(x,s)\, y_m\, ds, \quad m = 1, 2, \cdots \tag{75}$$

Multiplying by $y_n(x)$ and integrating over (a, b) we get

$$\begin{aligned}
\int_a^n y_m(a)\, y_n(x)\, dx &= \lambda_m \int_a^b y_n(x) dx \int_a^b g(x,s)\, y_m(s) ds, && (76)\\
&= \lambda_m \int_a^b y_m(s) ds \int_a^b y_n(x)\, g(x,s) dx, && (77)\\
&= \frac{\lambda_m}{\lambda_n} \int_a^b y_m(s)\, y_n(s) ds && (78)
\end{aligned}$$

which leads to

$$\int_a^b y_m(x)\, y_n(x)\, dx = 0 \tag{79}$$

when $\lambda_m \neq \lambda_n$

Remark 1: If there are more than one eigenfunctions corresponding to an eigenvalue λ_n then orthogonalization of such a set is performed by the standard Gram-Schmidt procedure. In the sequel we assume that such sets are orthogonalized.

Remark 2: In the case of complex functions we generalize the Prop. 7 in the following way

Proposition 7′. *Eigenfunctions of the homogeneous Fredholm integral equation with a hermitian kernel , $g(x,s) = \bar{g}(s,x)$ corresponding to different eigemvales are orthogonal.*

Note that the inner product in this case is defined by

$$< f, g >= \int_a^b f(x) \bar{g}(x) dx$$

where bar over a letter denotes complex conjugation. This proposition implies that

$$\int_a^b y_m(x)\, \bar{y}_n(x)dx = 0$$

for $m \neq n$. A Corollary of this proposition is that λ_n for such homogeneous Fredholm integral equation with hermition kernel and hence also for real symmetric kernels the eigenvales are real *(Prove these statements)*.

We have now a theorem which will be used very often in our future analaysis.

Theorem 8. *Let $g(x,s)$ be a real and symmetric continuous kernel over (a,b). Let $H(x)$ be any continuous function over (a,b) the any function defined as*

$$h(x) = \int_a^b g(x,s)H(s)ds \tag{80}$$

can be represented as a linear superposition of the eigenfunctions $y_n(x)$, $n = 1, 2, \cdots$, of the homogeneous Fredholm equations with same kernel over (a,b).

Hence we have

$$h(x) = \sum_{n=1} a_n\, y_n(x)$$

and

$$a_n = \frac{1}{\rho_n} \int_a^b h(x) y_n(x) ds$$

with $\rho_n = \int_a^b y_n(x)^2\, dx$.

Remark 3: If there arefinite number of eigenfunctions then the functions generated by the operation

$$\int_a^b g(x,s)\, H(s)ds$$

form a restiricted class of functions , irrespctive the form of the function $H(x)$.

Example 7. Let $g(x,s) = sin(x+s)$ with $(a,b) = (0, 2\pi)$ then it is easy to show that

$$\lambda_1 = \frac{1}{\pi}, \quad y_1 = \sin x + \cos x, \tag{81}$$

$$\lambda_2 = -\frac{1}{\pi}, \quad y_2 = \sin x - \cos x. \tag{82}$$

Hence we have finite number of eigenfunctions. Then any function $h(x)$ given as

$$h(x) = \int_0^{2\pi} g(x,s)\, H(s) ds$$

takes the form

$$g(x) = C_1 \sin x + C_2 \cos x$$

whatever the function $H(x)$ is. It is obvious that $h(x)$ can also be written as

$$h(x) = a_1 y_1(x) + a_2 y_2(x)$$

Remark 4. In some cases the eigenfunctions of the homogeneous Fredholm integral equation may not form a complete set (see DK for the definition). This means that any continuous function $f(x)$ defined in (a,b) may not be represented over the same interval by a series of y_n's. In the sequel we assume that y_n's form a complete set over the interval (a,b).

The essence of the eigenfunctions of the homogeneous Fredholm integral eequation with a real and symmetric kernel shows up when we wish to solve the inhomogeneous Fredholm integral equation

$$y(x) = f(x) + \lambda \int_a^b g(x,s) y(s) ds) \tag{83}$$

where $f(x)$ is a given continuous function over the interval (a,b). First we shall use the orthonormalized set of eigenfunctions $\phi_n(x)$ which are defined by

