

Lecture 6 : Integral Equations
and Green's Functions

MATH544: Spring 2012

Integral Equations, Green's Functions and Eigenfunction Expansions.

①

Let us start with the following linear
diffusion equation

$$r(x) u_t = \frac{\partial}{\partial x} \left(k(x) \frac{\partial u}{\partial x} \right) - \ell(x) u, \quad x \in (0,1) \\ t > 0$$

where r, k, ℓ are some given functions.

The boundary conditions are

$$u(0, t) - h_0 u_x(0, t) = 0, \quad t > 0$$

$$u(1, t) + h_1 u_x(1, t) = 0, \quad t > 0.$$

Using the method of separation of variables

$$u(x, t) = X(x) T(t)$$

we get for the function

$$(k(x) X')' + [r(x) \lambda - \ell(x)] X = 0, \quad x \in (0,1)$$

with the boundary conditions

$$X(0) - h_0 X'(0) = 0$$

$$X(1) + h_1 X'(1) = 0$$

$$h_0, h_1 > 0$$

$$k(x), r(x) > 0, \quad \ell(x) \geq 0 \quad \text{for } 0 \leq x \leq 1$$

The problem satisfied by X is called the "Sturm-Liouville ^{eigenvalue} problem". To find the solution of this problem, we have the operator L given by

$$L = \frac{1}{r} \frac{d}{dx} k(x) \frac{d}{dx} - l(x)/r$$

Eigenvalue problem

$$LX = \lambda X$$

with the BCs

$$X(0) - h_0 X'(0) = 0$$

$$X(1) + h_1 X'(1) = 0$$

$$h_0, h_1 > 0$$

Basic problems:

- 1) Establish that the Sturm-Liouville eigenvalue problem (SLEVP) has an infinite sequence of eigenfunctions and eigenvalues (X_n, λ_n) such that solution of the diffusion eqn takes the form

$$u_n(x, t) = e^{-\lambda_n t} X_n(x), \quad n=1, 2, \dots$$

Then

$$u(x,t) = \sum_1 c_n e^{-\lambda_n t} X_n(x)$$

and the coefficients are determined from the initial condition. Let

$$u(x,0) = f(x) \quad 0 \leq x \leq 1$$

Then

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x) \quad 0 \leq x \leq 1$$

2. Establish that any reasonable arbitrary function, $f(x)$, has an eigenfunction expansion, and determine the mode of convergence of this expansion

• This is the justification of the formal solution.

• A part of this problem involves finding the coefficients c_n . Here the key point is to discover the "orthogonality relations"

$$\int_0^1 r(x) X_n(x) X_m(x) dx = \delta_{nm}, \quad \forall m,n$$

$$= \|X\|^2 \delta_{mn}$$

(4)

Then

$$C_n = \frac{1}{\|X_n\|^2} \int_0^1 r(x) f(x) X_n(x) dx, \quad n=1,2,\dots$$

Steady state diffusion equation with a source term (inhomogeneous)

$$r(x) u_t = \frac{\partial}{\partial x} \left(k(x) \frac{\partial u}{\partial x} \right) - \ell(x) u + g(x, t) p(x)$$

Steady state : as $t \rightarrow \infty$ $g(x, t) \rightarrow g(x)$
and $u(x, t) \rightarrow u(x)$

Then we have

$$(k(x) u')' - \ell(x) u = g(x) p(x)$$

$$\begin{aligned} \text{BCs} \quad u(0) - h_0 u'(0) &= 0 \\ u(1) + h_1 u'(1) &= 0 \end{aligned}$$

This is the Sturm-Liouville boundary value problem (SLBVP)

3) Establish that this (above) Sturm-Liouville problem has a solution.

Combination of "Sturm-Liouville eigenvalue problem" (SLEVP) and "Sturm-Liouville boundary value problem" (SLBVP) is called the Sturm-Liouville Problem (SLP)

$$-[p(x)y']' + [q(x) - \lambda r(x)]y = f(x), 0 < x < 1$$

$$y(0) - h_0 y'(0) = 0$$

$$y(1) + h_1 y'(1) = 0$$

$$h_0, h_1 > 0$$

$$f(x) = 0 \quad (\text{SLEVP})$$

$$\lambda = 0 \quad (\text{SLBVP})$$

First we shall solve SLBVP by the GF method

(6)

Green's Functions for Sturm-Liouville

BV Problems

$$A \quad \begin{cases} -(p(x)y')' + q(x)y = f(x), & 0 < x < 1 \\ y(0) - h_0 y'(0) = 0 \\ y(1) + h_1 y'(1) = 0 \end{cases}$$

where $p(x) \neq 0$, $p'(x)$, $q(x)$ and $f(x)$ are continuous on $0 \leq x \leq 1$ and h_0, h_1 are nonnegative constants. Let L be the operator

$$L = - \frac{d}{dx} p(x) \frac{d}{dx} + q(x) \quad (1)$$

which is called the "Sturm-Liouville operator".

Then the above problem is rewritten as

$$B \quad \begin{cases} L y = f(x), & x \in (0,1) \\ y(0) - h_0 y'(0) = 0 \\ y(1) + h_1 y'(1) = 0 \end{cases}$$

We shall solve this BV problem by using the method of GF. Let $u, v \in C^2(I)$, then we have

(7)

$$v L u - u L v = \frac{d}{dx} [p(uv' - uv')] \quad (2)$$

let $v = G(x, y)$ such that

$$L_x G(x, y) = \delta(x - y) \quad (3)$$

Then (2) becomes.

$$G(x, y) f(x) - u(x) \delta(x - y) = \frac{d}{dx} [p(x) (u'(x) G(x, y) - u(x) G'(x, y))]$$

Integrating wrt x in $(0, 1)$ we get.

$$-u(y) + \int_0^1 f(x) G(x, y) dx = p(x) \left[G(x, y) \frac{d}{dx} u(x) - u(x) \frac{d}{dx} G(x, y) \right] \Big|_0^1$$

$$\Rightarrow u(y) = \int_0^1 f(x) G(x, y) dx - p(1) \left[G(1, y) u'(1) - u(1) \frac{d}{dx} G(x, y) \Big|_{x=1} \right] + p(0) \left[G(0, y) u'(0) - u(0) \frac{d}{dx} G(x, y) \Big|_{x=0} \right]$$

Since

$$\begin{aligned} u(0) - h_0 u'(0) &= 0 \\ u(1) + h_1 u'(1) &= 0 \end{aligned} \quad (3)$$

$$\begin{aligned} \Rightarrow G(0, y) - h_0 G_x(0, y) &= 0 \\ G(1, y) + h_1 G_x(1, y) &= 0 \end{aligned} \quad (4)$$

Hence

$$u(y) = \int_0^1 f(x) G(x, y) dx. \quad (5)$$

1) $G(x, y)$ is continuous at $x=y$

2) $G'(x, y)$ is discontinuous at $x=y$.

$$-\frac{d}{dx} \left[p \frac{d}{dx} G(x, y) \right] + q(x) G(x, y) = \delta(x-y)$$

$$-\int_{y-\varepsilon}^{y+\varepsilon} \frac{d}{dx} \left[p \frac{d}{dx} G(x, y) \right] dx + \int_{y-\varepsilon}^{y+\varepsilon} q(x) G(x, y) dx = 1.$$

Take $\varepsilon \rightarrow 0$ we get

$$-p(x) \frac{d}{dx} G(x, y) \Big|_{y-}^{y+} = 1$$

$$G_x(y_-, y) - G_x(y_+, y) = 1/p(y) \quad (6)$$

3) $G(x, y)$ satisfies the BCs (4)

Assume that $v_0(x)$ and $v_1(x)$ are the distinct solutions of the homogeneous eqn. $Lu=0$ then

$$G(x, y) = \begin{cases} a_1 v_0(x) + b_1 v_1(x) & , \quad x \leq y \\ a_2 v_0(x) + b_2 v_1(x) & , \quad x > y \end{cases} \quad (7)$$

with $Lv_i = 0$, $i=1,2$. We assume that

$$PW(v_0, v_1) \neq 0$$

Indeed one finds that

$$PW(v_0, v_1) = C \quad (\text{constant}) \quad (8)$$

1) continuity of G at y

$$a_1 v_0(y) + b_1 v_1(y) = a_2 v_0(y) + b_2 v_1(y)$$

$$(a_1 - a_2) v_0(y) = (b_2 - b_1) v_1(y) \quad (9)$$

$$2) \quad a_1 v_0'(y) + b_1 v_1'(y) - a_2 v_0'(y) - b_2 v_1'(y) = 1/p(y)$$

$$(a_1 - a_2) v_0'(y) + (b_1 - b_2) v_1'(y) = 1/p(y) \quad (10)$$

3) BCs

$$a_1 v_0(0) + b_1 v_1(0) - h_0 (a_1 v_0' + b_1 v_1') = 0$$

$$a_1 (v_0(0) - h_0 v_0'(0)) + b_1 (v_1(0) - h_0 v_1'(0)) = 0 \quad (11)$$

$$a_2 v_0(1) + b_2 v_1(1) + h_1 [a_2 v_0'(1) + b_2 v_1'(1)] = 0 \quad (15)$$

$$a_2 (v_0(1) + h_1 v_0'(1)) + b_2 (v_1(1) + h_1 v_1'(1)) = 0 \quad (12)$$

let $v_0(x)$ satisfy the left boundary condition

$$v_0(0) - h_0 v_0'(0) = 0 \quad (13)$$

and $v_1(x)$ satisfy the right boundary condition

$$v_1(1) + h_1 v_1'(1) = 0 \quad (14)$$

and take $b_1 = 0$, $a_2 = 0$. Then (9) gives

$$a_1 v_0(y) = b_2 v_1(y)$$

(10) gives

$$a_1 v_0'(y) - b_2 v_1'(y) = \frac{1}{p(y)}$$

Then let $b_2 = A v_0(y)$, $a_1 = A v_1(y)$

where A is a constant, $A \neq 0$

$$A p W(v_0, v_1) = 1, \quad A \neq 0 \quad (15)$$

$\Rightarrow$

$$G(x, y) = \begin{cases} A V_0(x) V_1(y) & x \leq y \\ A V_0(y) V_1(x) & x > y. \end{cases} \quad (16)$$

