

MATH 544. METHODS OF APPLIED MATHEMATICS

Spring 2012 Lecture 7

Stability and Bifurcation.

1. Stability in two dimensions
2. Stability in one dimension and bifurcation.

(1)

1. Stability in two dimensions:

let $(x, y) \in \mathbb{R}^2$ and $t \in I$ so that
 $x = x(t)$, $y = y(t)$ satisfying the system of
 ODE

$$\left. \begin{aligned} \frac{dx}{dt} &= \dot{x} = P(x, y) \\ \frac{dy}{dt} &= \dot{y} = Q(x, y) \end{aligned} \right\} \quad (1)$$

where P and Q are given functions of x and y .
 We assume that P and Q have continuous partial
 derivatives with respect to x and y at all orders in
 a domain $D \subseteq \mathbb{R}^2$ (the xy plane).

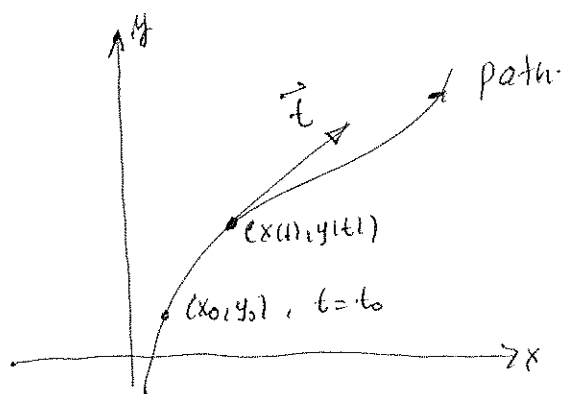
Lemma: Let P and Q have continuous partial
 derivatives in a region D of the xy -plane at all
 orders then the initial value problem

$$\left. \begin{aligned} \dot{x} &= P(x, y), \quad \dot{y} = Q(x, y), \quad t \in I \\ x(t_0) &= x_0, \quad y(t_0) = y_0, \quad t_0 \in I \end{aligned} \right\} \quad (2)$$

has a unique solution, where $t_0 \in I$ and
 $(x_0, y_0) \in D$

(2)

The variables x and y are called the "state variable".
 In general $x = x(t)$, $y = y(t)$ define a curve in the xy -plane which is called the "phase plane". The curve is called the path or trajectory.



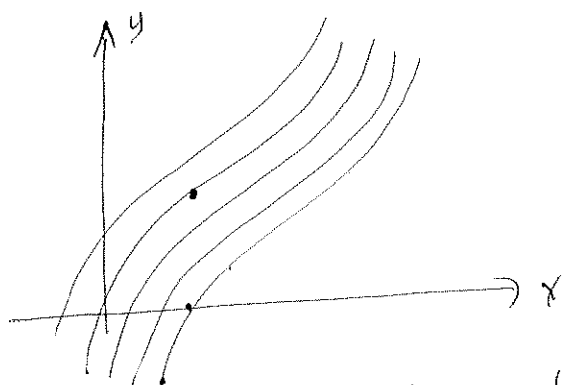
"phase plane"

$\vec{t}$ tangent vector

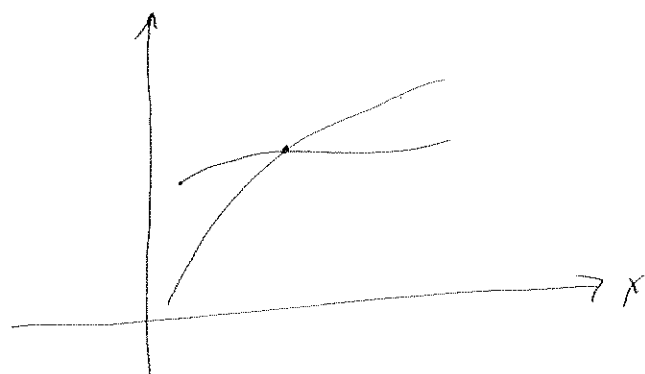
$$= \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = (P, Q)$$

at any point $(x(t), y(t))$ on the curve.

Corollary of the Lemma: At most one path passes through each point of the phase plane and all of the paths cover the entire phase plane without intersecting each other.



paths cover the entire phase plane



Two paths cannot intersect. It violates the uniqueness lemma. At the intersecting point there are two different solutions of (2).

Equilibrium or Steady state solutions:

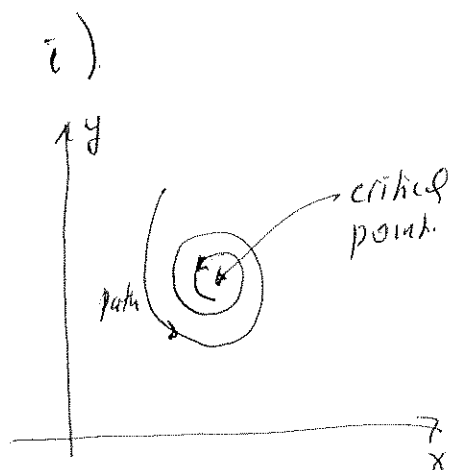
(3)

Let $x(t) = x_1$, $y(t) = y_1$, $\forall t \in I$ where x_1 and y_1 are constants and $Q(x_1, y_1) = 0$, $P(x_1, y_1) = 0$ at the same time then (x_1, y_1) is called a critical point.

Remark: It is clear that the critical point can not be on any paths otherwise uniqueness would be violated.

phase portrait: critical points of (1), and qualitative behaviour of all the paths in the phase plane is determined to a large extent by the location of the critical point and the local behaviour of the paths near those points.

We have the following behaviours:



A path can not approach a critical point in a finite time, that is, if a path approaches a critical point then necessarily $t \rightarrow \pm \infty$.

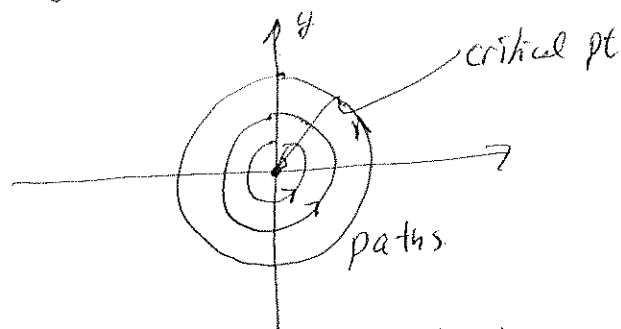
(4)

Examples (1) $\dot{x} = y$
 $\dot{y} = -x$

Solution: $\ddot{x} = \dot{y} = -x \Rightarrow x(t) = a \cos t + b \sin t$
 $\Rightarrow y(t) = -a \sin t + b \cos t$

Critical point: $(0,0)$

paths: $x^2 + y^2 = a^2 + b^2$



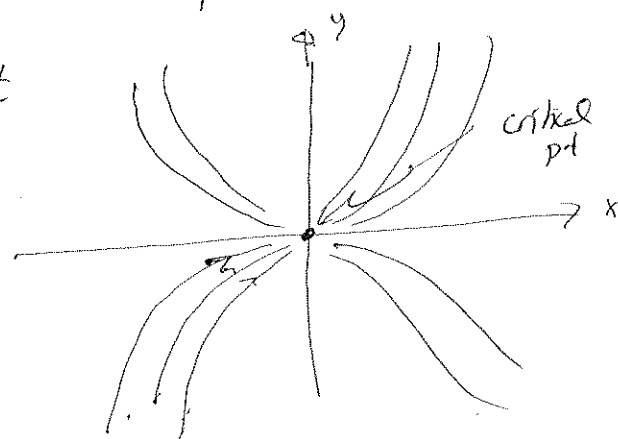
phase portrait

(2) $\dot{x} = 2x, \dot{y} = 3y$

$x = c_1 e^{2t}, y = c_2 e^{3t}$

paths: $y^2 = c x^3$

critical point: $(0,0)$



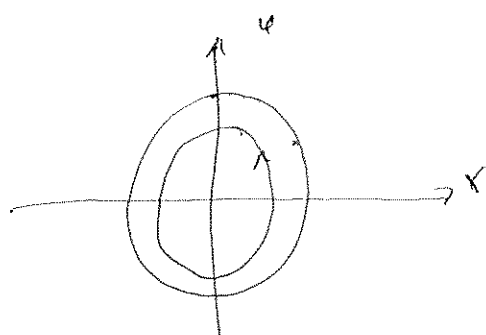
paths.

phase portrait
(a node)

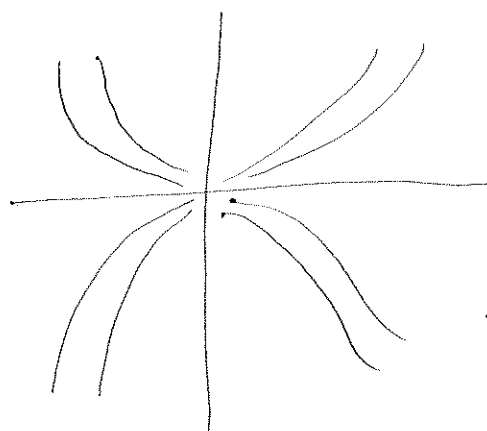
paths approach the critical point $(0,0)$ as $t \rightarrow -\infty$.

Types of Critical Points

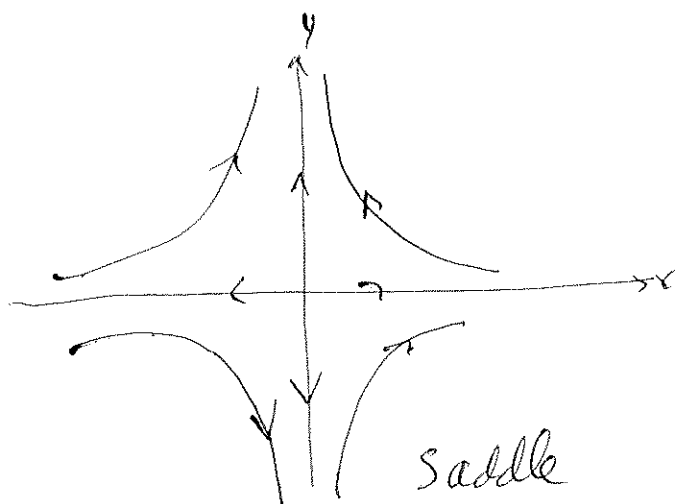
Center, node, saddle and spiral



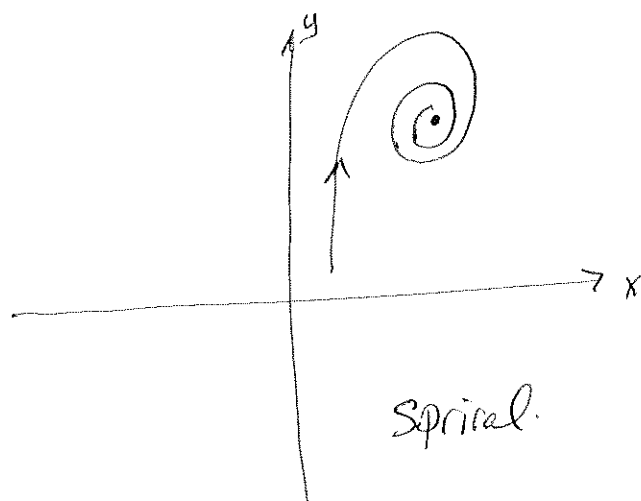
center



node



saddle



spiral.

A critical point is stable if all paths that start sufficiently close to the point remain close to the point.