$$y_n = C_n y_n(x)$$

where $C_n = \frac{1}{\rho_n}$ and

$$\int_a^b \phi_m(x)\,\phi_n(x)dx = 0, \quad m \neq n$$

Then from (83) by letting

$$\int_a^b \lambda\, g(x,s)\, y(s)ds = \sum_{n=1} a_n\, \phi_n(x)$$

Hence we get

$$y(x) = f(x) + \sum_{n=1} a_n\, \phi_n(x), \quad a \leq x \leq b$$

where

$$a_n = \int_a^b [y(s) - f(s)]\phi_n(s)ds$$

By defining

$$c_n = \int_a^b y(s)\,\phi_n ds, \quad \beta_n = \int_a^b f(s)\,\phi_n(s)ds$$

then

$$a_n = c_n - \beta_n$$

Multiplying (83) by $\phi_n(x)$ over (a,b) and using the symmetry of the kernel we obtain

$$[1 - \frac{\lambda}{\lambda_n}]\, c_n = \beta_n, \quad n = 1, 2, \cdots \tag{84}$$

case a. If $\lambda = \lambda_k$ where λ_k is of the eigenvalues then c_k becomes arbitrary and the solution and $\beta_k = 0$ or $\int_a^b f(s)\,\phi_k(s)ds = 0$ and the solution becomes

$$y(x) = f(x) + c_k\,\phi_k(x) + \lambda_k \sum_{n \neq k} \frac{\beta_n}{\lambda_n - \lambda}\,\phi_n(x) \tag{85}$$

case b. If $\lambda \neq$ any one of the eigenvalues. Then the solution is unique.

$$y(x) = f(x) + \lambda \sum_{n=1} \frac{\beta_n}{\lambda_n - \lambda}\phi_n(x)$$

4.4 Singular Integral Equations

So far we have assumed that **(i)** the interval (a,b) of the integral equations are finite was the kernel $g(x,s)$ was continiuous. If an integral equation has either an infinite interval or has a discontiniuous kernel then such an integral equation is called *singular integral equation.* The first two of the following are singular Fredholm equations of the first kind and the last one is the singular Volterra integral equation of the second kind (known also as the Abel's equation). Here $y(x)$ are unkown functions to be determined in each cases.

$$a) \quad F(x) = \int_0^\infty e^{-xs}\, y(s)ds, \tag{86}$$

$$b) \quad F(x) = \int_0^\infty \sin(xs)\, y(s)ds, \tag{87}$$

$$c) \quad F(x) = \int_0^x \frac{y(s)}{\sqrt{x-s}}ds \tag{88}$$

Here in each case $F(x)$ is a given function. The solutions of the homogeneous integral equations

$$y(s) = \lambda \int_a^b g(x,s)y(s)ds, \tag{89}$$

with

(i): $(a,b) = (0,\infty)$ or
(ii): $g(x,s)$ is not continuous in (a,b)

do not share the same properties as solutions of the homogeneous equations in the previous sections. For example the first singular equation above has continuous eigenvalues, the second one has two eigenvalues with infinite multiplicity and the last one can be solved exactly. Here we shall only present the first case. Recall the definition of the Gamma function

$$\int_0^\infty e^{-xs}\, s^{\alpha-1}\, ds = \Gamma(\alpha)\, x^{-\alpha}, \quad \alpha > 0 \tag{90}$$

Changing α to $1-\alpha$ and rewriting the above equation once more we get

$$\int_0^\infty e^{-xs}\, s^{-\alpha}\, ds = \Gamma(1-\alpha)\, x^{\alpha-1}, \quad \alpha < 1 \tag{91}$$

Dividing first one by $\sqrt{\Gamma(\alpha)}$ the second one by $\sqrt{\Gamma(1-\alpha)}$ and adding them we obtain

$$\int_0^\infty [\frac{1}{\sqrt{\Gamma(\alpha)}}\, s^{\alpha-1} + \frac{1}{\sqrt{\Gamma(1-\alpha)}}\, s^{-\alpha}] ds \tag{92}$$

$$= \sqrt{\Gamma(\alpha)}\, x^{-\alpha} + \sqrt{\Gamma(1-\alpha)}\, x^{\alpha-1}, \quad 0 < \alpha < 1 \tag{93}$$

This last equation gives us the eigenvalues and the eigenfunctions of the homogeneous singular Fredholm integral equatuion (89)

$$\lambda_\alpha = \frac{1}{\sqrt{\Gamma(\alpha)\,\Gamma(1-\alpha)}}, \tag{94}$$

$$y_\alpha = \sqrt{\Gamma(1-\alpha)}\, x^{\alpha-1} + \sqrt{\Gamma(\alpha)}\, x^{\alpha} \tag{95}$$

By using the identity

$$\Gamma(\alpha)\,\Gamma(1-\alpha) = \frac{\pi}{\sin\alpha\,\pi}, \quad 0 < \alpha < 1$$

then the eigenvalues become more simpler

$$\lambda_\alpha = \sqrt{\frac{\sin\alpha\pi}{\pi}}$$

As we observe that α takes any values in the interval $(0,1)$, hence eigenvalues take also continuous values.