or

$$G(x, y) = \begin{cases} \frac{V_0(x) V_1(y)}{C} & x \leq y \\ \frac{V_0(y) V_1(x)}{C} & x > y \end{cases} \quad (17)$$

$$C \neq 0.$$

Hence we have the following theorem

Theorem 1. The Sturm-Liouville problem (B)

$$Lu = f \quad x \in (0, 1)$$

$$u(0) - h_0 u'(0) = 0$$

$$u(1) + h_1 u'(1) = 0$$

(18)

has a Green's function if and only if the corresponding homogeneous problem ($f=0$) has only trivial solution, in which case the GF is given by

$$G(x, y) = \begin{cases} v_0(x) v_1(y) & 0 < x \leq y < 1 \\ v_1(x) v_0(y) & 1 > x \geq y > 0 \end{cases} \quad (17)$$

(by scaling $C=1$) with $L u_i = 0$, $i=1,2$ and

$$v_0(0) - h_0 v_0'(0) = 0 \quad \text{and} \quad v_1(1) + h_1 v_1'(1) = 0.$$

Proof. a) Suppose that the SLP has a Green's function, then

$$u(x) = \int_0^1 f(y) G(x, y) dy. \quad (20)$$

provides the unique solution for each continuous f in $(0, 1)$. It is clear that the homogeneous problem, $f=0$, has trivial solution $u=0$.

b). Assume that the homogeneous problem has only the trivial solution $u=0$. We can find nontrivial solutions of $L v = 0$, v_0, v_1 .

with

$$P W(v_0, v_1) = C = \text{const} \quad (21)$$

If $C=0$ then v_0 and v_1 are linearly dependent they both satisfy the both BCS. but this is a contradiction (homogeneous problem has only trivial solution) Hence $C \neq 0$ which can be scaled to unity. Therefore one has

$$\begin{aligned}
 L v_0 &= 0 \quad \text{with} \quad v_0(0) - h_0 v'_0(0) = 0 \quad \text{and} \quad (18) \\
 L v_1 &= 0 \quad \text{with} \quad v_1(1) + h_1 v'_1(1) = 0 \quad (22)
 \end{aligned}$$

and $pW = 1$ Then G is given in (19).

Thm 2 Assume that the homogeneous problem corresponding to the SL problem has only trivial solution. Let $G(x, y)$ be the function defined by

$$G(x, y) = \begin{cases} v_0(x) v_1(y) & x \leq y \\ v_1(x) v_0(y) & x \geq y \end{cases}$$

with

$$L v_i = 0, \quad i = 0, 1$$

satisfying the boundary conditions

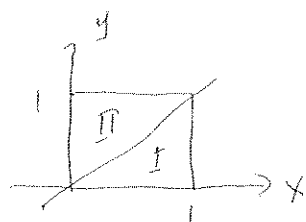
$$v_0(0) - h_0 v'_0(0) = 0$$

$$v_1(1) + h_1 v'_1(1) = 0$$

$$\text{so that} \quad pW(v_0, v_1) = 1$$

Then

(i) $G(x, y)$ is continuous on the square $0 \leq x, y \leq 1$



$$I: \quad x \geq y$$

$$II: \quad x \leq y$$

and has continuous second derivatives on each of the triangle I and II. On each triangle I and II

$$L G(x, y) = 0$$

ii) $G(x, y)$ satisfies the BCs

$$G(0, y) - h_0 G_x(0, y) = 0$$

$$G(1, y) + h_1 G_x(1, y) = 0$$

iii). For each $y \in (0, 1)$ we have

$$G_x(y^-, y) - G_x(y^+, y) = \frac{1}{p(y)}$$

Theorem 3. The Green's function for the SL problem is symmetric, provided that it exists. The Green's function does exist if $p(x) > 0$, $q(x) \geq 0$, $h_0, h_1 > 0$.

proof: From the first theorem, for the existence of the Green's function the homogeneous problem should have trivial solution $v = 0$. Suppose

that the homogenous problem has a nontrivial solution, i.e., let z satisfy

$$Lz = 0 \quad \text{or}$$

$$-\frac{d}{dx} \left(p \frac{dz}{dx} \right) + qz = 0 \quad (23)$$

$$z(0) - h_0 z'(0) = 0 \quad (24)$$

$$z(1) + h_1 z'(1) = 0$$

multiply (23) by z and integrate.

$$-\int_0^1 z [p z']' dx + \int_0^1 q(x) z^2 dx = 0$$

$$\Rightarrow \int_0^1 p z'^2 dx - p(x) z z' \Big|_0^1 + \int_0^1 q(x) z^2 dx = 0$$

$$\text{or } \int_0^1 p z'^2 dx + p(1) h_1 z'(1)^2 + p(0) h_0 z'(0)^2 + \int_0^1 q(x) z^2 dx = 0$$

each term in the above is a nonnegative term and sum equals zero which leads to $z(x) = 0 \forall x \in (0,1)$ which is a contradiction.

General Sturm-Liouville problem

$$Ly - \lambda r(x)y = f(x) \quad x \in (0,1)$$

$$y(0) - h_0 y'(0) = 0$$

$$y(1) + h_1 y'(1) = 0$$

can be expressed as

$$Ly = \lambda r(x)y + f(x) \equiv F(x)$$

Green's function method gives us

$$y(x) = \int_0^1 G(x, x') F(x') dx'$$

$$\begin{aligned} \text{or} \quad y(x) &= \frac{1}{\lambda} \int_0^1 r(x') y(x') G(x, x') dx' \\ &+ \int_0^1 f(x') G(x, x') dx' \end{aligned}$$

$$\text{let } y_1(x) = \int_0^1 f(x') G(x, x') dx'$$

$$\Rightarrow y(x) = y_1(x) + \lambda \int_0^1 r(x') y(x') G(x, x') dx'$$

(25)

This is a Fredholm integral equation of the second kind which can be simplified further.

$$\text{Let } z(x) = \sqrt{r(x)} y(x)$$

$$g(x, x') = \sqrt{r(x)r(x')} G(x, x')$$

$$z_1(x) = \sqrt{g(x)} y_1(x)$$

Multiply (25) by $\sqrt{r(x)}$ and use the above definitions we get

$$z(x) = z_1(x) + \lambda \int_0^1 z(x') g(x, x') dx' \quad (26)$$

Neumann Series.

We have seen that the general Sturm-Liouville equation can be written as a Linear Fredholm Integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x,s) y(s) ds \quad (27)$$

where K is called the kernel of the equation and $f(x)$ is a given function.

Let $a \leq x, s \leq b$ so that $-\infty < a < b < \infty$. $y(x)$ depends on K and λ . Let us seek a power series expansion in the form

$$y(x) = \sum_{n=0}^{\infty} \lambda^n y_n(x) \quad (28)$$

where $y_n(x)$, $n=0,1,2,\dots$ are to be determined. using (28) in (27) we get

$$\sum_{n=0}^{\infty} \lambda^n y_n = f(x) + \int_a^b K(x,s) \sum_{n=0}^{\infty} \lambda^{n+1} y_n(s) ds \quad (29)$$

Comparing the power of λ^n we get

$$y_0(x) = f(x) \quad (30)$$

$$y_1(x) = \int_a^b k(x,s) y_0(s) ds \quad (31)$$

$$y_2(x) = \int_a^b k(x,s) y_1(s) ds \quad (32)$$

$$y_{n+1}(x) = \int_a^b k(x,s) y_n(s) ds \quad (33)$$

$$n = 0, 1, 2, \dots$$

The recurrence formula (33) uniquely determines the coefficient y_n in

$$y(x) = \sum_{n=0}^{\infty} \lambda^n y_n(x) \quad (34)$$

Hence the Fredholm integral equation is solved provided that term by term integration in (27) can be justified. We justify it by showing the above series (34) is uniformly convergent in $[a,b]$ provided that the parameter λ is not too large. Let $M = \max_{x,s \in [a,b]} |k(x,s)|$, $N = \max_{x \in [a,b]} |f(x)|$, Then

where M and N are positive real numbers.

