

MATH544: Methods of Applied Mathematics II

Spring 2012 lecture 5

- Laplace Equation

- Green's Function for Laplace eqn.

- Maximum - Minimum Principles for the
Laplace Eqn.

(1)

Laplace Equation (The Potential Theory)

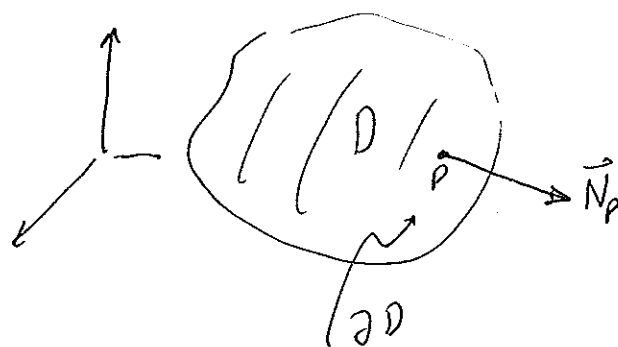
Boundary value problems of potential Theory

Both in two dimensions and in three dimensions

$\mathbb{R}^2$



$\mathbb{R}^3$



1) Dirichlet's problem

$$\nabla^2 u = 0 \quad \text{in } D$$

$$u = f \quad \text{on } \partial D$$

2) Neuman's problem

$$\nabla^2 u = 0 \quad \text{in } D$$

$$\frac{\partial u}{\partial n} = \vec{N} \cdot \vec{\nabla} u = g(x) \quad \text{on } \partial D$$

(2)

3) Robin's Problem

$$\nabla^2 u = 0 \quad \text{in } D$$

$$\frac{\partial u}{\partial n} + \alpha(x) u = \beta(x) \quad , \quad x \in \partial D$$

$$\text{where } \alpha(x) > 0 \quad \forall x \in \partial D$$

Some preliminaries: In addition to the Cartesian coordinates

$$\mathbb{R}^2 : \quad \nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \quad \text{in polar coordinates}$$

$$\mathbb{R}^3 : \quad \nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \quad \text{in cylindrical coordinates}$$

$$\begin{aligned} \nabla^2 u = & \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} \\ & + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) \end{aligned} \quad \text{in spherical coordinates}$$

In $\mathbb{R}^3$ there are other coordinate systems but we shall use those coordinate system in addition to the Cartesian coordinate system

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Some remarks: In $\mathbb{R}^2$ a function ϕ is said to be harmonic if it satisfies the Laplace eqn.

$$\phi_{xx} + \phi_{yy} = 0$$

Cauchy-Riemann equations

$$\psi_x = \phi_y$$

$$\psi_y = -\phi_x$$

relate solutions of the Laplace eqn. It means that the function ψ is also harmonic

$$\psi_{xx} + \psi_{yy} = 0$$

In $\mathbb{R}^n$ a spherically symmetric function $u = u(r)$ has the property that

$$\begin{aligned} \nabla^2 u &= u_{x_1 x_1} + u_{x_2 x_2} + \dots + u_{x_n x_n} \\ &= u'' + \frac{n-1}{r} u' \quad , \quad r = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \end{aligned}$$

$$\nabla^2 u = 0 \Rightarrow$$

$$u = a_1 \ln r + a_2 \quad n=2$$

$$= a_1 r^{2-n} + a_2 \quad \text{for } n \geq 3$$

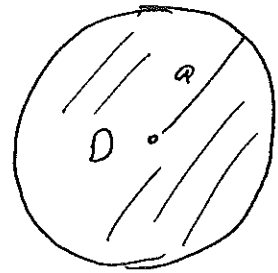
a_1 and a_2 are constants.

Exact Solutions

To solve boundary value problems we use separation of variable for finite regions and integral transforms for infinite regions

Example 1. A Dirichlet's problem where the region D is bounded by a circle with radius a .

$$\begin{aligned}\nabla^2 u &= 0 & \text{in } D \\ u &= f & \text{at } r=a\end{aligned}$$



Solution: $\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$

we use the method of separation of variables for $r < a, 0 \leq \theta \leq 2\pi$

$$u(r, \theta) = R(r) \Theta(\theta)$$

$$\frac{1}{Rr} \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) + \frac{1}{\Theta r^2} \frac{d^2 \Theta}{d\theta^2} = 0$$

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Let

$$\Theta'' = \lambda \Theta, \quad \lambda = \text{constant}$$

Then

$$\frac{d}{dr} \left(r \frac{dR}{dr} \right) + \frac{\lambda}{r} R = 0$$

To single valued solutions

$$\Theta(\theta + 2\pi m) = \Theta(\theta), \quad m = 1, 2, \dots$$

we must have $\lambda = -n^2$, because.

$$\Theta = A \cos \sqrt{-\lambda} \theta + B \sin \sqrt{-\lambda} \theta$$

$$\sqrt{-\lambda} 2\pi = 2\pi n \Rightarrow \lambda = -n^2$$

$$n = \pm 1, \pm 2, \dots$$

Hence we get

$$\Theta = A_n \cos n\theta + B_n \sin n\theta, \quad 0 \leq \theta \leq 2\pi$$

Radial equation

$$R'' + \frac{1}{r} R' - \frac{n^2}{r^2} R = 0$$

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$$\text{for } n=0 \quad R = A_0 \ln r + B_0$$

$$\text{for } n \neq 0 \quad R_n = A_n \left(\frac{r}{a}\right)^{-n} + B_n \left(\frac{r}{a}\right)^n$$

$$\left[\text{Take } r = e^x, \quad R' = R_x \frac{1}{r}, \quad R'' = \frac{1}{r^2} R_{xx} - \frac{1}{r^2} R_x \right. \\ \left. R_{xx} - n^2 R = 0 \right].$$