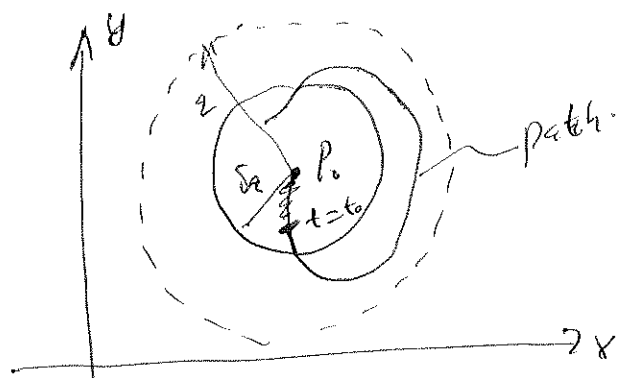
Let $P_0 = (0,0)$ be the isolated critical point. P_0 is stable if for each $\epsilon > 0$ there exists a positive number δ_ϵ such that

$$|u_0 - u(t)| < \delta_\epsilon \quad \text{whenever} \quad |u(t) - u_0| < \epsilon.$$

$\forall t \in I$. Every path inside the circle of

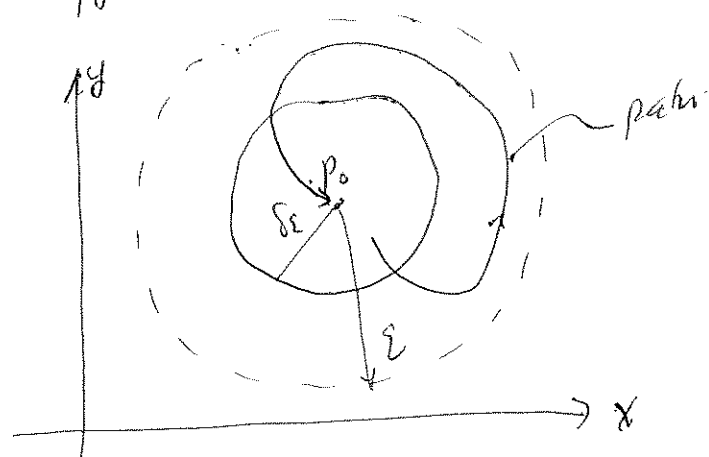
(6)

of radius δ_ϵ at $t=t_0$ remain inside the circle of radius ϵ for all $t \geq t_0$



Stable critical points

The critical point is asymptotically stable if it is stable and there exist a circle of radius δ_ϵ such that every path that is inside the circle at $t=t_0$ approaches P_0 as $t \rightarrow \infty$.



Asymptotically stable critical points

(7)

Center: stable but not asymptotically stable
 Saddle: unstable
 Spiral: either asymptotically stable or unstable
 node: " "
 (stable or unstable).

Stability Analysis:

Let $P_0 = (x_0, y_0)$ be an isolated critical point.

Let

$$x(t) = x_0 + \bar{x}(t), \quad y(t) = y_0 + \bar{y}(t)$$

Then (1) becomes

$$\frac{d\bar{x}}{dt} = P(\bar{x}, \bar{y}), \quad \frac{d\bar{y}}{dt} = Q(\bar{x}, \bar{y}).$$

Since P and Q have continuous partial derivatives in a region $D \subseteq \mathbb{R}^2$ at all orders then using

Taylor's expansion

$$P(\bar{x}, \bar{y}) = P(x_0, y_0) + P_x(x_0, y_0)\bar{x} + P_y(x_0, y_0)\bar{y} + O(\bar{x}^2 + \bar{y}^2)$$

$$Q(\bar{x}, \bar{y}) = Q(x_0, y_0) + Q_x(x_0, y_0)\bar{x} + Q_y(x_0, y_0)\bar{y} + O(\bar{x}^2 + \bar{y}^2)$$

Let $a = P_x(x_0, y_0)$, $b = P_y(x_0, y_0)$, $c = Q_x(x_0, y_0)$, $d = Q_y(x_0, y_0)$

$\Rightarrow$

Linearised perturbation equations are

(8)

$$\frac{d\bar{x}}{dt} = a\bar{x} + b\bar{y}, \quad \frac{d\bar{y}}{dt} = c\bar{x} + d\bar{y} \quad (4)$$

Let

$$u = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\Rightarrow$ (4) becomes

$$\frac{du}{dt} = A u, \quad (5)$$

We assume that $\det A \neq 0$. Hence at the critical point (x_0, y_0) $Au = 0 \Rightarrow u = 0$.

In another words the critical point of the linearised perturbation equation (4) is $u = 0$.
On the other hand the characteristic equation of A is

$$\det(A - \lambda I) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$$

From Cayley-Hamilton theorem this implies

$$A^2 - (\lambda_1 + \lambda_2)A + \lambda_1\lambda_2 I = 0$$

where λ_1 and λ_2 are the eigenvalues of A

Take, now, one more + derivative of (5)

(9)

$$\frac{d^2 u}{dt^2} = A \frac{du}{dt} = A^2 u$$

$$= [(\lambda_1 + \lambda_2) A - \lambda_1 \lambda_2 I] u$$

$$= (\lambda_1 + \lambda_2) \frac{du}{dt} - \lambda_1 \lambda_2 u$$

(6)

$$\ddot{u} - (\lambda_1 + \lambda_2) \dot{u} + \lambda_1 \lambda_2 u = 0$$

$$\text{Let } D = \frac{d}{dt} \Rightarrow$$

(7)

$$(D - \lambda_1)(D - \lambda_2) u = 0$$

$$a) \text{ If } \lambda_1 \neq \lambda_2 \Rightarrow u = u_1 + u_2 \quad \text{where}$$

$$(D - \lambda_1) u_1 = 0 \Rightarrow u_1 = \alpha_1 e^{\lambda_1 t}$$

$$(D - \lambda_2) u_2 = 0 \Rightarrow u_2 = \alpha_2 e^{\lambda_2 t}$$

Then

$$u = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t}$$

Here α_1 and α_2 parallel to the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ of A corresponding to the eigenvalues λ_1 and λ_2 respectively

$$u = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

(8)

(10)

v_1 and v_2 normalised eigenvectors of A corresponding to λ_1 and λ_2 respectively, i.e.

$$A v_1 = \lambda_1 v_1 \quad \|v_1\| = 1$$

$$A v_2 = \lambda_2 v_2 \quad \|v_2\| = 1$$

and c_1 and c_2 are constants. They are determined by the use of initial conditions.

b) If $\lambda_1 = \lambda_2 \Rightarrow$ (7) becomes

$$(D - \lambda_1)^2 u = 0$$

$$\text{let } (D - \lambda_1)u = w \Rightarrow$$

$$(D - \lambda_1)w = 0$$

which has the solution

$$w = a e^{\lambda_1 t}, \quad a \text{ constant vector}$$

Hence

$$(D - \lambda_1)u = a e^{\lambda_1 t}$$

$$\frac{d}{dt} (e^{-\lambda_1 t} u) = a$$

$$e^{-\lambda_1 t} u = at + b, \quad b \text{ is a constant vector} \quad (13)$$

or

$$u = (b + at)e^{\lambda_1 t}$$

This must satisfy $\frac{du}{dt} = Au$.

$$ae^{\lambda_1 t} + \lambda_1 (b + at)e^{\lambda_1 t} = (Ab + t(Aa))e^{\lambda_1 t}$$

$$a + \lambda_1 (b + at) = Ab + t(Aa)$$

$$\Rightarrow \text{ i) } Aa = \lambda_1 a$$

a is the eigenvector of A corresponding to λ_1

$$\text{ ii) } Ab = a + \lambda_1 b \Rightarrow$$

$$a = Ab - \lambda_1 b$$

There are two cases:

1) A has two independent eigenvectors v_1 and v_2 for a given eigenvalue λ_1

$$a = 0, \quad b = c_1 v_1 + c_2 v_2$$

$$u = (c_1 v_1 + c_2 v_2)e^{\lambda_1 t}$$

2) A has only one eigenvector for λ_1

$$Aa = \lambda_1 a, \quad a = Ab - \lambda_1 b$$

$$u = (b + at)e^{\lambda_1 t}$$

Classification of the critical point with respect to the eigenvalues λ_1 and λ_2 .

Case - 1

$$\lambda_2 > \lambda_1 > 0$$

(both distinct and positive).

$$u = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

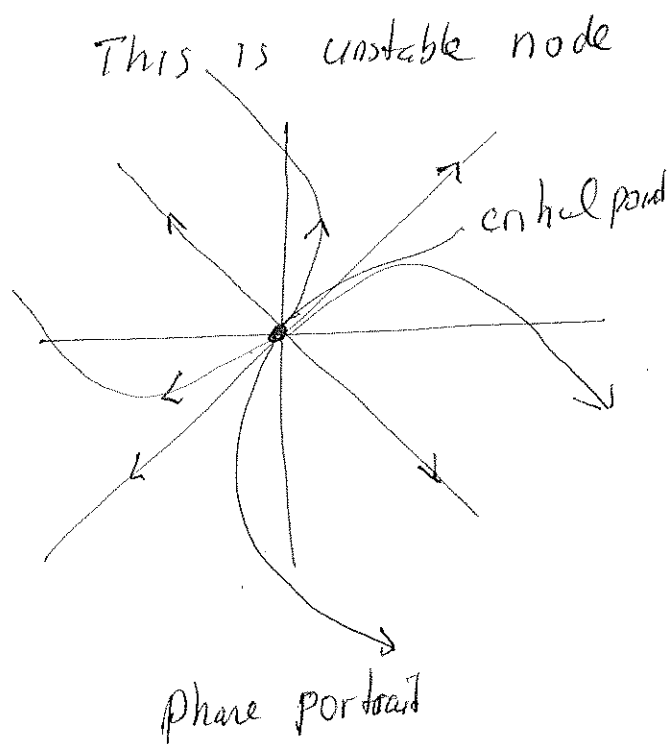
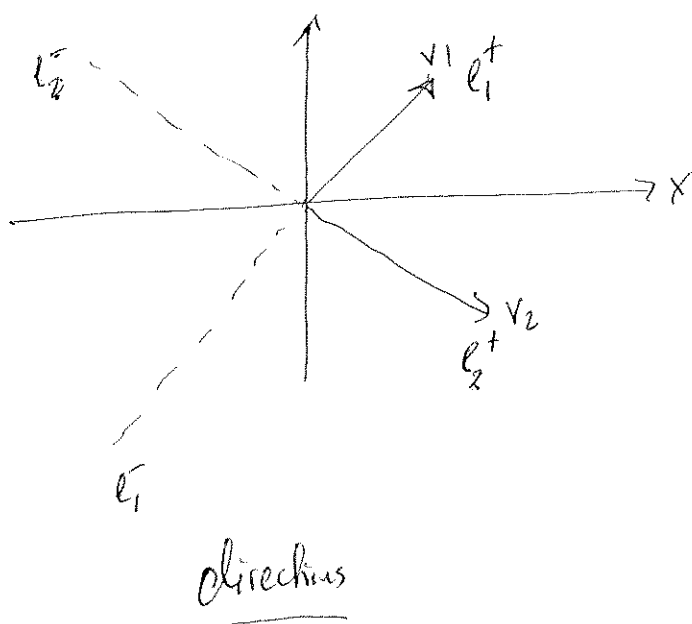
v_1 and v_2 are directions for $\pm \infty$.

as $t \rightarrow -\infty$ direction is v_1

as $t \rightarrow +\infty$ " " v_2 .