Then

Then

$$|y_0(x)| = |f(x)| \leq N$$

$$|y_1(x)| = \left| \int_a^b k(x,s) y_0(s) ds \right| \leq MN(b-a)$$

$$|y_2(x)| = \left| \int_a^b k(x,s) y_1(s) ds \right| \leq M^2 N (b-a)^2$$

$$\dots$$

$$|y_n(x)| = \left| \int_a^b k(x,s) y_{n-1}(s) ds \right| \leq N (M(b-a))^n$$

Hence

$$\begin{aligned} \left| \sum \lambda^n y_n(x) \right| &\leq \sum_{n=0}^{\infty} |\lambda|^n N (M(b-a))^n \\ &= \sum_{n=0}^{\infty} N (|\lambda| M(b-a))^n \end{aligned}$$

$$\text{If } M|\lambda|(b-a) < 1 \Rightarrow$$

$$\left| \sum \lambda^n y_n(x) \right| \leq \frac{N}{1 - M|\lambda|(b-a)} < \infty$$

Solution $y(x) = \sum \lambda^n y_n(x)$ is continuous, because the series is uniformly convergent and each $y_n(x)$ ($n=0,1,2,\dots$) is continuous. This series is called the Neumann Series.

The Neumann Series can be put in an alternative form.

$$y(x) = f(x) + \lambda \int_a^b k(x,s) y(s) ds.$$

$$y = \sum_{n=0}^{\infty} \lambda^n y_n(x) \Rightarrow$$

$$y_0(x) = f(x)$$

$$y_1(x) = \int_a^b k(x,s) f(s) ds$$

$$y_2(x) = \int_a^b k(x,s) y_1(s) ds$$

$$= \int_a^b k(x,s) \left(\int_a^b k(s,t) f(t) dt \right) ds$$

$$= \int_a^b \left(\int_a^b (k(x,s) k(s,t)) ds \right) f(t) dt$$

$$= \int_a^b g_2(x,t) f(t) dt$$

$$g_2(x,t) = \int_a^b k(x,s) k(s,t) ds.$$

similarly

$$y_n(x) = \int_a^b k_n(x,t) f(t) dt$$

where $k_n(x,t) = \int_a^b k(x,s) k_{n-1}(s,t) ds.$

$$n=0,1,2,\dots$$

Hence

$$\begin{aligned}
 y(x) &= \sum_{n=0}^{\infty} \lambda^n y_n(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n y_n(x) \\
 &= f(x) + \sum_{n=1}^{\infty} \lambda^n \int_a^b g_n(x,s) f(s) ds \\
 &= f(x) + \lambda \int_a^b \gamma(x,s) f(s) ds.
 \end{aligned}$$

with

$$\gamma(x,s) = \sum_{n=1}^{\infty} \lambda^{n-1} g_n(x,s).$$

Theorem : Let $g(x,s)$ and $f(x)$ be continuous on $a \leq x,s \leq b$, let $M = \max_{x,s \in [a,b]} |g(x,s)|$. If

$|\lambda| \frac{1}{M(b-a)}$ the solution

$$y(x) = f(x) + \lambda \int_a^b \gamma(x,s) f(s) ds$$

is unique and continuous.

proof. let us assume that there are two different solutions $z_1(x)$ and $z_2(x)$ of the integral equation. Then their difference $w(x) = z_2(x) - z_1(x)$ satisfies

(23)

$$w(x) = \lambda \int_a^b g(x,s) w(s) ds.$$

Hence for all $x \in [a,b]$ we have

$$|w(x)| \leq |\lambda| M W (b-a)$$

where $W = \max_{x \in [a,b]} |w(x)|$ Then.

$$[1 - |\lambda| M (b-a)] W \leq 0.$$

This implies that $W = 0$. Hence $z_1 = z_2$ everywhere in $[a,b]$.

Example: Find the solution of

$$y'(x) = f(x) + \lambda \int_a^b e^{x-s} y(s) ds$$

see next page:

Solution of the example.

$$y = f(x) + \lambda \int_a^b e^{x-s} y(s) ds$$

$$y = f(x) + \lambda e^x \int_a^b e^{-s} y(s) ds$$

$$= f(x) + \lambda e^x A$$

$$A = \int_a^b e^{-s} y(s) ds = \int_a^b e^{-s} [f(s) + \lambda e^s A] ds$$

$$= \int_a^b e^{-s} f(s) ds + \lambda A (b-a)$$

$$A [1 - \lambda (b-a)] = \int_a^b e^{-s} f(s) ds.$$

$$(i) \quad \lambda (b-a) \neq 1$$

$$A = \frac{1}{1 - \lambda (b-a)} \int_a^b e^{-s} f(s) ds$$

$$\Rightarrow y(x) = f(x) + \frac{\lambda e^x}{1 - \lambda (b-a)} \int_a^b e^{-s} f(s) ds$$



ii) if $1 = \lambda(b-a)$

$$\Rightarrow \int_a^b e^{-s} f(s) ds = 0 \quad (*)$$

otherwise there exist no soln

If $(*)$ is satisfied ~~that~~ is there are infinitely many soln

$$y(x) = f(x) + \lambda A e^x \quad A \text{ is arbitary}$$

Hilbert-Schmidt Theory

When the kernel $g(x, s)$ is not separable like

$$g(x, s) = \sum_i^N p_i(x) q_i(s)$$

there are in general infinitely many eigenvalues λ_n and eigenfunctions $y_n(x)$ of the homogeneous Fredholm integral equation. In some exceptional cases there are more than one eigenfunctions to a single λ .

Let $g(x, s)$ be a symmetric kernel, $g(x, s) = g(s, x)$ with respect to variables x and s . Let $y_n(x)$ and $y_m(x)$ be eigenfunctions of the homogeneous Fredholm integral equation corresponding to different eigenvalues λ_m and λ_n respectively of the homogeneous Fredholm equation

$$y(x) = \lambda \int_0^b g(x, s) y(s) ds$$

Remark: Eigenvalue λ can not be zero, because it corresponds to the trivial solution $y=0$. We ignore such solutions.

proposition 7. Eigen functions of the homogenous Fredholm integral equation with a symmetrical kernel corresponds to different eigenvalues are orthogonal

proof: By definition for the eigenvalue λ_n we have the eigenfunction $y_n(x)$ satisfying

$$y_n(x) = \lambda_n \int_a^b g(x,s) y_n(s) ds, \quad n=1,2,\dots$$

Multiplying by $y_m(x)$ and integrating

$$\begin{aligned} \int_{-\infty}^{\infty} y_n(x) y_m(x) dx &= \lambda_n \int_{-\infty}^{\infty} dx y_m(x) \int_{-\infty}^{\infty} g(x,s) y_n(s) ds \\ &= \lambda_n \int_{-\infty}^{\infty} ds y_n(s) \int_{-\infty}^{\infty} y_m(x) g(x,s) dx \\ &= \lambda_n \int_{-\infty}^{\infty} ds y_n(s) \frac{1}{\lambda_m} y_m(s) \\ &= \frac{\lambda_n}{\lambda_m} \int_{-\infty}^{\infty} dx y_n(x) y_m(x) \end{aligned}$$

$$\Rightarrow \left(1 - \frac{\lambda_n}{\lambda_m}\right) \langle y_n, y_m \rangle = 0, \quad \lambda_n \neq \lambda_m$$

$$\Rightarrow \langle y_n, y_m \rangle = 0 \quad n \neq m.$$

(23)

Remark: If there are more than one eigenfunction corresponding to one eigenvalue then we can orthogonalize them by using the Gram-Schmidt Orthogonalization procedure. In the sequel we assume that such sets are normalized.

Remark: In the case of complex functions $y_n(x)$ and when the kernel is hermitian $g(x,s) = \bar{g}(s,x)$ then the corresponding eigenvalues are real (prove it!).

Theorem 8. Let $g(x,s)$ be real and symmetric let $H(x)$ be continuous in (a,b) Then any function $h(x)$ given by

$$h(x) = \int_a^b g(x,s) H(s) ds$$

can be represented over (a,b) by a linear combination of the eigenfunctions $y_n(x)$, $n=1,2,\dots$ of the homogeneous Fredholm integral equation with $g(x,s)$ as its kernel.

Remark: When the number of eigenfunctions is infinite, the resulting infinite series is uniformly convergent in the interval (a,b) . This theorem applies, more generally, when g is any continuous complex kernel with $g(x,s) = \bar{g}(s,x)$ where bar over a letter is complex conjugation. Hence

Hence we have

$$h(x) = \sum_{n=0}^{\infty} a_n y_n(x) \quad , \quad a \leq x \leq b.$$

and
$$a_n = \frac{1}{\rho_n} \int_a^b h(x) y_n(x) dx \quad , \quad n=1,2,\dots$$

with
$$\rho_n = \int_a^b y_n^2(x) dx \quad n=1,2,\dots$$

Remark: If there exists finite number of eigenfunctions, then the function generated by the operation

$$\int_a^b g(x,s) H(s) ds$$

form a very restricted class of functions.

Example: As an example, let $g(x,s) = \sin(x+s)$ and $(a,b) = (0, 2\pi)$. Then.

$$\int_0^{2\pi} \sin(x+s) H(s) ds = B_1 \sin x + B_2 \cos x$$

where

$$B_1 = \int_0^{2\pi} H(s) \cos s \, ds, \quad B_2 = \int_0^{2\pi} H(s) \sin s \, ds.$$

Hence any function in this class is represented as

$$h(x) = B_1 \sin x + B_2 \cos x.$$

~~where the function~~ ~~that~~ where the function $H(x)$ is.
(It is assumed to be continuous over $(0, 2\pi)$, that's all).

