In the theory of extrema of functions of single variable, a necessary condition for a minimum, besides $f'(x) = 0$, is that $f'' \geq 0$ (if it exists). A condition somewhat analogous to this holds for functionals. Let us suppose that there is an admissible function $y$, for which $J(y) = \int_a^b L(x, y, y') dx$ is a minimum. Then $f(\epsilon) = J(y + \epsilon h)$ has a minimum at $\epsilon = 0$; accordingly $f'(0) = 0$ (from which follows the Euler-Lagrange equation $E_y(L) = 0$ and $f''(0) \geq 0$, assuming its existence. Hence for all all $h$,

$$f''(0) = \int_a^b \left[ L_{yy} h^2 + 2L_{yy'} h h' + L_{yy'} h'^2 \right] dx \geq 0$$

Choose a special variation

$$h = \begin{cases} 
0 & a \leq x \leq \xi - \epsilon \\
1 + (x - \xi) / \epsilon & \xi - \epsilon \leq x \leq \xi \\
1 - (x - \xi) / \epsilon & \xi \leq x \leq \xi + \epsilon \\
0 & \xi + \epsilon \leq x \leq b 
\end{cases}$$

If we substitute this function in the above expression an let $\epsilon \to 0$, then the term

$$\frac{1}{\epsilon^2} \int_{\xi-\epsilon}^{\xi+\epsilon} L_{yy'} dx$$

will dominate the left hand side of the above inequality and determine its sign. Thus the the sign of $L_{yy'}$ determines the sign of $f''(0)$, and for minimum, the Legendre condition $L_{yy'} \geq 0$ must hold. We have the Legendre test:

**Theorem 1.** The Legendre test. If

(i) Euler-Lagrange equation is satisfied,
(i) the range of integration is sufficiently small,

(iii) the sign of $L_y y'$ is constant throughout this range,
then $J(y)$ is a minimum or a maximum value of $J$ according as the sign of $L_y y'$ is positive or negative.

The second variation: Let $y = y(x)$ be the path of integration for which the integral $J(y) = \int_a^b L(x, y, y')\,dx$ is minimum or maximum (sometimes called the integral is stationary). Consider all admissible functions $y + \varepsilon h$ where both $y$ and $h$ belong to $C^2[a, b]$. Then

$$J(y + \varepsilon h) = J(y) + \frac{\varepsilon^2}{2} J_2 + O(\varepsilon^3)$$

where

$$J_2 = \int_a^b \left[ \varepsilon^2 \frac{\partial^2 L}{\partial y^2} + 2 \varepsilon h \frac{\partial^2 L}{\partial y \partial y'} + \varepsilon^2 \frac{\partial^2 L}{\partial y'^2} \right] \,dx$$

If $J(y)$ is minimum (or maximum) the sign of $J_2$ must be positive (or negative) without depending upon the choice of $h(x)$, for all sufficiently small $\varepsilon$. For simplicity we use the following notation: $L_0 = \frac{\partial L}{\partial y}$, $L_1 = \frac{\partial L}{\partial y'}$, $L_{00} = \frac{\partial^2 L}{\partial y^2}$, $L_{01} = \frac{\partial^2 L}{\partial y \partial y'}$, $L_{11} = \frac{\partial^2 L}{\partial y'^2}$. Then $J_2$ takes the form

$$J_2 = \int_a^b \left[ L_{00} \varepsilon^2 + 2 L_{01} \varepsilon h + L_{11} \varepsilon^2 \right] \,dx$$

We have the following Lemma. Proof is straightforward (done in the class)

**Lemma 2.** Let $h(a) = h(b) = 0$, then

$$J_2 = \int_a^b \left\{ \varepsilon^2 \left( L_{00} - \frac{d}{dx} L_{01} \right) - \varepsilon \frac{d}{dx} \left( L_{11} \varepsilon^2 \right) \right\} \,dx$$

**Definition.** (Jacobi equation and Jacobi field). Let $u \in C^2[a, b]$. Then the following equation is called the Jacobi equation.
\[
\{L_{00} - \frac{d}{dx}(L_{01})\} u - \frac{d}{dx}(L_{11} \frac{du}{dx}) = 0
\]

the function \( u \) satisfying this second order ODE is called the Jacobi field

**Lemma 3.** If \( h(a) = h(b) = 0 \) and is a Jacobi field then

\[
J_2 = \int_a^b L_{11} [h' - h \frac{u'}{u}]^2 \, dx,
\]

If the term \( \{h' - h (\frac{u'}{u})\} \neq 0 \) and \( L_{11} \) has constant sign for all points of the extremal arc \( AB \), where \( A = (a, y(a)) \) and \( B = (b, y(b)) \), then \( J_2 \) must have a sign which is independent of the choice of \( h \). Now in the extremal case

\[
J(y + \varepsilon h) - J(y) = \frac{\varepsilon^2}{2} J_2 + O(\varepsilon^3)
\]

Hence \( J \) takes its minimum value (or maximum value) if \( L_{11} \) is positive (or negative) at all points on the extremal curve \( y(x) \) with \( x \in [a, b] \). This is essentially the Legendre test stated above.