$$x(t) = a_1 e^{\lambda_1 t} + b_1 e^{\lambda_2 t}$$

$$y(t) = a_2 e^{\lambda_1 t} + b_2 e^{\lambda_2 t}$$

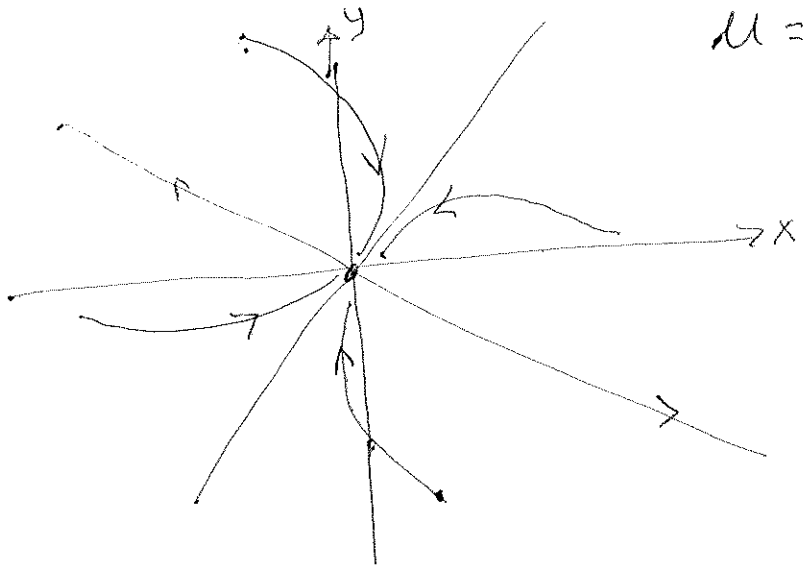


Case II. $\lambda_2 < \lambda_1 < 0$

both real distinct and negative

This is a stable node

$$u = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

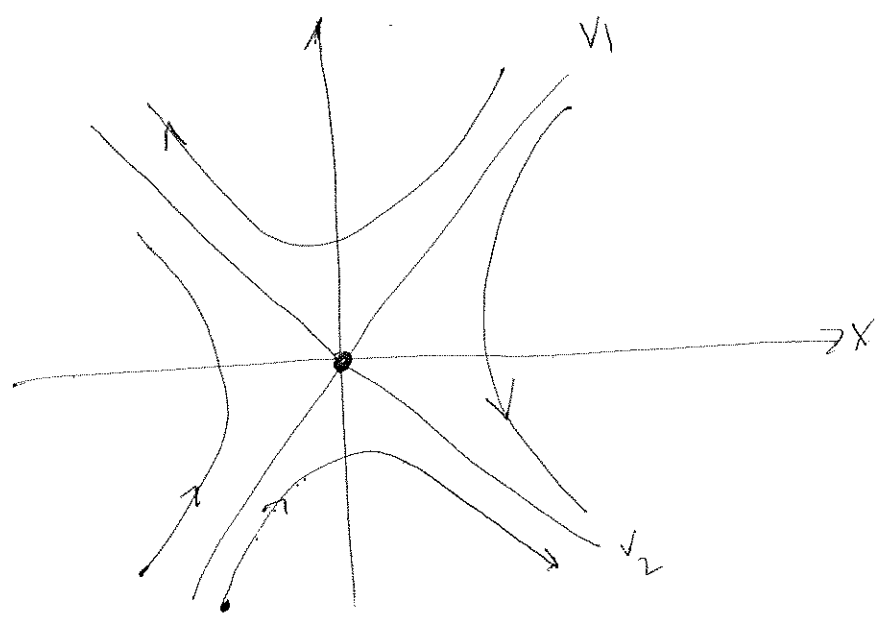


Case III. $\lambda_1 < 0 < \lambda_2$

real distinct, opposite sign

$$u = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

Saddle point



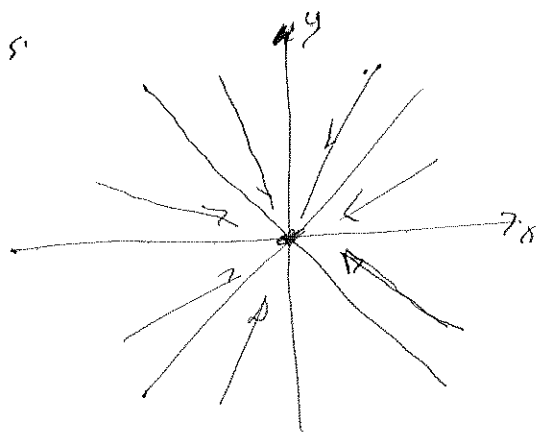
Solution $v_1 e^{\lambda_1 t}$ is asymptotically stable all others are asymptotic to the half lines.

Case IV: $\lambda_1 = \lambda_2 < 0$

a) There exist two eigenvectors:

$$u = (c_1 v_1 + c_2 v_2) e^{\lambda_1 t}$$

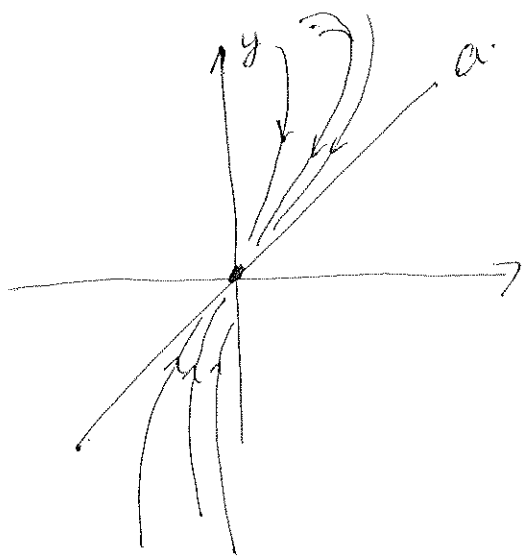
"stable node"



b) Only one eigenvector

$$u = (b + at) e^{\lambda_1 t}$$

$$Aa = \lambda_1 a, \quad \dot{a} = Aa - \lambda_1 a$$

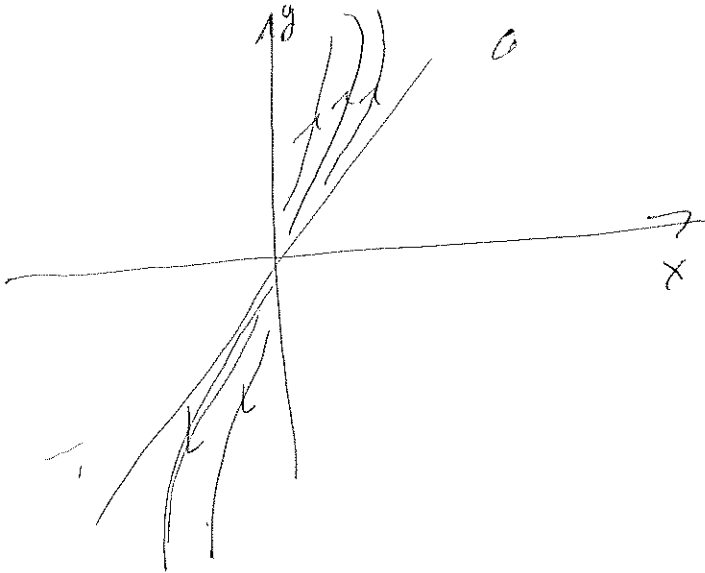


stable node

Case V. $\lambda_1 = \lambda_2 > 0$ (real, equal, positive) (17)

like case CIV but the directions are reversed

unstable node



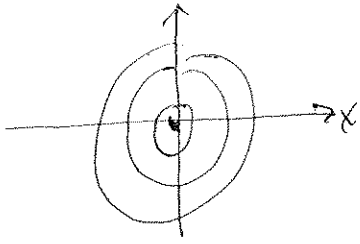
Case VI complex eigenvalues $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \bar{\lambda}_1$

eigen vectors $\lambda_1 \rightarrow (\omega + i\nu)$
 $\lambda_2 \rightarrow (\omega - i\nu)$

$$u = c_1 e^{\alpha t} (\omega \cos \beta t - \nu \sin \beta t) + c_2 e^{\alpha t} (\omega \sin \beta t + \nu \cos \beta t)$$

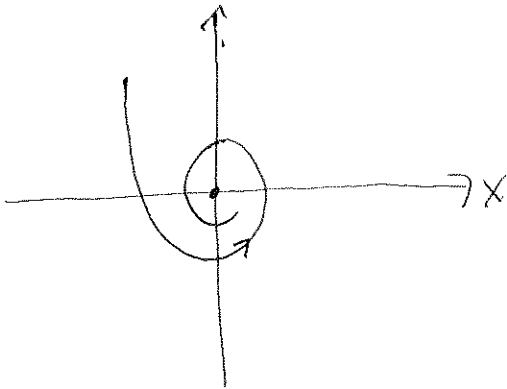
proof: soln. is $u = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$

where $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, $v_1 = \omega + i\nu$
 $v_2 = \omega - i\nu$ then the result follows

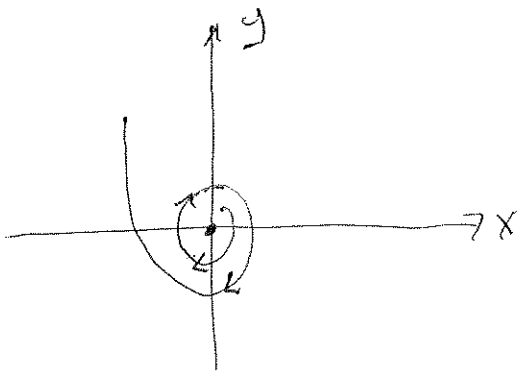


$\alpha = 0$ closed pat with
period $2\pi/\beta$.

P_0 is a center



$\alpha < 0$ stable spirals wind
about the origin.



$\alpha > 0$ unstable spiral

Theorem: The critical point $P_0(0,0)$ of the linear system $\frac{du}{dt} = Au$, $\det A \neq 0$ is stable iff the eigenvalues of A have nonpositive real parts. If it is asymptotically stable if and only if the eigenvalues have negative real parts.

Example (1).

$$\dot{x} = 3x - 2y$$

$$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

$$\dot{y} = 2x - 2y$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} = (3-\lambda)(-2-\lambda) + 4$$

$$= \lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = -1$$

$$A v_1 = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{cases} 3a - 2b = 2a \\ 2a - 2b = 2b \end{cases} \Rightarrow a = 2b$$

$$v_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$A v_2 = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -1 \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{cases} 3a - 2b = -a \\ 2a - 2b = -b \end{cases} \Rightarrow b = 2a$$

$$v_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$u = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

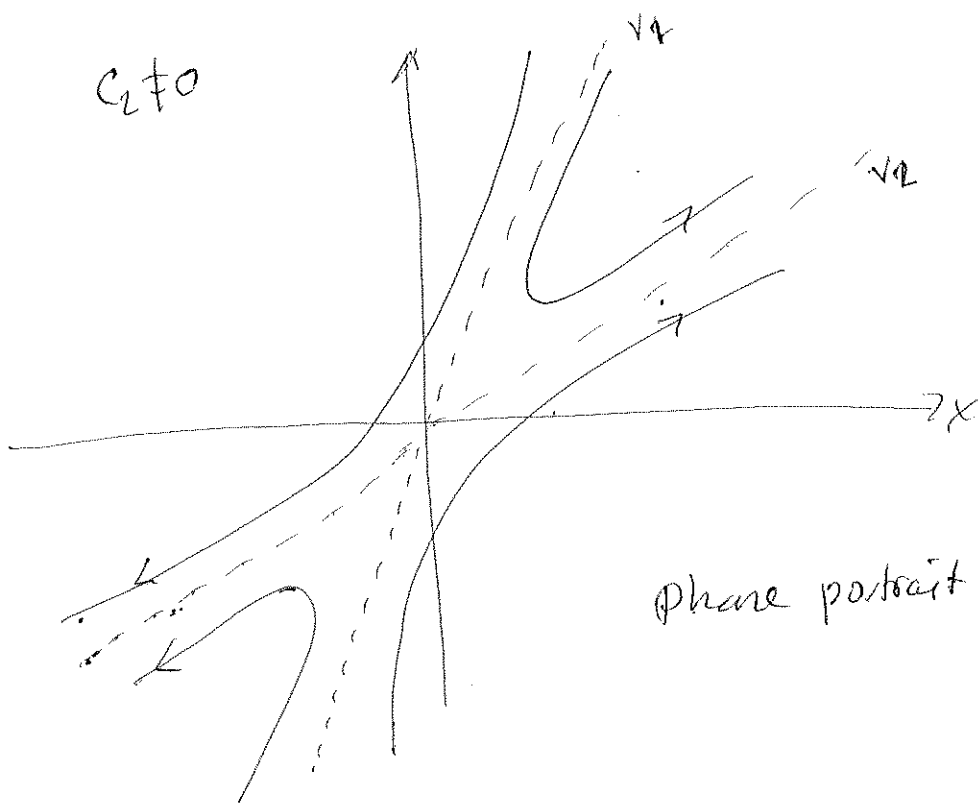
$$x(t) = c_1 e^{-t} + c_2 e^{2t}$$

$$y(t) = 2c_1 e^{-t} + c_2 e^{2t}$$

$$\text{if } c_2 = 0$$

(0,0) is a stable node

$c_2 \neq 0$



$(0,0)$ is a saddle point.