Characteristic functions of the homogeneous Fredholm integral equation can be found as follows:

$$\begin{aligned} y(x) &= \lambda \int_0^{2\pi} \sin(x+s) y(s) \, ds \\ &= \lambda \int_0^{2\pi} [\sin x \cos s + \cos x \sin s] y(s) \, ds \\ &= \lambda [C_1 \sin x + C_2 \cos x] \quad (*) \end{aligned}$$

where

$$C_1 = \int_0^{2\pi} \cos s \, y(s) \, ds, \quad C_2 = \int_0^{2\pi} \sin s \, y(s) \, ds.$$

using (*) in C_1 and C_2 we get

$$\begin{aligned} C_1 &= \int_0^{2\pi} \cos s \, \lambda [C_1 \sin s + C_2 \cos s] \, ds = \lambda C_1 \int_0^{2\pi} \sin s \cos s \, ds \rightarrow 0 \\ &\quad + \lambda C_2 \int_0^{2\pi} \cos^2 s \, ds \end{aligned}$$

$$C_1 = \lambda \pi C_2$$

$$C_2 = \lambda \int_0^{2\pi} \sin s [C_1 \sin s + C_2 \cos s] ds$$

$$= \lambda C_1 \pi$$

$$\Rightarrow C_1 = \lambda \pi C_2 \quad C_2 = \lambda \pi C_1$$

$$\Rightarrow (\lambda \pi)^2 = 1$$

we have two eigenvalues $\lambda_1 = \frac{1}{\pi}$, $\lambda_2 = -\frac{1}{\pi}$

Hence

$$1) \quad \lambda_1 \pi = 1 \quad C_2 = C_1$$

$$y_1(x) = \frac{C_1}{\pi} (\sin x + \cos x)$$

$$2) \quad \lambda_2 \pi = -1 \quad C_2 = -C_1$$

$$y_2(x) = \frac{C_2}{\pi} (\sin x - \cos x)$$

It is clear that any function $h(x)$ in the form of $\int_0^{2\pi} g_m(x+s) H(s) ds$ is expressed as

$$h(x) = a_1 y_1 + a_2 y_2$$

Remark: (Completeness) In some cases the eigenfunctions $y_n(x)$ of the homogeneous Fredholm integral equation may not form a complete set. This means that any continuous function $h(x)$ defined in (a,b) may not be represented over the interval (a,b) by a series of $y_n(x)$'s. In the sequel we assume that y_n 's form a complete set over interval (a,b) .

When the kernel is real and symmetric we shall solve the inhomogeneous Fredholm integral equation.

$$y(x) = f(x) + \lambda \int_a^b g(x,s) y(s) ds$$

by the use of the eigenfunctions of the homogeneous equation. Here f is a given function (continuous over (a,b)).

Let us first normalize the orthonormal functions $y_n(x)$, $n=1,2,\dots$ as

$$\phi_n(x) = \frac{y_n(x)}{\sqrt{p_n}}, \quad p_n = \int_a^b y_n^2(x) dx$$

$$\Rightarrow \langle \phi_n, \phi_n \rangle = 1 \quad \forall n=1,2,3,\dots$$

Let
$$y(x) = f(x) + \lambda \int_a^b g(x,s) y(s) ds$$

be an inhomogeneous Fredholm integral equation and the difference function $y(x) - f(x)$ is generated by

$$\int_a^b \lambda g(x,s) y(s) ds$$

Hence by Theorem 7 it should be expressed as linear sum of the functions $\phi_n(x)$

$$y(x) - f(x) = \sum a_n \phi_n(x), \quad a \leq x \leq b$$

where

$$a_n = \int_a^b (y(x) - f(x)) \phi_n(x) dx$$

$$= C_n - \beta_n.$$

$$\forall n = 1, 2, \dots$$

where

$$C_n = \int_a^b y(x) \phi_n(x) dx$$

$$n = 1, 2, \dots$$

$$\beta_n = \int_a^b f(x) \phi_n(x) dx$$

$$n = 1, 2, \dots$$

multiply

$$y(x) = f(x) + \lambda \int_a^b g(x,s) y(s) ds$$

by $\phi_n(x)$ and integrate, we get

$$\begin{aligned} c_n &= \beta_n + \lambda \int_a^b dx \phi_n \int_a^b g(x,s) y(s) ds \\ &= \beta_n + \lambda \int ds y(s) \int \phi_n(x) g(x,s) dx \\ &= \beta_n + \frac{\lambda}{\lambda_n} \int ds \phi_n(s) y(s) = \beta_n + \frac{\lambda}{\lambda_n} c_n. \end{aligned}$$

$$\Rightarrow c_n \left(1 - \frac{\lambda}{\lambda_n}\right) = \beta_n \quad \forall n=1,2,\dots$$

$$c_n = \frac{\lambda_n \beta_n}{\lambda_n - \lambda}$$

If $\lambda \neq \lambda_n$ Then.

$$a_n = c_n - \beta_n = \frac{\lambda_n \beta_n}{\lambda_n - \lambda} - \beta_n = \beta_n \frac{\lambda}{\lambda_n - \lambda}$$

Then

$$y(x) = f(x) + \lambda \sum_{n=0}^{\infty} \frac{\beta_n \phi_n}{\lambda_n - \lambda}$$

Remark We have set

$$\beta_n = \int_a^b f(x) \phi_n(x) dx, \quad n=1,2,\dots$$

but it is not obvious that

$$f(x) = \sum_{n=1}^{\infty} \beta_n \phi_n(x)$$

which is valid only functions represented as

$$\int_a^b g(x,s) H(s) ds$$

ii) If λ equals to any one of λ_n

Then, going back to previous page

$$c_n \left(1 - \frac{\lambda}{\lambda_n}\right) = \beta_n \quad \forall n=1,2,\dots$$

but $n \neq k$.

$n=k$ $\beta_k = 0$ which means that

$$\int_a^b f(x) \phi_k(x) dx = 0 \quad \text{a constraint.}$$

and c_k is left arbitrary. Then

$$y(x) = f(x) + c_k \phi_k + \lambda_k \sum_{n \neq k} \frac{\beta_n \phi_n}{\lambda_n - \lambda_k}$$

Infinite many solutions. If the condition is not satisfied then there exists NO solution.

$$2) \quad F(x) = \int_0^{\infty} \sin(xs) y(s) ds$$

$F(x)$ can be considered as the Fourier transform of $y(x)$.
If $F(x)$ is piecewise differentiable when $x > 0$, and
if $\int_0^{\infty} |F(x)| dx$ exists, then it is known that
the above integral can be ~~converted~~ inverted uniquely
in the form

$$y(x) = \frac{2}{\pi} \int_0^{\infty} \sin(xs) F(s) ds, \quad x > 0$$

Hence this leads that the homogeneous
Fredholm eqn of the second kind

$$y(x) = \lambda \int_0^{\infty} \sin(xs) y(s) ds$$

when converted

$$y(x) = \frac{2}{\pi \lambda} \int_0^{\infty} \sin(xs) y(s) ds$$

$$\Rightarrow \quad \lambda = \frac{2}{\pi \lambda} \Rightarrow \quad \lambda = \pm \sqrt{\frac{2}{\pi}} \quad (y \neq 0)$$

The corresponding eigenfunctions follow from
the identities

$$\sqrt{\frac{\pi}{2}} e^{-ax} \pm \frac{x}{x^2+a^2} = \pm \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin(xs) \left[\sqrt{\frac{\pi}{2}} e^{-as} \pm \frac{s}{a^2+s^2} \right] ds$$

where $x > 0$, $a > 0$. Therefore

$$\lambda_1 = \sqrt{\frac{2}{\pi}} \Rightarrow y_1(x) = \sqrt{\frac{\pi}{2}} e^{-ax} + \frac{x}{a^2+x^2}, \quad (x > 0, a > 0)$$

$$\lambda_2 = -\sqrt{\frac{2}{\pi}} \Rightarrow y_2(x) = \sqrt{\frac{\pi}{2}} e^{-ax} - \frac{x}{a^2+x^2}, \quad (x > 0, a > 0)$$

y_1 and y_2 carry an arbitrary constant a , hence there are infinitely many eigenfunction corresponding to λ_1 and λ_2 . We say that there are infinite "multiplicity".

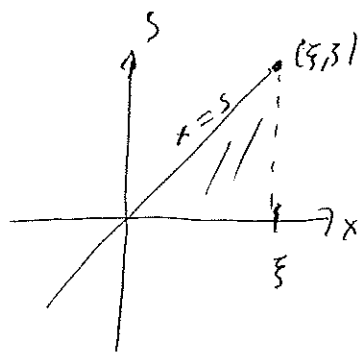
Abel's equation :

$$F(x) = \int_0^x \frac{y(s) ds}{\sqrt{x-s}}$$

The kernel is singular at one of the endpoints.