If, however, \( L_{11} \), does not keep its sign constant at all \( x \in [a, b] \) on the curve \( y(x) \), then the value \( J(y) \) is neither a minimum nor a maximum.

If the term \( \{h' - h (\frac{u'}{u})\} \) vanishes at all points of the extremal curve \( y(x) \). Then it is clear that \( h(x) = \alpha u(x) \), where \( \alpha \) is an arbitrary constant. Along the extremal curve \( y(x) \) the first variation vanishes. If in addition \( h(x) = \alpha u(x) \) is chosen then the second variation, \( J_2 \) vanishes as well. The sign of

\[
\delta J = J(y + \varepsilon h) - J(y)
\]

will depend on the third variation \( \delta J = \frac{\varepsilon^3}{6} J_3 \), where

\[
J_3 = \int_a^b [h^3 L_{000} + 3h^2 h' L_{001} + 3hh'^2 L_{011} + h'^3 L_{111}] \, dx
\]

Since the sign of \( \delta J \) depends on that of \( \varepsilon \) there can be no maximum or minimum value of \( J \) unless \( J_3 \) vanishes, in which case the sign of \( \delta J \) will depend on that of \( J_4 \), the fourth variation. In order to avoid this difficulty
Jacobi proposed a test. This test provides whether the $h(x) = \alpha u(x)$ at all points of the extremal curve $y(x)$.

**Definition:** Let $u(x)$ be a solution of the Jacobi equation. Let $u(a) = 0$. This means that the Jacobi field vanishes at the point $A = (a, y(a))$. The all other points on the extremal curve $y(x)$ at which $u(x)$ vanishes are called the *conjugate points* to the point $A$.

The jacobi field (a function satisfying the Jacobi equation with $u(a) = 0$) may be given as the linear sum of the fundamental solutions $u_1$ and $u_2$ of the Jacobi equation.

$$u(x) = a_1 u_1(x) + a_2 u_2(x)$$

where $a_1$ and $a_2$ are constants. Since $u(a) = 0$, then

$$\frac{u_1(a)}{u_2(a)} = \frac{a_1}{a_2}$$

Hence if $x$ is the abscissa of the conjugate point to $A$ ($u(x) = 0$) then

$$\frac{u_1(x)}{u_2(x)} = \frac{a_1}{a_2}$$

It shows that this ratio is the same for all conjugate points. This is the way obtaining the conjugate points. We now give a nice way of determining the fundamental solutions $u_1$ and $u_2$. Since the function $y(x)$, the solution of the Euler-Lagrange equation is second order ODE then it contains two independent constants of integration $y(x, c_1, c_2)$. It is easy to prove the following Lemma.

**Lemma 4.** $u_i = \frac{\partial y}{\partial c_i}$, $i = 1, 2$.

Hence the conjugate points are easily determined from the following formula. Let $Y = (\frac{\partial y}{\partial c_1})/(\frac{\partial y}{\partial c_2})$ then

$$Y(x) = Y(a)$$
Let $A'$ be the first conjugate point to $A$ when moving on the extremal curve $y(x)$ in the direction from $A = (a, y(a))$ to $B = (b, y(b))$. There are three possibilities

(i) $B$ lies between $A$ and $A'$,
(ii) $B$ coincides with $A'$,
(iii) $B$ lies beyond $A'$.

In all three cases $u(x)$ vanishes at $A$, but in case (i) it can not vanish again at the point $B$, in case (ii) it vanishes again at $B$, and in case (iii) it vanishes again at some point of the arc $AB$ lying between $A$ and $B$. In case (i) we have $u(a) = h(a) = 0$ but $u(b) \neq 0$, $h(b) = 0$. Hence $u(x)$ can not be proportional to $h(x)$ at all points of the extremal arc $AB$. Therefore we must have $(h' - h\frac{u'}{u})^2 > 0$ at all points of $AB$, except, possibly, at finite number of points where $h(x)$ and $h'(x)$ vanish simultaneously. Then we have the following theorem *Jacobi test.*

**Theorem 5.** Let $y = y(x)$ be the curve connecting the points $A$ and $B$ for which the functional $J(y) = \int_a^b L(x, y, y')dx$ takes its extremum value. Let $A$ and $A'$ be the conjugate points on this curve. If (i) $B$ lies between $A$ and $A'$ and (ii) $L_{11}$ has constant sign for all points on the arc $AB$, then $J(y)$ is a minimum (maximum) when $L_{11} > 0$ ($L_{11} < 0$)

For all the other cases we need higher variations for extremum tests.