Phase portrait of the example

(20)

Stability and Bifurcations

(21)

One dimensional case

We shall consider a first order differential equation

$$\frac{du}{dt} = f(\mu, u), \quad t > 0 \quad (1)$$

where μ is a real parameter, f is a given function having continuous partial derivatives at all orders (Here three times will be enough)

The unknown function is $u = u(t, \mu)$. The one value of parameter μ may correspond to the several equilibrium solutions (critical points)

$$f(\mu, u) = 0 \quad (2)$$

Each solution of this equation gives the equilibrium solution of (1). To each point (μ_0, u_0) on the locus of (2) there corresponds an equilibrium solution $u = u_0$ at the particular value of μ_0 (Bifurcation)

A graph of the locus of (2) in the μu plane is called the bifurcation diagram or branching

The interesting branches of (2) are the bifurcating solutions and the intersecting points which are called the bifurcating points

Example

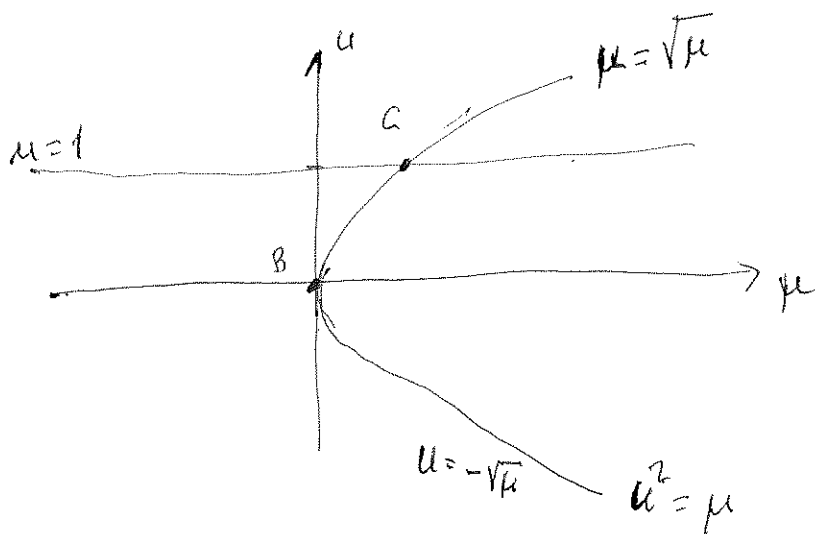
$$\frac{du}{dt} = (1-u)(u^2-\mu), \quad \mu \in \mathbb{R}, t > 0$$

$$f(\mu, u) = (1-u)(u^2-\mu)$$

Equilibrium solutions

$$f(\mu, u) = 0 \Rightarrow u = 1, u^2 = \mu$$

Bifurcating diagram



Bifurcating solutions

$$u = 1,$$

$$u = \sqrt{\mu}$$

$$u = -\sqrt{\mu}.$$

Bifurcating points are B and C (intersecting points)

stability: let u_0 be an equilibrium solution of (1). This solution is stable if every solution $u(t)$ of (1) starting sufficiently close to u_0 at $t=0$, $|u_0 - u(0)| < \delta_\varepsilon$, remains close to u_0 for all $t > 0$, i.e.,

given $\varepsilon > 0$ there exists a parameter $\delta_\varepsilon > 0$ such that for all $t > 0$, $|u(t) - u_0| < \varepsilon$ whenever $|u_0 - u(0)| < \delta_\varepsilon$

Asymptotic stability: If in addition to being stable

$$\lim_{t \rightarrow \infty} |u(t) - u_0| = 0 \quad (3)$$

for every solution $u(t)$ starting sufficiently close to u_0 at $t=0$, then we say that u_0 is asymptotically stable

(24)

An equilibrium solution is called unstable if it is not stable. In all cases above we fix μ .

Stability Analysis

let μ be a fixed parameter and let u_0 be an equilibrium solution of (1) for this fixed value of μ .

consider a small perturbation $\bar{u}(t)$ about the equilibrium solution, i.e.,

$$u(t) = u_0 + \bar{u}(t) \quad (4)$$

Then (1) becomes

$$\frac{d\bar{u}}{dt} = f(\mu, u_0 + \bar{u}(t)) \quad (5)$$

since f is differentiable then we expand it about $u = u_0$,

$$f(\mu, u_0 + \bar{u}) = f(\mu, u_0) + \bar{u} f_u(\mu, u_0) + \frac{1}{2} \bar{u}^2 f_{uu}(\mu, u_0) + \dots$$

Linearized perturbation: Neglecting the higher order terms we get

(25)

$$\frac{d\bar{u}}{dt} = \alpha \bar{u}, \quad \alpha = f_u(\mu, u_0), \quad t > 0 \quad (6)$$

Here, since u_0 is the equilibrium solution for fixed value of μ , then $f(\mu, u_0) = 0$ by definition.

Eq. (6) has the solution

$$\bar{u}(t) = C e^{\alpha t}, \quad t > 0 \quad (7)$$

where $C = \bar{u}(0)$ which is a constant.

i) if $\alpha > 0$ u_0 is asymptotically unstable (hence unstable)

ii) if $\alpha < 0$ u_0 is asymptotically stable (hence stable)

α is called the stability indicator.

Theorem . Let u_0 be an equilibrium solution of (1) and assume that

$$f(\mu, u_0 + \bar{u}) = f_u(\mu, u_0) \bar{u} + R(u_0, \bar{u})$$

where the remainder term $R(u_0, \bar{u})$ is $O(\bar{u}^2)$

or $|R(u_0, \bar{u})| \leq K|\bar{u}|^2$, $K > 0$ then u_0

is asymptotically stable if $\alpha < 0$ and unstable if $\alpha > 0$ with $\alpha = f_u(\mu, u_0)$ (26)

Example:

$$\frac{du}{dt} = \mu u - u^2, \quad t > 0$$

Equilibrium solutions: $u=0, u=\mu$

1) $u_0 = 0, \quad f(\mu, u) = \mu u - u^2, \quad f_u = \mu - 2u$

$$f_u(\mu, u_0) = \mu - 2u_0 \quad \text{when } u_0 = 0 \Rightarrow f_u(\mu, u_0) = \mu$$

hence $\alpha = \mu$.

The equilibrium solution $u=0$ is

stable if $\mu < 0$

unstable if $\mu > 0$

2) $u_0 = \mu \Rightarrow \alpha = -\mu$.

Then the equilibrium solution $u=\mu$ is

stable if $\mu > 0$

unstable if $\mu < 0$

If $\mu = 0$, then

$$\frac{du}{dt} = -u^2, \quad u=0 \text{ is the equilibrium solution. } \Rightarrow \alpha=0 \text{ (stable!)}$$

Solution:

$$u(t) = \frac{1}{t-t_0}$$

$$\lim_{t \rightarrow \infty} |u(t) - u_0| = \lim_{t \rightarrow \infty} \frac{1}{|t-t_0|} = 0 \quad \text{stable}$$

but singular at $t = t_0$.

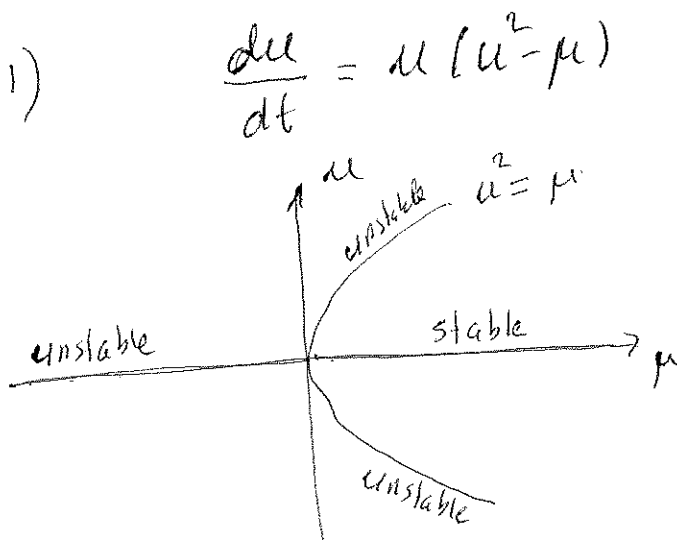
(28)

Classification of Bifurcating Points.

Exchange of stability

Is there a critical value μ_c of the parameter μ where $\mu < \mu_c$ and $\mu > \mu_c$ correspond to different stability properties?

Examples



Bifurcation Portrait

$$\begin{aligned} u &= 0 \\ f_u &= 3u^2 - \mu \\ \alpha &= -\mu \end{aligned}$$

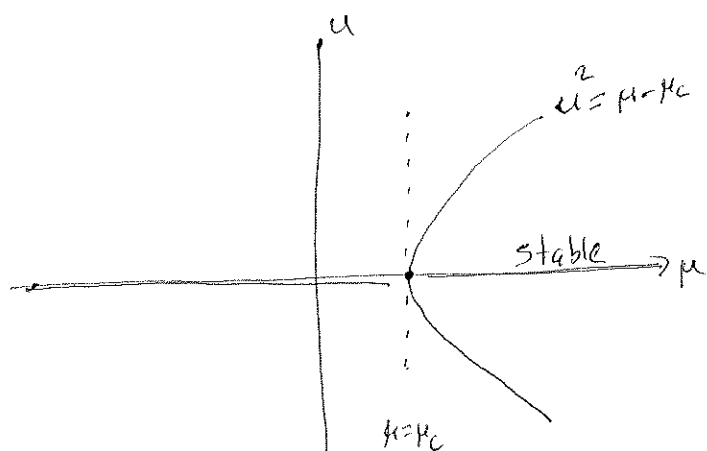
Equilibrium solution $u=0$
is stable for $\mu > 0$
unstable for $\mu < 0$

$$u^2 = \mu \quad (\mu > 0)$$

$$\begin{aligned} f_u &= 2\mu \\ \alpha &= 2\mu \end{aligned}$$

both branches are unstable

$$ii) \quad \frac{du}{dt} = u(u^2 - \mu + \mu_c)$$



$$u = 0$$

$$f_u = 3u^2 - \mu + \mu_c$$

$$\alpha = -\mu + \mu_c$$

stable for $\mu > \mu_c$

unstable for $\mu < \mu_c$

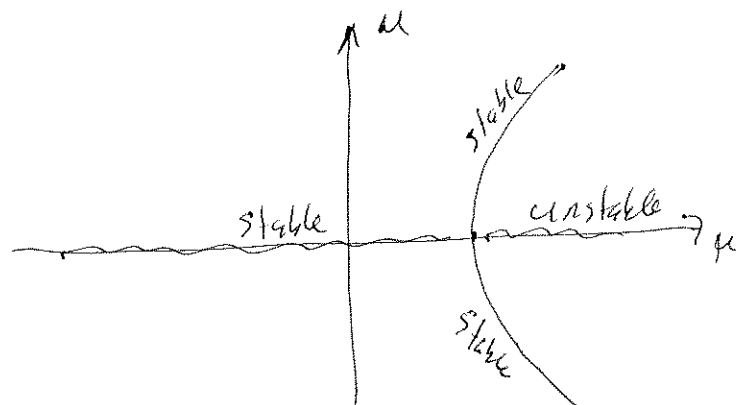
$$u^2 = \mu - \mu_c$$

$$\alpha = 2(\mu - \mu_c)$$

unstable because $\mu > \mu_c$

~~To find the equilibrium solutions we need some requirements on the function f.~~

$$iii) \quad \frac{du}{dt} = u(\mu - \mu_c - u^2)$$



$$u = 0$$

$$f_u = -3u^2 + \mu - \mu_c$$

$$\alpha = \mu - \mu_c$$

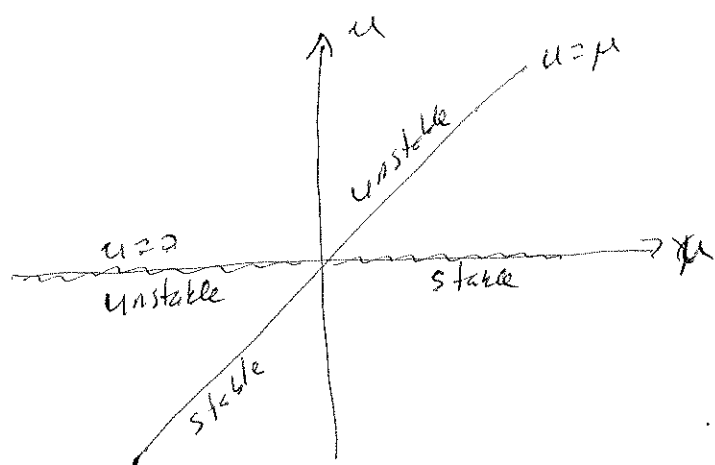
$$u^2 = \mu - \mu_c$$

$$\alpha = -2(\mu - \mu_c)$$

stable because $\mu > \mu_c$

iv)