Dividing both sides by $\sqrt{\xi-x}$ and integrating over $(0, \xi)$ we get

$$\int_0^{\xi} \frac{F(x) dx}{\sqrt{\xi-x}} = \int_0^{\xi} \frac{dx}{\sqrt{\xi-x}} \int_0^x \frac{y(s) ds}{\sqrt{x-s}}$$



$$= \int_0^{\xi} \int_0^x \frac{y(s) ds dx}{\sqrt{\xi-x} \sqrt{x-s}}$$

$$= \int_0^{\xi} \int_s^{\xi} \frac{y(s) dx ds}{\sqrt{x-s} \sqrt{\xi-x}}$$

$$= \int_0^{\xi} ds y(s) \left(\int_s^{\xi} \frac{dx}{\sqrt{\xi-x} \sqrt{x-s}} \right)$$

by letting $x = (\xi-s)t + s$ in

$$\int_s^{\xi} \frac{dx}{\sqrt{(\xi-x)(x-s)}} = \int_0^1 \frac{dt}{\sqrt{t(1-t)}}$$

and letting $\frac{1}{2} - t = \frac{1}{2} \sin \theta$

$$\int_s^{\xi} \frac{dx}{\sqrt{x-s} \sqrt{\xi-x}} = \int_{-\pi/2}^{\pi/2} d\theta = \pi$$

Then.

$$\int_0^{\xi} \frac{F(x) dx}{\sqrt{\xi-x}} = \pi \int_0^{\xi} ds y(s).$$

or

$$\int_0^x y(s) ds = \frac{1}{\pi} \int_0^x \frac{F(s)}{\sqrt{x-s}} ds.$$

Differentiating both sides wrt x we get.

$$y(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x \frac{F(s)}{\sqrt{x-s}} ds.$$

provided that $F(x)$ satisfies certain conditions
so that the RHS exist and continuous.

Lecture 6. INTEGRAL EQUATIONS AND GREEN'S FUNCTIONS

Ronald B Guenther and John W Lee, Partial Differential Equations of Mathematical Physics and Integral Equations.

Hildebrand, Methods of Applied Mathematics, second edition

In the study of the partial differential equations of hyperbolic and parabolic types we solved several initial and boundary value problems. While solving these equations we used the method separation of variables which reduces the problem to one of the following types of *Sturm-Liouville problems*

Sturm-Liouville Eigenvalue problem: Let $p(x) > 0, q(x) \geq 0, r(x) \geq 0$ in $I = (a, b)$. Here we assume $b > a$. Let $X \in C^2(I) \cap C^1(\bar{I})$, then the set of equations given below

$$-\frac{d}{dx} \left[p(x) \frac{dX}{dx} \right] + q(x)X = \lambda r(x)X, \quad (1)$$

$$X(a) - h_0 X'(a) = 0, \quad (2)$$

$$X(b) + h_1 X'(b) = 0. \quad (3)$$

is called the *The Sturm-Liouville Eigenvalue Problem* (SLEVP). Here $h_0 \geq 0$ and $h_1 \geq 0$ are constants. Let the Sturm-Liouville operator L be defined by

$$L = -\frac{d}{dx} p(x) \frac{d}{dx} + q(x), \quad (4)$$

then the *Sturm-Liouville Eigenvalue equation* becomes

$$LX = \lambda r(x)X, \quad X(a) - h_0 X'(a) = 0, \quad X(b) + h_1 X'(b) = 0. \quad (5)$$

Note that L is self-adjoint, i.e., $L = L^*$

The Sturm-Liouville Boundary Value Problem: With the same p, q, r defined above we have the Sturm-Liouville Boundary Value Problem (SLBVP)

$$L X = f(x), \text{ in } I, \quad (6)$$

$$X(a) - h_0 X'(a) = 0, \quad (7)$$

$$X(b) + h_1 X'(b) = 0. \quad (8)$$

Here $f(x)$ is a continuous function in I .

Sturm-Liouville Problem: Combination of SLEVP and SLBVP may be given as

$$L X = \lambda r(x)X + f(x), \text{ in } I, \quad (9)$$

$$X(a) - h_0 x'(a) = 0, \quad (10)$$

$$X(b) + h_1 x'(b) = 0. \quad (11)$$

4.1. The Sturm Liouville Boundary Value Problem and Green's Functions

Let us formulate the SLBVP as

$$L u(x) = f(x), \quad x \in I, \quad (12)$$

$$u(a) - h_0 u'(a) = 0, \quad (13)$$

$$u(b) + h_1 u'(b) = 0. \quad (14)$$

We shall solve this problem by the use of Green's function technique. Let $u, v \in C^2(I) \cap C^1(\bar{I})$ then it is easy to prove the *Lagrange identity*

$$vLu - uLv = -\frac{d}{dx}[p(x)(vu' - uv')], \quad (15)$$

Now letting $v = G(x, y)$ and

$$L_x G(x, y) = \delta(x - y), \quad x, y \in I, \quad (16)$$

$$G(a, y) - h_0 \frac{dG(x, y)}{dx} \Big|_{x=a} = 0, \quad (17)$$

$$G(b, y) + h_1 \frac{dG(x, y)}{dx} \Big|_{x=b} = 0. \quad (18)$$

then we have

$$u(x) = \int_a^b G(x, y) f(y) dy, \quad (19)$$

Hence the SLBVP reduces to the construction of the Green's function $G(x, y)$ satisfying (16)-(18). In addition to these properties $G(x, y)$ satisfy also the following properties

1. $G(x, y)$ is continuous in $x, y \in I$ (*prove it*) in particular we have as $\varepsilon \rightarrow 0_+$

$$G(x, y)|_{x=y-\varepsilon} = G(x, y)|_{x=y+\varepsilon}. \quad (20)$$

2. $\frac{dG(x, y)}{dx}$ is discontinuous at $x = y$. As $\varepsilon \rightarrow 0_+$ we have the jump condition *prove it*

$$\frac{dG(x, y)}{dx} \Big|_{x=y-\varepsilon} - \frac{dG(x, y)}{dx} \Big|_{x=y+\varepsilon} = p(y) \quad (21)$$

Since $G(x, y)$ satisfies the homogeneous equation for $x < y$ and $x > y$ we then have

$$G(x, y) = \begin{cases} a_1(y) v_0(x) + a_2(y) v_1(x), & a \leq x < y \leq b \\ b_1(y) v_0(x) + b_2(y) v_1(x), & a \leq y < x \leq b \end{cases} \quad (22)$$

where $v_0(x)$ and $v_1(x)$ are the solutions of the homogeneous SL-equation and a_1, a_2, b_1 and b_2 are functions of y to be determined through the conditions (17), (18), (20), and (21). If we choose v_0 as a solution of the homogeneous SL equation satisfying the *left boundary condition*

$$v_0(a) - h_0 v_0'(a) = 0, \quad (23)$$

and $v_1(x)$ as the solution of the SL equation satisfying the *right boundary condition*

$$v_1(b) + h_1 v_1'(b) = 0, \quad (24)$$

then $a_2 = b_1 = 0$ and due to the symmetry $G(x, y) = G(y, x)$ and remaining conditions we show that

$$G(x, y) = \begin{cases} v_0(x) v_1(y), & a \leq x < y \leq b \\ v_0(y) v_1(x), & a \leq y < x \leq b \end{cases} \quad (25)$$

The jump condition (21) becomes

$$p(x) W(v_0, v_1) = -1 \quad (26)$$

where $W(u, v) = uv' - vu'$ is the Wronskian of u and v . We then have the following theorem:

Theorem 1. *The SLBVP (12) has a Green's function if and only if the corresponding homogeneous SLEVP [with $f(x) = 0$] has only the trivial solution, in which case the Green's function is given in (25) with the boundary conditions in (23) and (24).*

proof: (a) If we assume the existence of the Green's function we have the solution (19) for each function $f(x)$ in I . The corresponding homogeneous solution (with $f = 0$ in (19)) goes to the trivial solution $u = 0$. (b) Let us assume that the only solution of homogeneous SLBVP is only the trivial one. Hence we should be able to find nontrivial solutions so that $pW = C$. Let $C = 0$. This means v_0 and v_1 are proportional hence, for example v_0 satisfies the homogeneous SLBVP which is nontrivial, hence we obtain a contradiction. This means $C \neq 0$. By a simple scaling $C = -1$ (by redefining

v_0 and v_1

v_0 for instance). Hence $pW = -1$. hence we have v_0 and v_0 solving the homogeneous SL equation, v_0 satisfying the left boundary condition (23) and v_1 the right boundary condition (24). Then there exist a unique Green's function given in (25).

To summarize all properties of the Green's function we formulate the following theorem

Theorem 2. *Assume that the homogeneous SLBVP ($f = 0$ problem has only the trivial solution. Let $G(x, y)$ be the function given in (25) and with the conditions (23) and (24). Then*

i $G(x, y)$ is continuous on the square $a \leq x, y \leq b$ and has continuous second derivatives on each of the triangles $a \leq x < y \leq b$ and $a \leq y < x \leq b$. On each triangle $L_x G(x, y) = 0$

(ii) $G(x, y)$ satisfies the boundary conditions at $x = a$ and $x = b$,

(iii) For each $y \in (a, b)$, $G(x, y)$ satisfies the jump condition (21)

Conversely, properties (i)-(iii) uniquely determine G such that (19) solves the SLBVP in (19).

We have remarked that, since L is self adjoint, the Green's function is symmetric. If $p(x) > 0, q \geq 0$ and $h_0, h_1 \geq 0$ then we have the following result

Theorem 3. *The Green's function of the SLBVP exists if $p(x) > 0, q(x) \geq 0$, and $h_0, h_1 \geq 0$.*

proof. Th.1 says that the Green's function exists if and only if the homogeneous SLBVP has only the trivial solution. The homogeneous SLBVP is given as follows. Let $z \in C^2(I) \cap C^1(\bar{I})$ satisfy

$$-[p(x)z']' + q(x)z = 0, \quad x \in I, \quad (27)$$

$$z(a) - h_a z'(a) = 0, \quad z(b) + h_1 z'(b) = 0. \quad (28)$$

Multiplying the differential equation by Z and integrating over I and using integration parts we obtain

$$p(b)h_1z'(b)^2 + p(a)h_0z'(a)^2 + \int_a^b [p(x)z'^2 + q(x)z^2]dx = 0 \quad (29)$$

Here, since $p > 0$ in I then $z'(x) = 0$ in I . Hence $z =$ a constant everywhere in I , but this constant should vanish at the boundary points a and b , then $z(x)$ must vanish at all pints in I . This proves that under the given circumstances the only solution of the SLBVP is the trivial one , $z = 0$.