$$\frac{du}{dt} = u(u - \mu) = u^2 - \mu u$$



$$u = 0$$

$$f_u = 2u - \mu$$

$$\alpha = -\mu$$

stable for $\mu > 0$

unstable for $\mu < 0$

$$u = \mu$$

$$f_u = 2u - \mu$$

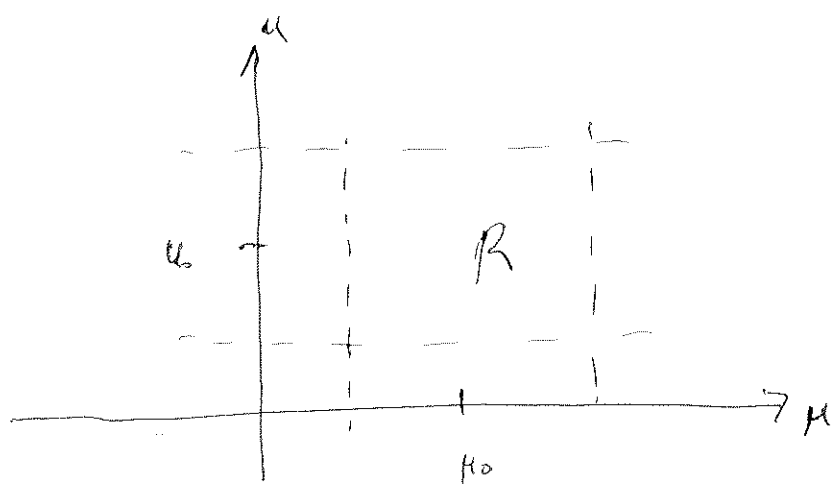
$$\alpha = \mu$$

stable for $\mu < 0$

unstable for $\mu > 0$

To find the equilibrium solutions we need some requirement on the functions f .

Theorem . Let $f(\mu, u)$ be a continuously differentiable function in some open region U of the μu plane containing the point (μ_0, u_0) . If $f(\mu_0, u_0) = 0$ and $f_u(\mu_0, u_0) \neq 0$ then there is a rectangle R about the point (μ_0, u_0) : $|\mu - \mu_0| < b$, $|u - u_0| < a$ contained in U such that



- i) The function $f(\mu, u) = 0$ has a unique solution: $u = u(\mu)$
- ii) The function u is continuously differentiable on $|\mu - \mu_0| < b$ and its derivative is given by

$$\frac{du}{d\mu} = - \frac{f_{\mu}(\mu, u)}{f_u(\mu, u)} \quad \text{or} \quad \frac{d\mu}{du} = - \frac{f_u(\mu, u)}{f_{\mu}(\mu, u)}$$

let f has continuous partial derivatives up to third order in a neighborhood of a point $P_0 = (\mu_0, u_0)$

regular point: A point P_0 which is in the locus of $f(\mu, u) = 0$, i.e. $f(P_0) = 0$ and $f_u(P_0) \neq 0$, $f_{\mu}(P_0) \neq 0$

Singular point. A point P_0 with

$$f(P_0) = 0, f_u(P_0) = 0, f_\mu(P_0) = 0$$

At the singular points, the implicit function theorem does not apply

$$\frac{du}{d\mu} = \frac{0}{0} \quad , \quad \text{or} \quad \frac{d\mu}{du} = \frac{0}{0} \quad \text{at } P_0$$

Singular points of $f(\mu, P) = 0$ can be studied systematically by assuming further that second derivatives of f do not vanish simultaneously.

At the singular points: Taylor's expansion about P_0

$$\begin{aligned} f(P) = & f(P_0) + f_u(P_0) \Delta u + f_\mu(P_0) \Delta \mu \\ & + \frac{1}{2} [f_{uu}(P_0) \Delta u^2 + 2 f_{\mu u}(P_0) \Delta u \Delta \mu \\ & + f_{\mu\mu}(P_0) \Delta \mu^2] + O(\Delta u^3 + \Delta \mu^3) \end{aligned}$$

where $\Delta u = u - u_0$, $\Delta \mu = \mu - \mu_0$. If P_0 is a singular pt

$$\begin{aligned} f(P) = & \frac{1}{2} [f_{uu}(P_0) \Delta u^2 + 2 f_{\mu u}(P_0) \Delta u \Delta \mu \\ & + f_{\mu\mu}(P_0) \Delta \mu^2] + O(\Delta u^3 + \Delta \mu^3) \end{aligned}$$

As $\Delta\mu, \Delta u \rightarrow 0$ we get $f(p) = 0$ and

$$f_{uu}(p_0) du^2 + 2f_{\mu u}(p_0) du d\mu + f_{\mu\mu}(p_0) d\mu^2 = 0$$

let $du = g d\mu$, then

$$g^2 f_{uu}(p_0) + 2g f_{\mu u}(p_0) + f_{\mu\mu}(p_0) = 0$$

let $\Delta = (f_{\mu u}(p_0))^2 - f_{uu}(p_0) f_{\mu\mu}(p_0)$.

Then

$$g = \frac{-f_{\mu u}(p_0) \pm \sqrt{\Delta}}{f_{uu}(p_0)}, \quad (f_{uu}(p_0) \neq 0)$$

or let $d\mu = \xi dp$, then

$$f_{uu}(p_0) + 2\xi f_{\mu u}(p_0) + \xi^2 f_{\mu\mu}(p_0) = 0$$

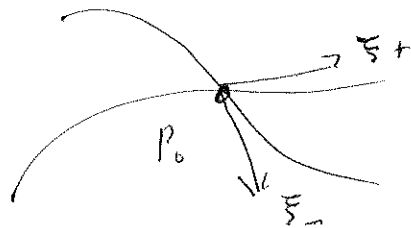
$$\Rightarrow \xi = \frac{-f_{\mu u}(p_0) \pm \sqrt{\Delta}}{f_{\mu\mu}(p_0)}, \quad (f_{\mu\mu}(p_0) \neq 0)$$

(34)

such points are called the "double points".

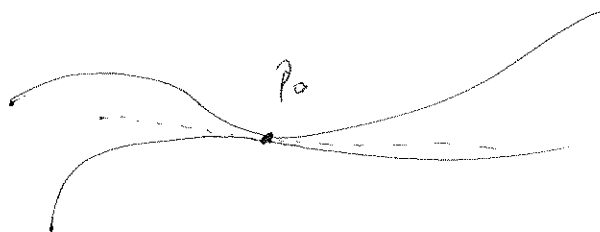
The numbers ξ and ζ give the slopes of the tangents of the bifurcating curves at the double points.

If $\Delta > 0$ there are two tangents



If $\Delta < 0$ there are no real roots of ξ or ζ . Such points P_0 are called "isolated points".

if $\Delta = 0$, then two curves through P_0 have the same tangent



Tangents at P_0

$$\frac{du}{d\mu} = - \frac{f_{u\mu}(P_0)}{f_{uu}(P_0)} \pm \sqrt{\frac{\Delta}{f_{uu}(P_0)^2}}$$

or

$$\frac{d\mu}{du} = - \frac{f_{u\mu}(P_0)}{f_{\mu\mu}(P_0)} \pm \sqrt{\frac{\Delta}{f_{\mu\mu}(P_0)^2}}$$

Tangents change sign:

If P_0 is a double point of $f(\mu, u) = 0$,
then either

i) $f_{\mu\mu}(P_0) \neq 0$ and the corresponding tangent is
given above $\frac{d\mu}{du}(P_0)$

ii) $f_{uu}(P_0) \neq 0$ and the corresponding tangent
is given above $\frac{du}{d\mu}(P_0)$

iii) If $f_{\mu\mu}(P_0) = 0$ then

$$(f_{uu}(P_0) du + 2f_{\mu u}(P_0) d\mu) du = 0$$

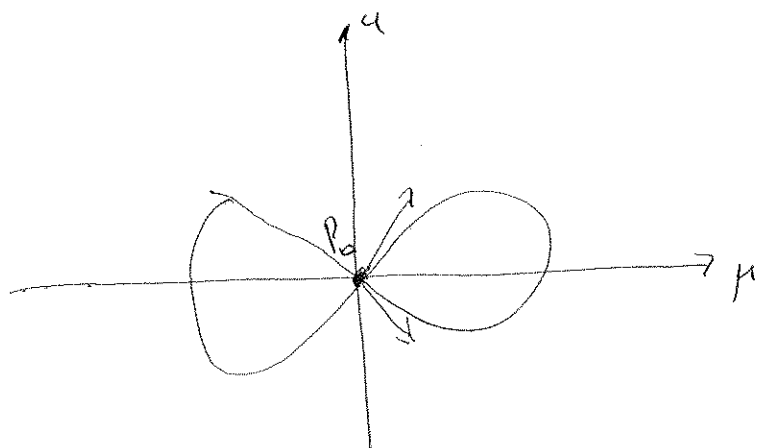
$$\Rightarrow \frac{du}{d\mu} = 0, \quad \frac{d\mu}{du} = - \frac{f_{uu}(P_0)}{2f_{\mu u}(P_0)} \quad \text{at } P_0$$

Example

(36)

$$\frac{du}{dt} = f(\mu, u) = (\mu^2 + u^2)^2 - 2(\mu^2 - u^2)$$

equilibrium solutions : $f(\mu, \mu) = (\mu^2 + u^2)^2 - 2(\mu^2 - u^2) = 0$
the locus of these solutions is a "lemniscate"



$P_0(0,0)$ is an equilibrium point. Furthermore

$$f(P_0) = 0, \quad f_u(P_0) = 0$$

$$f_\mu(P_0) = 0$$

Hence P_0 is a singular point "a double point".

$$f_u = 4u(\mu^2 + u^2) + 4u, \quad f_{uu} = 12u^2 + 4\mu^2 + 4$$

$$f_\mu = 4\mu(\mu^2 + u^2) - 4\mu, \quad f_{\mu\mu} = 12\mu^2 + 4u^2 - 4$$

$$f_{u\mu} = 8u\mu$$

$$f_{uu}(P_0) = 4, \quad f_{\mu\mu}(P_0) = -4, \quad f_{u\mu}(P_0) = 0$$

$$\Rightarrow \Delta = (f_{u\mu})^2 - f_{uu} f_{\mu\mu} = 16 > 0$$

$$\beta = \pm \frac{4}{4} = \pm 1$$

$\Rightarrow P_0(0,0)$ is a double point. Has two different tangents

(31)

Example: $f(\mu, u) = u^3 - \mu^2$

equilibrium solutions: $u^3 = \mu^2$

$P_0(0,0)$ an equilibrium point.

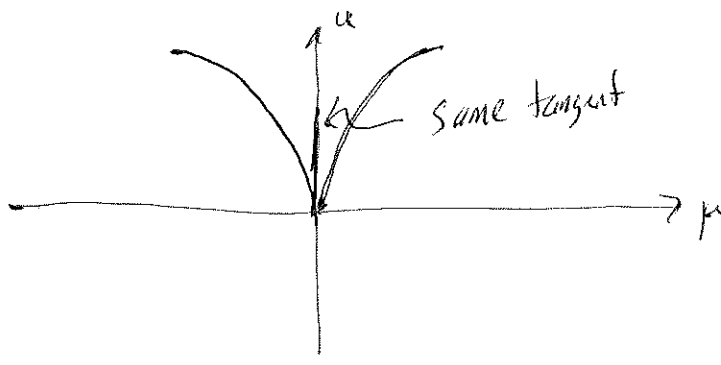
$$f_u = 3u^2, \quad f_\mu = -2\mu$$

Hence $f_u(P_0) = f_\mu(P_0) = 0$. Then P_0 is a singular point.

$$f_{uu} = 6u, \quad f_{\mu\mu} = -2, \quad f_{u\mu} = 0$$

$$f_{uu}(P_0) = 0, \quad f_{\mu\mu}(P_0) = -2, \quad f_{\mu u}(P_0) = 0$$

$$\Delta = 0 \Rightarrow \xi = 0 \quad \frac{d\mu}{du}(P_0) = 0$$



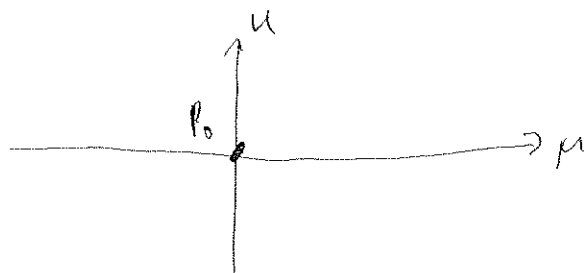
Example: $f(\mu, u) = u^2 + \mu^2$, $P_0(0,0)$ is a ~~critical~~ point
singular

$$f_u = 2u, \quad f_\mu = 2\mu$$

$$f_{uu} = 2, \quad f_{\mu\mu} = 2, \quad f_{\mu u} = 0$$

$$\Delta = -4 < 0$$

a single ^{isolated} point
no tangent.