Example 1. Solve $u'' = f(x)$, $x \in (a, b)$ and $u(a) = u(b) = 0$.

solution. Let

$$G(x, y) = \begin{cases} a_1x + a_2, & a \leq x < y \leq b \\ b_1x + b_2, & a \leq y < x \leq b \end{cases} \quad (30)$$

where the coefficient are to be determined through the conditions (17), (18), (20) and (21). We find

$$G(x, y) = \begin{cases} \frac{(x-a)(y-b)}{b-a}, & a \leq x < y \leq b \\ \frac{(y-a)(x-b)}{b-a}, & a \leq y < x \leq b \end{cases} \quad (31)$$

Hence the solution is given by

$$u(x) = \int_a^b G(x, y)f(y)dy = \quad (32)$$

$$\int_a^x \frac{(y-a)(x-b)}{b-a} f(y)dy + \int_x^b \frac{(y-b)(x-a)}{b-a} f(y)dy, \quad (33)$$

$$= \frac{x-b}{b-a} \int_a^x (y-a) f(y)dy + \frac{x-a}{b-a} \int_x^b (y-b) f(y)dy \quad (34)$$

Example 2. Solve the same problem by changing the boundary conditions to $u(a) = 0$, $u'(b) = 0$.

Example 3. Solve the same problem by changing the boundary conditions to $u(a) = 0, \quad u'(a) = 0$.

Example 4. Solve the same problem by changing the boundary conditions to $u(a) - \alpha^2 u'(a) = 0$ and $u(b) + \beta^2 u'(b) = 0$. Here α and β are some constants.

4.2 The Neumann Series

Let us now consider the general SL problem

$$L u = \lambda r(x) u + f(x), \quad x \in (a, b), \quad (35)$$

$$u(a) - h_0 u'(a) = 0, \quad u(b) + h_1 u'(b) = 0. \quad (36)$$

Since the Green's function $G(x, y)$ outlined in the first section belongs to the operator L we then have

$$u(x) = \int_a^b G(x, y) [f(y) + \lambda r(y) u(y)] dy \quad (37)$$

or, equivalently we have

$$u(x) = g(x) + \lambda \int_a^b G(x, y) r(y) u(y) dy, \quad (38)$$

where

$$g(x) = \int_a^b G(x, y) f(y) dy$$

Eq(38) is called the *The Fredholm integral equation of the second kind* which can be put into a more symmetrical one. Let

$$y(x) = \sqrt{r(x)} u(x), \quad g(x, s) = \sqrt{r(x)} \sqrt{r(s)} G(x, s), \quad f_1(x) = \sqrt{r(x)} g(x). \quad (39)$$

then

$$y(x) = f_1(x) + \lambda \int_a^b g(x, s) y(s) ds, \quad (40)$$

In our future analysis we let $f_1 \rightarrow f$. Here $g(x, s)$ is called the *kernel* of the integral equation, $f(x)$ is given and λ is in general a complex parameter. In most cases it is real. And also we can assume that $g(x, s)$ is continuous in $a \leq x, s \leq b$ and $f(x)$ is continuous in $a \leq x \leq b$. The integral equation given in (40) may be solved by using several methods. Let us assume that y is a power series in λ

$$y(x) = \sum_{n=0}^{\infty} \lambda^n y_n(x), \quad (41)$$

where the coefficients $y_n(x)$ will be determined by the use of the integral equation (40) (we let $f_1 = f$). We obtain

$$y_0(x) = f(x), \quad (42)$$

$$y_1(x) = \int_a^b g(x, s) y_0(s) ds, \quad (43)$$

$$y_2(x) = \int_a^b g(x, s) y_1(s) ds, \quad (44)$$

$$y_{n+1} = \int_a^b g(x, s) y_n(s) ds, \quad (45)$$

for $n = 0, 1, \dots$. Hence all $y_n(x)$ can be calculated recursively. We have shown that the series (41) must be uniformly convergent to justify the term by term integration in (40).

Proposition 4. *Assume that $g(x, s)$ is continuous in $a \leq x, s \leq b$ and $f(x)$ is continuous in $a \leq x \leq b$. The series in (40) is uniformly convergent if $|\lambda|M(b-a) \leq 1$*

proof. Since both g and f are continuous we have

$$M = \max |g(x, s)|, \quad N = \max |f(x)|$$

in $a \leq x, s \leq b$. Using (42), (43), (44), and (45) we obtain

$$|y_0(x)| \leq N, \quad (46)$$

$$|y_1(x)| \leq MN(b-a), \quad (47)$$

$$|y_2(x)| \leq NM^2(b-a)^2, |y_n(x)| \leq NM^n(b-a)^n \quad (48)$$

which leads to

$$|\sum_{n=0}^{\infty} \lambda^n y^n(x)| \leq N \sum_{n=0}^{\infty} [|\lambda|M(b-a)|]^n < \infty$$

This result provides also a solution of the integral equation (40)

Proposition 5. *Let*

$$y(x) = f(x) + \lambda \int_a^b g(x, s) y(s) ds \quad (49)$$

be the integral equation. Let $g(x, s)$ and $f(x)$ be continuous in $a \leq x, s \leq b$. If $|\lambda| < \frac{1}{M(b-a)}$ then

$$y(x) = f(x) + \lambda \int_a^b \gamma(x, s) f(s) ds \quad (50)$$

where $\gamma(x, s)$ is called the resolvent kernel and given by

$$\gamma(x, s) = \sum_{n=1}^{\infty} \lambda^{n-1} k_n(x, s) \quad (51)$$

proof: Using (42)-(45) we obtain

$$y_0(x) = f(x), \quad (52)$$

$$y_1(x) = \int_a^b g(x, s) f(s) ds, \quad (53)$$

$$y_2(x) = \int_a^b g(x, s) y_1 ds, \quad (54)$$

$$= \int_a^b g_2(x, s) f(s) ds, \quad (55)$$

$$(56)$$

where $g_2(x, s) = \int_a^b g(x, t) g(t, s) dt$. Defining, in general

$$g_1(x, s) = g(x, s), \quad (57)$$

$$g_n(x, s) = \int_a^b g(x, t) g_{n-1}(t, s) dt, \quad n = 2, 3, \dots \quad (58)$$

then it is straightforward to establish the relation

$$y_n(x) = \int_a^b g_n(x, s) f(s) ds, \quad n = 1, 2, \dots$$

and hence

$$y(x) = \sum_{n=0}^{\infty} \lambda^n y_n(x) \quad (59)$$

$$= f(x) + \sum_{n=1}^{\infty} \lambda^n y_n(x) \quad (60)$$

$$= f(x) + \sum_{n=1}^{\infty} \lambda^n \int_a^b g_n(s) f(s) ds \quad (61)$$

$$= f(s) + \lambda \int_a^b \gamma(x, s) f(s) ds, \quad (62)$$

where

$$\gamma(x, s) = \sum_{n=1}^{\infty} \lambda^{n-1} g_n(x, s) \quad (63)$$

Convergence of the series in (62) is guaranteed by Prop.4, and the solution given in Prop.5 is unique.

Theorem 6. *Let $g(x, s)$ and $f(x)$ be continuous on $a \leq x, s \leq b$. Let $M = \max|g(x, s)|$ for all $(x, s) \in [a, b]$. If $|\lambda| < \frac{1}{M(b-a)}$ the the solution*

$$y(x) = f(x) + \lambda \int_a^b \gamma(x, s) f(s) ds$$

is unique and continuous.

proof: Let us assume that there are two different solutions $z_1(x)$ and $z_2(x)$ of the integral equation (49). Then their difference $w(x) = z_2(x) - z_1(x)$ satisfies

$$w(x) = \lambda \int_a^b g(x, s) w(s) ds$$

Hence we have for all $x \in [a, b]$ we have

$$|w(x)| \leq |\lambda| M W (b - a)$$

where $W = \max|w(x)|$. Then

$$[1 - |\lambda| M (b - a)] W \leq 0$$

This implies that $W = 0$. Hence $z_1 = z_2$ everywhere in $[a, b]$.

Example 5. Find the solution of

$$y(x) = f(x) + \lambda \int_a^b e^{x-s} y(s) ds$$

Solution: Try to find the following solution. Given a continuous $f(x)$ for $x \in [a, b]$ we have

$$y(x) = f(x) + \frac{\lambda e^x}{1 - \lambda(b-a)} \int_a^b e^{-s} f(s) ds$$

where $\lambda \neq \frac{1}{b-a}$. (i) Discuss the case $\lambda = \frac{1}{b-a}$ (ii) and as an example let $f(x) = x^2$. Discuss also the application of (iii) Prop.4, (iv) Prop.5 and (v) Thm.6 to this example. (vi) find also the Neumann series corresponding to this example.

4.3. Fredholm Equation with Separable Kernels.

A separable kernel is given as

$$g(x, s) = \sum_{n=1}^N p_n(x) q_n(s) \quad (64)$$

Here we assume that the N functions p_n are linearly independent. Then the integral equation (49) takes the form

$$y(x) = f(x) + \lambda \int_a^b g(x, s) y(s) ds, \quad (65)$$

$$= f(x) + \lambda \int_a^b \left[\sum_{n=1}^N p_n(x) q_n(s) \right] y(s) ds, \quad (66)$$

$$= f(x) + \lambda \sum_{n=1}^N p_n(x) \left[\int_a^b q_n(s) y(s) ds \right], \quad (67)$$

$$= f(x) + \lambda \sum_{n=1}^N c_n p_n(x), \quad (68)$$

where

$$c_n = \int_a^b q_n(s) y(s) ds, \quad n = 1, 2, \dots \quad (69)$$

Using (68) in (69) we obtain the following algebraic linear equations for the constants c_n 's

$$c_n = \beta_n + \lambda \sum_{m=1}^N \alpha_{nm} c_m, \quad (70)$$

where for all $n, m = 1, 2, \dots$

$$\beta_n = \int_a^b q_n(s) f(s) ds, \quad (71)$$

$$\alpha_{mn} = \int_a^b p_n(s) q_m(s) ds \quad (72)$$

let A denote the $N \times N$ matrix corresponding to α_{mn} , B be the column N -vector corresponding to β_n , and C be the column unknown N -vector to be determined then (70) simply becomes

$$[I - \lambda A]C = B, \quad (73)$$

where I is the $N \times N$ unit matrix. The above linear equation for C is easily solved, but we have to consider all possible cases. In the above equation (70) we need to determine the unknown coefficients c_n 's in terms of the known coefficients β_n and α_{mn} for all $m, n = 1, 2, \dots$

case (a). $f(x) = 0$ or $B = 0$ the equation (73) becomes homogeneous. For nontrivial solutions $\det[I - \lambda A]$ must vanish. Otherwise there is only the trivial solution $c_n = 0$, for all $n = 1, 2, \dots$. If $\det[I - \lambda A] = 0$ at least (depending upon the rank of matrix A) one of the c_n 's is left arbitrary. In such a case there are infinitely many solutions. To remind you the terminology: Those values of λ where $\det[I - \lambda A] = 0$ are called characteristic or eigenvalues and any nontrivial solution of the homogeneous integral equation is called the corresponding characteristic or eigenfunction. If there are k number of constants c_n 's $n = 1, 2, \dots, k$ for a given eigenvalue λ , then k number of linearly independent eigenfunctions are obtained.

case (b). $f(x) \neq 0$ but $\beta_n = 0$, $n = 1, 2, \dots$ this means that $f(x)$ is orthogonal to all functions $q_n(x)$, $n = 1, 2, \dots$. Hence $B = 0$. The case (a) applies also here except for the fact that here the solution (68) contains the function $f(x)$. Hence the trivial solution $C = 0$ correspond to the solution

$y(x) = f(x)$. Solutions corresponding to eigenvalues of λ should be expressed as the sum of $f(x)$ and linear sum of the corresponding eigenfunctions.

case (c). $B \neq 0$. We assume that at least for some n , $\beta_n \neq 0$. if $\det[I - \lambda A] \neq 0$ a unique nontrivial solution of (73) exists, leading to a unique nontrivial solution $y(x)$ of the integral equation (68). if $\det[I - \lambda A] = 0$ either there is no solution or the solution is not unique meaning that there are infinitely many solutions.

Example 6. Let $g(x, s) = 1 - 3xs$. Solve the corresponding integral equation by considering all three cases above

4.4. Hilbert-Schmidt Theory

When the kernel $g(x, s)$ is not of type (64) there are, in general, infinitely many eigenvalues and eigenfunctions of the homogeneous Fredholm equation. In addition, there may also be more than one eigenfunctions corresponding to one eigenvalue. This is called the degeneracy and the number of eigenfunctions corresponding to a single eigenvalue is called the multiplicity. In this section we assume that the kernel $g(x, s)$ is symmetric with respect to the variables x and s . We also assume that multiplicity is one.

Remark: The homogeneous Fredholm equation

$$y(x) = \lambda \int_a^b g(x, s) y(s) ds \quad (74)$$

can not have zero eigenvalue, because the corresponding eigenfunction is also zero. We then have the following result.

Proposition 7. *Eigenfunctions of the homogeneous Fredholm integral equation with a symmetric kernel corresponding to different eigenvalues are orthogonal.*

proof: By its definition we have that

$$y_m = \lambda_m \int_a^b g(x, s) y_m ds, \quad m = 1, 2, \dots \quad (75)$$

Multiplying by $y_n(x)$ and integrating over (a, b) we get

$$\int_a^b y_m(x) y_n(x) dx = \lambda_m \int_a^b y_n(x) dx \int_a^b g(x, s) y_m(s) ds, \quad (76)$$

$$= \lambda_m \int_a^b y_m(s) ds \int_a^b y_n(x) g(x, s) dx, \quad (77)$$

$$= \frac{\lambda_m}{\lambda_n} \int_a^b y_m(s) y_n(s) ds \quad (78)$$

which leads to

$$\int_a^b y_m(x) y_n(x) dx = 0 \quad (79)$$

when $\lambda_m \neq \lambda_n$

Remark 1: If there are more than one eigenfunctions corresponding to an eigenvalue λ_n then orthogonalization of such a set is performed by the standard Gram-Schmidt procedure. In the sequel we assume that such sets are orthogonalized.

Remark 2: In the case of complex functions we generalize the Prop. 7 in the following way

Proposition 7'. *Eigenfunctions of the homogeneous Fredholm integral equation with a hermitian kernel , $g(x, s) = \bar{g}(s, x)$ corresponding to different eigenvalues are orthogonal.*

Note that the inner product in this case is defined by

$$\langle f, g \rangle = \int_a^b f(x) \bar{g}(x) dx$$

where bar over a letter denotes complex conjugation. This proposition implies that

$$\int_a^b y_m(x) \bar{y}_n(x) dx = 0$$

for $m \neq n$. A Corollary of this proposition is that λ_n for such homogeneous Fredholm integral equation with hermitian kernel and hence also for real symmetric kernels the eigenvalues are real (*Prove these statements*).

We have now a theorem which will be used very often in our future analysis.

Theorem 8. *Let $g(x, s)$ be a real and symmetric continuous kernel over (a, b) . Let $H(x)$ be any continuous function over (a, b) the any function defined as*

$$h(x) = \int_a^b g(x, s) H(s) ds \quad (80)$$

can be represented as a linear superposition of the eigenfunctions $y_n(x)$, $n = 1, 2, \dots$, of the homogeneous Fredholm equations with same kernel over (a, b) .

Hence we have

$$h(x) = \sum_{n=1} a_n y_n(x)$$

and

$$a_n = \frac{1}{\rho_n} \int_a^b h(x) y_n(x) dx$$

with $\rho_n = \int_a^b y_n(x)^2 dx$.

Remark 3: If there are finite number of eigenfunctions then the functions generated by the operation

$$\int_a^b g(x, s) H(s) ds$$

form a restricted class of functions, irrespective the form of the function $H(x)$.

Example 7. Let $g(x, s) = \sin(x + s)$ with $(a, b) = (0, 2\pi)$ then it is easy to show that

$$\lambda_1 = \frac{1}{\pi}, \quad y_1 = \sin x + \cos x, \quad (81)$$

$$\lambda_2 = -\frac{1}{\pi}, \quad y_2 = \sin x - \cos x. \quad (82)$$

Hence we have finite number of eigenfunctions. Then any function $h(x)$ given as

$$h(x) = \int_0^{2\pi} g(x, s) H(s) ds$$

takes the form

$$g(x) = C_1 \sin x + C_2 \cos x$$

whatever the function $H(x)$ is. It is obvious that $h(x)$ can also be written as

$$h(x) = a_1 y_1(x) + a_2 y_2(x)$$

Remark 4. In some cases the eigenfunctions of the homogeneous Fredholm integral equation may not form a complete set (see DK for the definition). This means that any continuous function $f(x)$ defined in (a, b) may not be represented over the same interval by a series of y_n 's. In the sequel we assume that y_n 's form a complete set over the interval (a, b) .