Lemma. Let P_0 be a double point of $f(\mu, u) = 0$. Then either (33)

(i) $f_{\mu\mu}(P_0) \neq 0$ and two tangents are given by

$$\frac{d\mu}{du} = \xi = \frac{-f_{\mu u} \pm \sqrt{\Delta}}{f_{\mu\mu}(P_0)}$$

or

(ii) $f_{\mu\mu}(P_0) = 0$ and two tangents are given by

$$(f_{uu} du + 2f_{\mu u} d\mu) du = 0$$

$$\Rightarrow \frac{du}{d\mu}(P_0) = 0 \quad \text{and} \quad \frac{d\mu}{du} = - \frac{f_{uu}(P_0)}{2f_{\mu u}(P_0)}$$

Higher order singular points: If all of the second partial derivatives $f_{uu}, f_{\mu u}, f_{\mu\mu}$ vanish at P_0 , i.e.

$$f(P_0) = f_u(P_0) = f_\mu(P_0) = 0$$

$$f_{uu}(P_0) = f_{\mu u}(P_0) = f_{\mu\mu}(P_0) = 0$$

such a point is called a triple point of $f(\mu, u) = 0$

Exchange of stability

(39)

- Since two different branches cross a double point, such points are of interest because a change of stability can occur.



- Stability can also change at a regular points. A regular point P_0 is called a turning point (with respect to μ) if

$$f_\mu(P_0) \neq 0 \text{ or } f_u(P_0) \neq 0 \quad \text{and} \quad \frac{d\mu}{du} \text{ changes sign at } P_0$$

Example: $\frac{du}{dt} = u^2 - \mu$

$$f(\mu, u) = u^2 - \mu, \quad f_u = 2u$$

$$\text{Equilibrium soln. : } u^2 = \mu$$

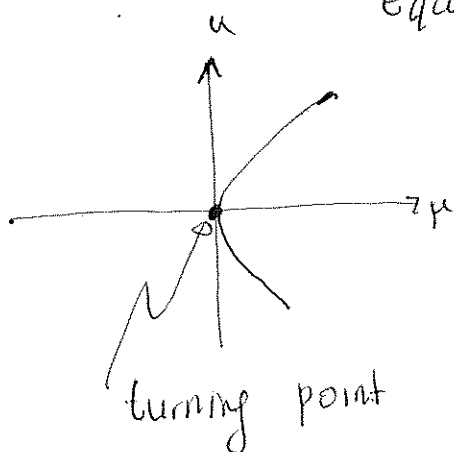
$$u = +\sqrt{\mu} \rightarrow f_u = 2\sqrt{\mu}$$

$$u = -\sqrt{\mu} \rightarrow f_u = -2\sqrt{\mu}$$

$$P_0 = (0, 0)$$

$$f_\mu(P_0) = -2 \quad \text{is a regular point}$$

$$\frac{d\mu}{du} = 2u \quad \text{changes sign at } P_0$$

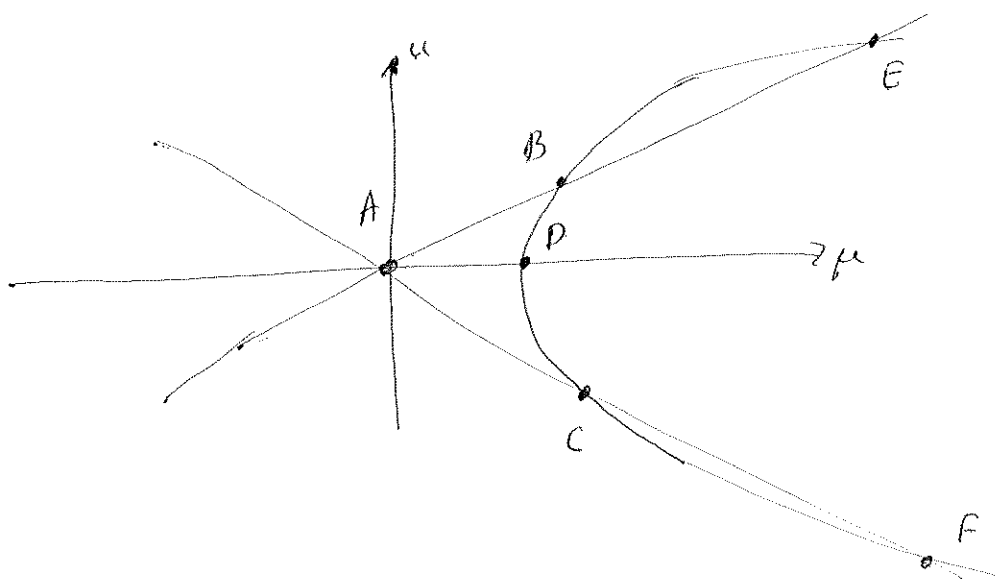


Example:

$$\frac{du}{dt} = f(\mu, u) = (1+u^2-\mu)(\mu^2-25u^2)$$

Equilibrium solutions

$$\mu = 1+u^2, \quad \mu = 5u, \quad \mu = -5u$$



All points are regular points except the points A, B, C, D, E, F

- D is a regular turning point.
- A, B, C, E, F are double points.

$$f(A) = f(B) = f(C) = f(E) = f(F) = 0$$

$$f_u(A) = \dots \quad = f_u(F) = 0$$

$$f_\mu(A) = \dots \quad = f_\mu(F) = 0$$

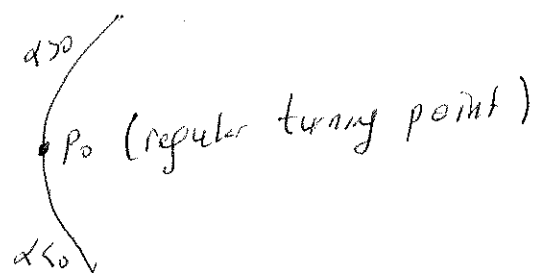
(41)

Theorem: Let P_0 be a regular turning point of $f(\mu, u) = 0$. Then equilibrium solution on side is stable, on the other side is unstable

proof: stability indicator $\alpha = f_u(\mu(u), u)$

$$= - \frac{d\mu}{du} f_{\mu}(\mu(u), u).$$

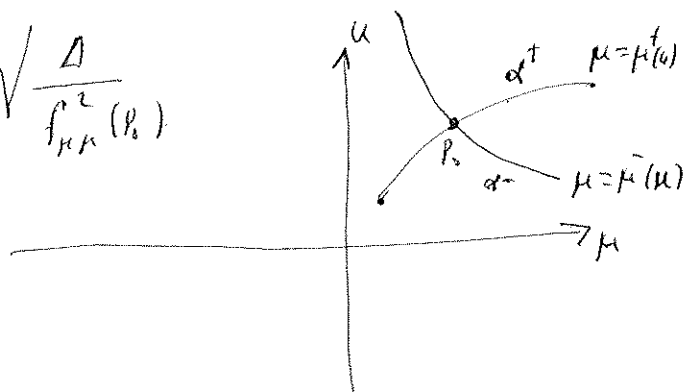
At the regular turning point $\frac{d\mu}{du}$ changes sign, hence α changes sign



Case I. Two curves $\mu^+(u)$ and $\mu^-(u)$ below passing through P_0 have tangents given by

$$\frac{du}{d\mu} = - \frac{f_{\mu u}(P_0)}{f_{uu}(P_0)} \pm \sqrt{\frac{\Delta}{f_{uu}^2(P_0)}}$$

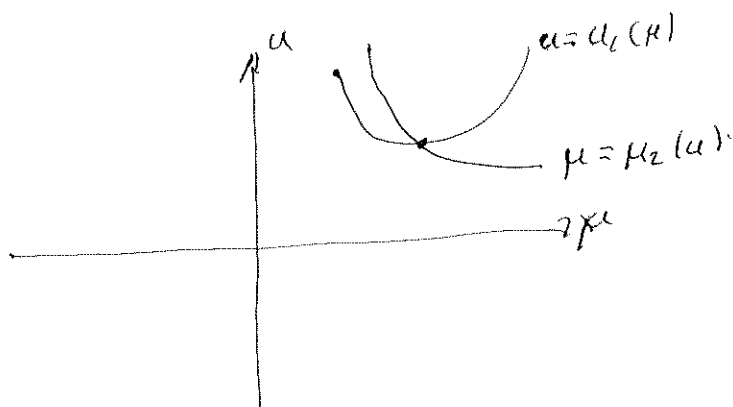
$$\frac{d\mu}{du} = - \frac{f_{\mu u}(P_0)}{f_{\mu\mu}(P_0)} \pm \sqrt{\frac{\Delta}{f_{\mu\mu}^2(P_0)}}$$



(40)

Case II. One of the curves is $u = u_1(\mu)$ with $\frac{du_1}{d\mu} = 0$ (parallel to the μ -axis at P_0) at P_0 and the other is $\mu = \mu_2(u)$ with

$$\frac{d\mu_2}{du} = - \frac{f_{uu}(P_0)}{2 f_{\mu u}(P_0)} \quad \text{at } P_0$$



We consider case I and $\alpha^+(u)$ and $\alpha^-(u)$ denote the stability indicators along the curves $\mu^+(u)$ and $\mu^-(u)$ respectively.

$$\alpha^+(u) = f_u(\mu^+(u), u)$$

$$\alpha^-(u) = f_u(\mu^-(u), u)$$

Theorem: Let $P_0(\mu_0, u_0)$ be a double point with $f_{\mu\mu}(P_0) \neq 0$. Then

$$\alpha^+(u) = + \frac{d\mu^+}{du} \left[\text{sign}(f_{\mu\mu}) \Delta (u - u_0) + O(u - u_0) \right]$$

$$\alpha^-(u) = - \frac{d\mu^-}{du} \left[\text{sign}(f_{\mu\mu}) \Delta (u - u_0) + O(u - u_0) \right]$$

proof : $\alpha^+ = \int_u (\mu^+(u), u)$

$$= \frac{d\mu}{du} \int_{\mu}$$

$$= \frac{d\mu^+}{du} \left[\cancel{f_{\mu\mu}(P_0)} + \cancel{f_{\mu u}(P_0)}(u-u_0) + f_{\mu\mu}(P_0)(\mu-\mu_0) + O(u-u_0)^2 \right].$$

as $4u \rightarrow 0$
 $4\mu \rightarrow 0$

$$= \frac{d\mu^+}{du} \left[\int_{\mu u} du + \int_{\mu\mu} d\mu \right]$$

$$= d\mu^+ \left[\int_{\mu u} + \int_{\mu\mu} \frac{d\mu^+}{du} \right]$$

$$= d\mu^+ \left[\cancel{f_{\mu u}} + \left(-\cancel{f_{\mu\mu}} + \sqrt{\frac{\Delta}{f_{\mu\mu}^2}} f_{\mu\mu}(P) \right) \right]$$

$$= d\mu^+ \left(\text{sign } f_{\mu\mu}(P) \sqrt{\Delta} \right)$$

$$= \frac{d\mu^+}{du} \left[\text{sign } f_{\mu\mu}(P) \sqrt{\Delta} (u-u_0) + O(u-u_0)^2 \right]$$

Theorem: let $P_0(\mu_0, u_0)$ be a double point with

$f_{\mu\mu}(P_0) \neq 0$, Then

$$\alpha^+(u) = \text{sign } f_{\mu\mu}(P_0) \sqrt{\Delta} (\mu - \mu_0) + O(u-u_0)^2$$

$$\alpha^-(u) = -\text{sign } f_{\mu\mu}(P_0) \frac{d\mu_2}{du} \sqrt{\Delta} (u-u_0) + O(u-u_0)^2$$