The essence of the eigenfunctions of the homogeneous Fredholm integral equation with a real and symmetric kernel shows up when we wish to solve the inhomogeneous Fredholm integral equation

$$y(x) = f(x) + \lambda \int_a^b g(x, s) y(s) ds \quad (83)$$

where $f(x)$ is a given continuous function over the interval (a, b) . First we shall use the orthonormalized set of eigenfunctions $\phi_n(x)$ which are defined by

$$\phi_n = C_n y_n(x)$$

where $C_n = \frac{1}{\sqrt{\rho_n}}$ and

$$\int_a^b \phi_m(x) \phi_n(x) dx = 0, \quad m \neq n$$

Then from (83) by letting

$$\int_a^b \lambda g(x, s) y(s) ds = \sum_{n=1} a_n \phi_n(x)$$

Hence we get

$$y(x) = f(x) + \sum_{n=1} a_n \phi_n(x), \quad a \leq x \leq b$$

where

$$a_n = \int_a^b [y(s) - f(s)] \phi_n(s) ds$$

By defining

$$c_n = \int_a^b y(s) \phi_n(s) ds, \quad \beta_n = \int_a^b f(s) \phi_n(s) ds$$

then

$$a_n = c_n - \beta_n$$

Multiplying (83) by $\phi_n(x)$ over (a, b) and using the symmetry of the kernel we obtain

$$\left[1 - \frac{\lambda}{\lambda_n}\right] c_n = \beta_n, \quad n = 1, 2, \dots \quad (84)$$

case a. If $\lambda = \lambda_k$ where λ_k is of the eigenvalues then c_k becomes arbitrary and the solution and $\beta_k = 0$ or $\int_a^b f(s) \phi_k(s) ds = 0$ and the solution becomes

$$y(x) = f(x) + c_k \phi_k(x) + \lambda_k \sum_{n \neq k} \frac{\beta_n}{\lambda_n - \lambda} \phi_n(x) \quad (85)$$

case b. If $\lambda \neq \lambda_k, k = 1, 2, \dots$, any one of the eigenvalues. Then the solution is unique.

$$y(x) = f(x) + \lambda \sum_{n=1}^{\infty} \frac{\beta_n}{\lambda_n - \lambda} \phi_n(x)$$

4.4 Singular Integral Equations

So far we have assumed that (i) the interval (a, b) of the integral equations are finite and (ii) the kernel $g(x, s)$ was continuous. If an integral equation has either an infinite interval or has a discontinuous kernel then such an integral equation is called *singular integral equation*. The first two of the following are singular Fredholm equations of the first kind and the last one is the singular Volterra integral equation of the second kind (known also as the Abel's equation). Here $y(x)$ are unknown functions to be determined in each cases.

$$a) \quad F(x) = \int_0^{\infty} e^{-xs} y(s) ds, \quad (86)$$

$$b) \quad F(x) = \int_0^{\infty} \sin(xs) y(s) ds, \quad (87)$$

$$c) \quad F(x) = \int_0^x \frac{y(s)}{\sqrt{x-s}} ds \quad (88)$$

Here in each case $F(x)$ is a given function. The solutions of the homogeneous integral equations

$$y(s) = \lambda \int_a^b g(x, s) y(s) ds, \quad (89)$$

with

(i): $(a, b) = (0, \infty)$ or

(ii): $g(x, s)$ is not continuous in (a, b)

do not share the same properties as solutions of the homogeneous equations in the previous sections. For example the first singular equation above has

continuous eigenvalues. the second one has two eigenvalues with infinite multiplicity and the last one can be solved exactly. Here we shall only present the first case. Recall the definition of the Gamma function

$$\int_0^{\infty} e^{-xs} s^{\alpha-1} ds = \Gamma(\alpha) x^{-\alpha}, \quad \alpha > 0 \quad (90)$$

Changing α to $1 - \alpha$ and rewriting the above equation once more we get

$$\int_0^{\infty} e^{-xs} s^{-\alpha} ds = \Gamma(1 - \alpha) x^{\alpha-1}, \quad \alpha < 1 \quad (91)$$

Dividing first one by $\sqrt{\Gamma(\alpha)}$ the second one by $\sqrt{\Gamma(1 - \alpha)}$ and adding them we obtain

$$\int_0^{\infty} \left[\frac{1}{\sqrt{\Gamma(\alpha)}} s^{\alpha-1} + \frac{1}{\sqrt{\Gamma(1 - \alpha)}} s^{-\alpha} \right] ds \quad (92)$$

$$= \sqrt{\Gamma(\alpha)} x^{-\alpha} + \sqrt{\Gamma(1 - \alpha)} x^{\alpha-1}, \quad 0 < \alpha < 1 \quad (93)$$

This last equation gives us the eigenvalues and the eigenfunctions of the homogeneous singular Fredholm integral equation (89)

$$\lambda_{\alpha} = \frac{1}{\sqrt{\Gamma(\alpha) \Gamma(1 - \alpha)}}, \quad (94)$$

$$y_{\alpha} = \sqrt{\Gamma(1 - \alpha)} x^{\alpha-1} + \sqrt{\Gamma(\alpha)} x^{\alpha} \quad (95)$$

By using the identity

$$\Gamma(\alpha) \Gamma(1 - \alpha) = \frac{\pi}{\sin \alpha \pi}, \quad 0 < \alpha < 1$$

then the eigenvalues become more simpler

$$\lambda_{\alpha} = \sqrt{\frac{\sin \alpha \pi}{\pi}}$$

As we observe that α takes any values in the interval $(0, 1)$, hence eigenvalues take also continuous values.

THE INTEGRAL EQUATIONS AND THE GREEN'S FUNCTION

Ronald B Guenther and John W Lee, Partial Differential Equations of Mathematical Physics and Integral Equations

See also Ch.4 of the lecture notes.

1. Show that the Green's function is unique, if it exists. *Hint.* If both $G(x, y)$ and $H(x, y)$ provides the same $u(x)$ given by

$$u(x) = \int_a^b G(x, y)f(y)dy, \quad (96)$$

$$u(x) = \int_a^b H(x, y)f(y)dy, \quad (97)$$

then

$$\int_a^b [G(x, y) - H(x, y)]f(y)dy = 0$$

2. Find Green's function of the following problem

$$-k u'' + l u = 0, \quad u(0) - u'(0) = 0, \quad u(1) = 0.$$

3. Discuss the existence and the construction of the Green's function of the following problem

$$-[p(x)u']' + q(x)u = f(x), \quad 0 < x < 1, \quad (98)$$

$$u(0) = u(1), \quad u'(0) = u'(1) \quad (99)$$

4. Find Green's function for the initial value problem

$$Lu = f(x), \quad u(0) = 0, \quad u'(0) = 0.$$

Show that the solution has the form

$$u(x) = \int_0^x G(x, y) f(y) dy.$$

5. Let $u'' - u = f(x)$. Find Green's function for the boundary conditions $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

THE INTEGRAL EQUATIONS, GREEN'S FUNCTION AND THE NEUMANN SERIES

1. Ronald B Guenther and John W Lee, 2. Hildebrandt
See also Ch.4 of the lecture notes.

1. An integral equation of the form

$$y(x) = f(x) + \lambda \int_a^x g(x, s) y(s) ds$$

is called a *Volterra equation*.

a. Show that , if $y(x)$ satisfies the differential equation

$$y'' + xy = 1$$

and the conditions $y(0) = y'(0) = 0$, then y satisfies the Volterra equation

$$y(x) = \int_0^x (s - x) s y(s) ds + \frac{x^2}{2}$$

b. Prove that the converse of the preceding statement is also true.

2. Transform the problem

$$y'' + xy = 1, \quad y(0) = y(1) = 0$$

to the integral equation

$$y(x) = -\frac{1}{2}s(1 - x) + \int_0^1 G(x, s) s y(s) ds$$

where $G(x, s) = x(1 - s)$ when $x < s$ and $G(x, s) = s(1 - x)$ when $x > s$.

3. Consider the integral equation

$$y(x) = f(x) + \lambda \int_a^b e^{x-s} y(s) ds.$$

(i) Find the iterated kernels $g_n(x, s)$ defined in the lecture (ii) the resolvent kernel $\gamma(x, s)$ (iii) Find the solution (iv) find the cases (restrictions on λ where there is no solution and (v) the solution is not unique (vi) find the solution when $f(x) = x^2$.

4. a. Consider the integral equation

$$y(x) = f(x) + \lambda \int_a^b q(x) q(s) y(s) ds$$

where $q(x)$ with $x \in [a, b]$ is a continuous function. Answer all questions (i)-(v) as in Pr.3. Consider also b. let $q(x) = x$ discuss the cases (i)-(v) and c. let $f(x) = \sin x$ then consider part b.

5. Consider the integral equation

$$y(x) = f(x) + \lambda \int_0^1 (1 - kxs) y(s) ds$$

where $f(x)$ is a continuous function in $[0, 1]$, k is a constant and λ is the parameter of the integral equation. Follow the cases (i)-(v) in Problem 3 and answer them. Discuss all possible cases.

6. Solve the following equation

$$y(x) = \lambda \int_0^{2\pi} \sin(x + s) y(s) ds.$$

Discuss all possibilities

7. Obtain the most general solution of the integral equation

$$y(x) = f(x) + \lambda \int_0^{2\pi} \sin(x + s) y(s) ds.$$

when $f(x) = x$ and $f(x) = 1$. Discuss all cases.

8. Prove that the equation

$$y(x) = f(x) + \frac{1}{\pi} \int_0^{2\pi} \sin(x+s) y(s) ds$$

has no solution when $f(x) = x$, but that it possesses infinitely many solutions when $f(x) = 1$. Determine all such solutions

THE INTEGRAL EQUATIONS AND THE GREEN'S FUNCTION

Ronald B Guenther and John W Lee, Partial Differential Equations of Mathematical Physics and Integral Equations

See also Ch.4 of the lecture notes.

1. Show that the Green's function is unique, if it exists. *Hint.* If both $G(x, y)$ and $H(x, y)$ provides the same $u(x)$ given by

$$u(x) = \int_a^b G(x, y) f(y) dy, \quad (100)$$

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then

$$\int_a^b [G(x, y) - H(x, y)] f(y) dy = 0$$

2. Find Green's function of the following problem

$$-k u'' + l u = 0, \quad u(0) - u'(0) = 0, \quad u(1) = 0.$$

3. Discuss the existence and the construction of the Green's function of the following problem

$$-[p(x)u']' + q(x)u = f(x), \quad 0 < x < 1, \quad (102)$$

$$u(0) = u(1), \quad u'(0) = u'(1) \quad (103)$$

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