[one of the curves is $u = u_1(\mu)$ with $\frac{du_1}{d\mu}(P_0) = 0$

and other is $\mu = \mu_2(u)$ with $\frac{d\mu_2}{du} = -\frac{f_{\mu u}(P_0)}{2 f_{\mu\mu}(P_0)}$

proof. :1) $\mu = \mu_1(\mu)$ so that $\frac{d\mu_1}{d\mu}(P_0) = 0$

$$\alpha^1 = f_{\mu\mu}(P_0) = \int_{\mu\mu} d\mu + f_{\mu\mu}(P_1) d\mu$$

$$= \left(\int_{\mu\mu} \frac{d\mu}{d\mu} + f_{\mu\mu} \right) d\mu + \text{higher orders}$$

$$= f_{\mu\mu}(P_1) d\mu + \text{higher order } (\mu - \mu_0)$$

$$= \text{sign}(f_{\mu\mu}(P_0)) \sqrt{\Delta} (\mu - \mu_0)$$

$$\Delta = f_{\mu\mu}(P_1)^2 \Rightarrow f_{\mu\mu}(P_1) = \text{sign}(f_{\mu\mu}(P_0)) \sqrt{\Delta}$$

$$2) \mu = \mu_2(u) \text{ with } \frac{d\mu_2}{du} = - \frac{f_{u\mu}(P_1)}{2 f_{\mu\mu}(P_1)}$$

$$\alpha^2 = f_u(P_0) = \int_{u\mu}(P_0) du + \int_{\mu\mu}(P_0) d\mu + \text{higher ord}$$

$$f_{u\mu}(P_0) = -2 \frac{d\mu}{du} f_{\mu\mu}(P_1)$$

$$d^2 = -2 \frac{d\mu}{du} \int_{\mu\mu}(P_1) du + \int_{\mu\mu}(P_1) d\mu + h.o.$$

$$= - \int_{\mu\mu}(P_1) d\mu + h.o.$$

$$= - \text{sign}(f_{\mu\mu}(P_1)) \sqrt{\Delta} (\mu - \mu_1) + h.o.$$

Example:

$$\frac{du}{dt} = (\mu - \mu_c)u - u^3.$$

$$f = (\mu - \mu_c)u - u^3$$

$$P_0(\mu_c, 0)$$

$$f_u = \mu - \mu_c - 3u^2$$

$$f(P_0) = 0$$

$$f_\mu = u$$

$$f_u(P_0) = 0$$

$$f_{uu} = -6u$$

$$f_\mu(P_0) = \mu_c$$

$$f_{\mu\mu} = 0$$

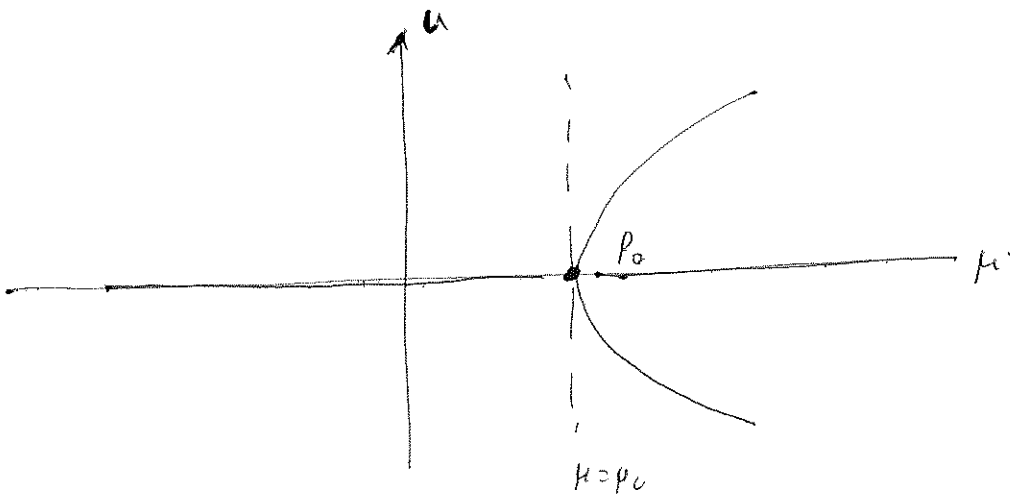
$$f_{uu}(P_0) = 0$$

$$f_{\mu u} = 1$$

$$f_{\mu\mu}(P_0) = 0$$

$$\left. \begin{array}{l} f_{uu} = -6u \\ f_{\mu\mu} = 0 \\ f_{\mu u} = 1 \end{array} \right\} \Delta = 1.$$

Equilibrium solns: $u=0$, $u^2 = \mu - \mu_c$



$$a^1 = f_u(u=0) = \mu - \mu_c$$

$$a^2 = f_{uu} = -2(\mu - \mu_c) = -2u^2$$

ASSIGNED EXERCISES OF MATH544:

Stability and Bifurcations: set 10

May, 2012

STABILITY AND BIFURCATIONS: One dimensional case

Classification of points of $f(\mu, u) = 0$

1. *Regular point* P_0 is the point where the implicit function theorem works: $f_\mu(P_0) = f_u(P_0) = 0$. Hence there exists a unique curve passing through this point.
2. *A regular turning point* P_0 where the α indicator or the slope of the tangent at P_0 changes its sign. This means that the stability is different on both sides of this point
3. *Singular point* P_0 is a point at which $f_\mu(P_0) = f_u(P_0) = 0$.
4. *A double point* P_0 is a singular point through which pass two and only two branches of $f = 0$ possessing distinct tangents. We shall assume that $f_{uu}(P_0)$, $f_{\mu u}(P_0)$, and $f_{\mu\mu}(P_0)$ are not vanishing simultaneously at the double point.
5. *A singular turning (double) point* P_0 is a double point at which $\frac{d\mu}{du}$ changes sign on one branch.
6. *A cusp point* P_0 is a point of second order contact between two branches of the curve. The two branches have the same tangent at a cusp point.
7. *A conjugate point* P_0 is an isolated point.
8. *higher-order singular point* P_0 is a singular point at which all second partial derivatives vanish simultaneously.

Summary of what we did so far.

Theorem 1. *Let u_0 be an equilibrium solution of*

$$\frac{du}{dt} = f(\mu, u), \quad t > 0, \quad (1)$$

where μ is a parameter, and assume that

$$f(\mu, u_0 + \bar{u}) = f_u(\mu, u_0)\bar{u} + R(u_0, \bar{u})$$

where the remainder satisfies $|R(u_0, \bar{u})| \leq K|\bar{u}|^2$ for $\bar{u}$ sufficiently small where K is a positive constant. Then u_0 is asymptotically stable if $\alpha = f_u(\mu, u_0) < 0$ and unstable if $\alpha > 0$.

Hence the indicator α is important in the study of stability. At the critical points defined above the determination of the α indicator depend on the slopes of the tangents about these points on the branches of equilibrium solutions. An important theorem (the implicit function theorem) for such a purpose is given by

Theorem 2. Let $f(\mu, u)$ be a continuously differentiable function in a region U of the μu plane containing the point $P_0 = (\mu_0, u_0)$. If $f(P_0) = 0$ and $f_u(P_0) \neq 0$, then there is a rectangle R , defined by $R = \{(\mu, u) | |u - u_0| < a, |\mu - \mu_0| < b\}$ contained in U such that

(i) The equation $f(\mu, u) = 0$ has a unique solution $u = u(\mu)$ on R . (ii) The function $u(\mu)$ is continuously differentiable on the interval $|\mu - \mu_0| < b$ and its derivative is given by

$$\frac{du}{d\mu} = -\frac{f_\mu(\mu, u(\mu))}{f_u(\mu, u(\mu))}$$

On the regular points of the bifurcating curves we can use above theorems and find the α indicator and investigate the stability of the model. Let us now assume that a point P_0 on the branches is a singular point. To study the stability about such points we have the following theorem. First we need the slopes of the curves about these points. For this purpose we have the following results.

Theorem 3. Let P_0 be a double point of $f(\mu, u) = 0$. Then either

(i) $f_{\mu\mu}(P_0) \neq 0$ and two tangents are given by

$$\frac{d\mu}{du} = -\frac{f_{\mu u}}{f_{\mu\mu}} \pm \frac{1}{|f_{\mu\mu}|} \sqrt{\Delta} \quad (2)$$

where

$$\Delta = f_{\mu u}^2 - f_{\mu\mu} f_{uu}$$

Remark: If $\Delta > 0$ then the curves intersecting at P_0 have different tangents. If $\Delta = 0$ then these curves have the same tangent line at the point P_0 . If $\Delta < 0$ then there exists no tangent line passing through P_0 such points are called *isolated points*.

(ii) If $f_{\mu\mu} = 0$ and the two tangents are given by

$$\frac{du}{d\mu}(P_0) = 0, \quad \text{and} \quad \frac{d\mu}{du} = -\frac{f_{uu}(P_0)}{2 f_{\mu u}(P_0)} \quad (3)$$

For stability analysis we need the α indicator about the point P_0 . For this purpose, in the neighborhood of a regular turning point we have

Theorem 4. *Let P_0 be a regular turning point of $f = 0$. Then equilibrium solutions on one side are stable and on the other side are not stable.*

Remark: The α indicator, $\alpha = f_u(\mu(u), u) = -\frac{d\mu}{du}(u) f_\mu(\mu(u), u)$. Since the slope changes its sign then α changes its sign on different sides of the point P_0 .

For double points to find the α indicator we consider two distinct cases.

case 1. The two curves $\mu = \mu^+(u)$ and $\mu = \mu^-(u)$ passing through (intersecting at) P_0 have tangents given by (2). Let the corresponding stability indicators are α^+ and α^- . They are given by

Theorem 5. *Let P_0 be a double point with $f_{\mu\mu}(P_0) \neq 0$. Then*

$$\alpha^+ = -\frac{d\mu^+}{du}(u) \left[\frac{f_{\mu\mu}(P_0)}{|f_{\mu\mu}(P_0)|} \sqrt{\Delta}(u - u_0) + o(|u - u_0|) \right], \quad (4)$$

$$\alpha^- = \frac{d\mu^-}{du}(u) \left[\frac{f_{\mu\mu}(P_0)}{|f_{\mu\mu}(P_0)|} \sqrt{\Delta}(u - u_0) + o(|u - u_0|) \right] \quad (5)$$

case 2. One of the curves is $u = u_1(\mu)$ with $\frac{du}{d\mu}(P_0) = 0$ and other is $\mu = \mu_2(u)$ with $\frac{d\mu_2}{du} = -\frac{f_{uu}(P_0)}{2f_{\mu u}(P_0)}$. Their corresponding α indicators are given by

Theorem 6. *Let P_0 be a double point with $f_{\mu\mu}(P_0) = 0$. Then*

$$\alpha^1(\mu) = \frac{f_{\mu u}(P_0)}{|f_{\mu u}(P_0)|} \sqrt{\Delta}(u - u_0) + o(|u - u_0|), \quad (6)$$

$$\alpha^2(\mu) = -\frac{f_{\mu u}(P_0)}{|f_{\mu u}(P_0)|} \sqrt{\Delta}(u - u_0) + o(|u - u_0|). \quad (7)$$

where $\alpha^1(\mu) = f_u(\mu, u_1(\mu))$ and $\alpha^2(u) = f_u(\mu_2(u), u)$ are the stability indicators of the curves $u_1(\mu)$ and $\mu_2(u)$

In addition to the solved problems and exercises given in Logan page 434-436 solve also the following problems

QUESTIONS

1. Check the stability of all equilibrium solutions of the equation (2) with

$$f = u(9 - \mu u)(\mu + 2u - u^2)[(\mu - 10)^2 + (u - 3)^2 - 1]$$

and draw a branching (bifurcation) diagram using the dashed lines for unstable equilibria.

2. Consider the integral equation

$$\lambda y(t) = \frac{2}{\pi} \int_0^\pi (3 \sin \tau \sin t + 2 \sin 2\tau \sin 2t) (y(\tau) + y^3(\tau)) d\tau$$

Solutions $y(t, \lambda)$ of this integral equation depend on the independent variable t and on the parameter λ .

(a) Show that every solution is of the form

$$y = A(\lambda) \sin t + B(\lambda) \sin 2t$$

(b) Are there nontrivial solutions for $\lambda = 0$?

(c) Calculate all solutions for $\lambda > 0$ (for $\lambda = 10$ there are nine different solutions).

(d) For each solution, calculate

$$[y] = \int_0^\pi y^2(t, \lambda) dt$$

and sketch a branching diagram

3. Cusp point Bifurcation At the cusp points it is not possible in general to obtain u as a function of μ or μ as a function of u . We then introduce a new parameter η in order to obtain a parametric representations $u(\eta)$ and $\mu(\eta)$. Let $P_0 = (0, 0)$ be a singular point

$$f(P_0) = 0, \quad f_\mu(P_0) = f_u(P_0) = 0 \quad \text{and} \quad \Delta = 0$$

We assume that the second partial derivatives of f are not vanishing simultaneously at the point P_0 . Here we assume that $f_{uu}(P_0) \neq 0$. Show that in the neighborhood of the point P_0 we have

$$\mu = \eta^2 \quad \text{or} \quad \mu = -\eta^2, \tag{8}$$

$$u = \frac{1}{2} u_{\eta\eta} \eta^2 + \frac{1}{6} u_{\eta\eta\eta} \eta^3 + O(\eta^4), \tag{9}$$

Hint: The above representations can be obtained by differentiating $f(\mu(\eta), u(\eta)) = 0$ with respect η in the neighborhood of the point P_0 and letting $u(0) = \eta(0) = 0$. For instance differentiating twice we obtain

$$f_{uu} (u_\eta)^2 + 2f_{\mu u} u_\eta \mu_\eta + f_{\mu\mu} (\mu_\eta)^2 = 0$$

and so on.

STABILITY AND BIFURCATIONS: Two dimensional case

Let x and y be functions of $t \in I$ satisfying the system of ODEs

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \quad t \in I \quad (10)$$

where P and Q are functions of x, y having continuous partial derivatives of all orders in some domain of the xy plane which is called the *phase plane*.

Let us consider the following initial value problem

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad t \in I \quad (11)$$

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad t_0 \in I \quad (12)$$

From

~~From~~ the theory of ODEs we have

Lemma 6. *If P and Q have continuous partial derivatives in some domain D the the initial value problem stated above has unique solutions*

Some definitions

1. *Phase Plane.* The xy plane.
2. *Phase variables or state variables :* x and y
3. *Trajectory or path.* A Curve C defined by the solution $x = x(t), y = y(t)$ where $t \in I$ of (10)
4. *Phase portrait.* The totality of all paths and critical points of (10) graphed in the phase plane.
5. *Critical Points.* All solution of $P(x_0, y_0) = Q(x_0, y_0) = 0$. Critical points are the equilibrium or steady state solutions of (10).

A corollary of Lemma 6. *There exists only one path passing through each point of the phase plane. (uniqueness of 10)*

This means that the paths corresponding to different initial values do not intersect each other. Critical points are also solutions of (10) which are called the equilibrium solutions. According to the above Corollary of Lemma 6 none of the trajectories pass through such critical points. Critical points are called isolated if there exists a neighborhood which contains only this critical point. Without proof we state the following results

- (a). If a path approaches a critical point then $t \rightarrow \pm\infty$
- (b). As $t \rightarrow \pm\infty$ a path approaches *a critical point, moves on a closed path, approaches a closed path, or leaves every bounded set*

Stability Analysis. In two dimensional case there stability definitions are given as follows

Definition(of stability). Let $u = (x(t), y(t))$ define the state vector for $t \in I$. Let $P_0 = (x_0, y_0)$ be a critical point and $u(t_0) = (x(t_0), y(t_0))$, $t_0 \in I$ be the initial value of the state vector. Then for given positive number $\epsilon > 0$ there exists a positive number δ_ϵ such that $\|u(t) - u_0\| \leq \epsilon$ for all $t \in I$ whenever $\|u(0) - u_0\| \leq \delta_\epsilon$

This implies that any path starting sufficiently closer to the critical point remains closer to the point.

Definition(of asymptotical stability). A critical point is asymptotically stable if it is stable and if all paths starting sufficiently closer to this point approach this point asymptotically (as $t \rightarrow \infty$). It means $\lim_{t \rightarrow \infty} \|u(t) - u_0\| = 0$.

This means that a critical point is *asymptotically stable* if all paths starting sufficiently closer to it asymptotically reaches that point. There are four types of Critical point, *center, node, saddle, and spiral*.

Remark. The critical points of center type are stable but not asymptotically stable

Linearized Stability. Let $P_0 = (x_0, y_0)$ be a critical point. Stability of this critical point is investigated by using linear perturbations about this equilibrium solution, $x = x_0 + \bar{x}, y = y_0 + \bar{y}$. Let us choose $P_0 = (0, 0)$ for simplicity then by using the differentiability properties of P and Q at all orders we get

$$\dot{x} = ax + by, \quad \dot{y} = cx + dy, \quad (13)$$

where a, b, c, d are arbitrary constants which are given by

$$a = P_x(0, 0), \quad b = P_y(0, 0), \quad c = Q_x(0, 0), \quad d = Q_y(0, 0)$$

This system of equations (13) may also be written by

$$\dot{u} = Au, \quad (14)$$

where A is the coefficient matrix in (13) and u is the column state vector. Let λ_1 and λ_2 be eigenvalues of the matrix A and v_1 and v_2 be the corresponding eigenvectors. For the solution of (14) we ^{have} two distinct cases

A. Two eigenvalues are different ($\lambda_1 \neq \lambda_2$). then the solution is given by

$$u(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} \quad (15)$$

where c_1 and c_2 are arbitrary constants to be determined by the initial conditions.

B. Eigenvalues are the same ($\lambda_1 = \lambda_2$). We have two sub cases. (B_1): $A = \lambda_1 I$, then

$$u(t) = [c_1 v_1 + c_2 v_2] e^{\lambda_1 t}. \quad (16)$$

Here v_1 and v_2 correspond to two different eigenvectors of corresponding to the same eigenvalue λ_1 . (B_2): $(A - \lambda_1 I)^2 = 0$, then

$$u(t) = [vt + w] e^{\lambda_1 t}, \quad (17)$$

where $v = Aw - \lambda_1 w$.

The classification of the critical point $(0, 0)$ is achieved by the eigenvalues of the matrix A

(1). Eigenvalues have the same sign. Critical point is a **node**. Node is stable if the eigenvalues are negative and unstable if the eigenvalues are positive. Solution is given (15).

(2). Eigenvalues have opposite sign. Critical point is a **saddle point**. Solution is given (15).

(3). Eigenvalues are equal. It is a **stable node** if the eigenvalue is negative. It is an **unstable node** otherwise. The solutions is given in (16) and (17).

(4). Eigenvalues are complex. Let $\lambda_1 = \alpha + i\beta$ then $\lambda_2 = \alpha - i\beta$. Let the eigenvector v_1 corresponding to eigenvalue be $v_1 = w + iv$. We have two types. Path about the critical point is a **spiral** if $\alpha \neq 0$ and **center** type of critical points if $\alpha = 0$ where the path is a closed curve (periodic solutions). The solutions are found from (15) by replacing the eigenvalues and taking the real part. We obtain $u(t) = c_1 u_1 + c_2 u_2$ where

$$u_1 = e^{\alpha t}(w \cos \beta t - v \sin \beta t)$$

$$u_2 = e^{\alpha t}(w \cos \beta t + v \sin \beta t)$$

As a summary we have the following theorem

Theorem 7. *The critical point of the system (13) is stable if and only if , the eigenvalues of A have non positive real parts, it is asymptotically stable if and only if the eigenvalues have negative real parts.*

So far we investigated the stability of the linear systems (13). For the non-linear systems we let

$$\dot{x} = ax + by + f(x, y), \tag{18}$$

$$\dot{y} = cx + dy + g(x, y) \tag{19}$$

where a, b, c, d are constant with (i). $ad - bc \neq 0$ and (ii).

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x,y)}{\sqrt{x^2 + y^2}} = 0, \quad \lim_{(x,y) \rightarrow (0,0)} \frac{g(x,y)}{\sqrt{x^2 + y^2}} = 0$$

It is obvious that $(0,0)$ is the critical point. Then we have the following theorem for the general case with the above conditions

Theorem 8. (Poincare' and Liapunov) *Let $(0,0)$ be a critical point of the linear system (13) and the nonlinear system (19) with the conditions (i) and (ii) stated above. The critical point $(0,0)$ of the nonlinear system is the same type as of the linear case if*

- a) *The eigenvalues of A are real, distinct, and have the same sign (node).*
- b) *The eigenvalues of A are real and have opposite signs (saddle).*
- c) *The eigenvalues of A are complex, but not purely imaginary (spiral).*

In addition if $(0,0)$ is asymptotically stable for the linear system (13), then it is asymptotically stable for the nonlinear system (19)

To understand whether the path is a closed curve we have the theorem of Bendixon and Dulac

Theorem 9. (Bendixon & Dulac) *If $P_x + Q_y$ has a fixed sign in a region of phase plane then (19) can not have a closed path in that region*

Assuming there exists a closed a path C and consider the form $Pdy - Qdx$ and integrating over C one obtains a contardiction. We have another useful theorems due to Poincare' and Bendixon.

Theorem 10. (Poincare') *A closed path of the system (19) surrounds at least one critical point of the system.*

Theorem 11. (Poncare' and Bendixon) *Let R be a closed and bounded region in the phase plane containing no critical points of (19). If C is a path of (19) that lies in R for some time t_0 and remains in R for all $t > t_0$, then C is either a closed path or spirals toward a closed path as $t \rightarrow \infty$*

Exercises. In addition to the assigned and solved exercises in Logan solve also the following problems.

1. (Duffing equation). Let u be a function of $t \in I$ satisfying the second order equation

$$\ddot{u} + \dot{u} - u + u^3 = 0$$

which is called the Duffing equation. The stability analysis of this equation is easily studied by letting $x = u$, $y = \dot{u}$. We then obtain the following nonlinear system

$$\dot{x} = P(x, y) = y, \quad (20)$$

$$\dot{y} = Q(x, y) = x - x^3 - y \quad (21)$$

We have the following critical points $(0, 0)$, $(1, 0)$ and $(-1, 0)$. For the first critical point we have $a = 0, b = 1, c = 1, d = -1$ then the eigenvalues are $\lambda_{1,2} = \frac{1}{2}(-1 \pm \sqrt{5})$ (calculate the eigenvectors). This critical point is a **saddle point**. For the other critical points $(\pm 1, 0)$ we have $a = 0, b = 1, c = -2, d = -1$. Hence $\lambda_{1,2} = \frac{1}{2}(-1 \pm i\sqrt{7})$. Hence $\alpha < 0$, then these critical points are *stable foci* (spiraling paths)

2. Discuss the stability of the equilibrium solutions of the Van der Pol equation

$$\ddot{u} - \lambda(1 - u^2) + \dot{u} + u = 0$$

for all possible values of λ .

3. Consider the system of ODEs

$$\dot{x} = -y + x \frac{1 - x^2 - y^2}{\sqrt{x^2 + y^2}}, \quad (22)$$

$$\dot{y} = x + y \frac{1 - x^2 - y^2}{\sqrt{x^2 + y^2}} \quad (23)$$

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

$$\dot{r} \cos \theta - r \sin \theta \dot{\theta} = -r \sin \theta + \cos \theta \left(\frac{1 - r^2}{r} \right)$$

$$\dot{r} \sin \theta + r \cos \theta \dot{\theta} = r \cos \theta + \sin \theta \left(\frac{1 - r^2}{r} \right)$$

$$\dot{r} = \frac{1 - r^2}{r}, \quad -r \dot{\theta} = -r \quad \dot{\theta} = -1$$

Show that the unit circle in the phase plane is a stable cycle. **Hint.** Use polar coordinates.

4. Consider the ordinary differential system

$$\dot{x} = -y + x(x^2 + y^2 - 1), \quad (24)$$

$$\dot{y} = x + y(x^2 + y^2 - 1). \quad (25)$$

Show that the periodic path is unstable.