At the center of D , $r=0$, the solution must exist hence $A_n = 0$. Then we get $(n=0, 1, 2, \dots)$.

$$u(r, \theta) = B_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (C_n \sin n\theta + D_n \cos n\theta) \quad (1)$$

Boundary condition

$$u(a, \theta) = B_0 + \sum_{n=1}^{\infty} (C_n \sin n\theta + D_n \cos n\theta) = f(\theta) \quad (2)$$

$\Rightarrow$

$$C_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta \quad n=1, 2, \dots$$

$$D_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta$$

$$B_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta = A_V(f).$$

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We obtained the "formal solution" of the boundary value problem. We have two infinite sums which should be uniformly convergent. The first one is

$$\left| \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (C_n \sin n\theta + D_n \cos n\theta) \right| \\ \leq \sum \left(\frac{r}{a}\right)^n [|C_n| + |D_n|]$$

It is enough to have $|C_n|$ and $|D_n|$ be bounded for all $n \geq 1$. This requires $f(\theta)$ to be continuous and periodic in θ , i.e. $f(\theta + 2\pi) = f(\theta)$. ~~The second one (2)~~

$$\left[\text{Since } \sum \left(\frac{r}{a}\right)^n \text{ is a power series} \right. \\ \left. \text{and converges} = \frac{1}{1 - r/a}, \quad r/a < 1 \right]$$

The second one (2)

$$\left| \sum C_n \sin(n\theta) + D_n \cos n\theta \right| \leq \sum (|C_n| + |D_n|)$$

This needs $C_n \sim \frac{1}{n^2}$ and $D_n \sim \frac{1}{n^2}$

which requires that f has continuous second derivative.

Remark: We may not need the twice differentiability of f to have the boundary condition.

$$u(a, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (C_n \cos n\theta + D_n \sin n\theta).$$

using C_n and D_n in page (6) we get.

$$\begin{aligned} a_n \cos n\theta + b_n \sin n\theta &= \frac{1}{n\pi} \int_0^{2\pi} f'(\theta') [-\sin n\theta' \cos n\theta \\ &\quad + \cos n\theta' \sin n\theta] d\theta' \\ &= \frac{M_n}{n} \end{aligned}$$

where

$$M_n(\theta) = \frac{1}{\pi} \int_0^{2\pi} f'(\theta') \sin n(\theta - \theta') d\theta'$$

Then

$$u(a, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{M_n(\theta)}{n}.$$

The series in the right hand is an alternating series which converges if f' is continuous in $[0, 2\pi]$.

Then we have the theorem

Theorem: Suppose that $f(\theta)$ is continuous and 2π periodic. Then $u(r, \theta)$ defined by (1) ~~satisfies~~ satisfies $\nabla^2 u = 0$, $0 \leq r < a$. In addition if f has a continuous second derivative, $u(r, \theta)$ satisfies also the boundary condition $u(a, \theta) = f(\theta)$.

Here, in the theorem the differentiability condition might be so stronger. Consider the following argument.

Inserting B_0 , C_n , and D_n into the expression of $u(r, \theta)$ in (1)

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta') d\theta' \\ &\quad + \frac{1}{\pi} \int_0^{2\pi} \sum \left(\frac{r}{a}\right)^n f(\theta') [\cos n\theta' \cos n\theta + \sin n\theta' \sin n\theta] d\theta' \\ &= \frac{1}{\pi} \int_0^{2\pi} f(\theta') d\theta' \left[\frac{1}{2} + \sum \left(\frac{r}{a}\right)^n \cos n(\theta - \theta') \right] d\theta' \end{aligned}$$

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An identity

$$\frac{1}{2} + \sum_{n=1}^{\infty} x^n \cos n\alpha = \frac{1}{2} \frac{1-x^2}{1-2x\cos\alpha+x^2}$$

Then

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\frac{1}{2}(1-r^2/a^2) f(\theta') d\theta'}{1-2(r/a)\cos(\theta-\theta')+r^2/a^2}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2-r^2) f(\theta') d\theta'}{r^2+a^2-2ar\cos(\theta-\theta')}$$

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$$r < a$$

This is called the "Poisson formula". In this expression we don't need the differentiability of the function $f(\theta)$.

Example 2 $\nabla^2 u = 0 \quad z > 0, (x, y) \in \mathbb{R}^2$

$$u(x, y, 0) = f(x, y), \quad (x, y) \in \mathbb{R}^2$$

We solve the problem for a special case
 $f(x, y) = f(r)$ "spherically symmetric data"

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} = 0 \quad (5)$$

cylindrical coordinates. (θ independent). $r > 0$
 $z > 0$

$$u(r, z) \Big|_{z=0} = f(r).$$

We use an "integral transform". The one which is suitable for this problem is the Hankel transform

$$u(r, z) = \int_0^\infty \hat{u}(s, z) J_0(sr) s ds$$

$$\hat{u}(s, z) = \int_0^\infty u(r, z) J_0(sr) r dr$$

Multiplying (5) by $r J_0(sr)$ and integrating over $(0, \infty)$ and using integrating by parts

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$$\int_0^{\infty} (u_{rr} r J_0 + u_r J_0 + u_{zz} J_0 r) dr = 0$$

Integrating by parts and using the fact that $u, u_r, u_{rr} \rightarrow 0$ as $r \rightarrow \infty$ we get

$$\int_0^{\infty} r (J_0'' + \frac{1}{r} J_0') u dr + \int_0^{\infty} r u_{zz} J_0 dr = 0$$

$$J_0'' + \frac{1}{r} J_0' = -\rho^2 J_0$$

$$\Rightarrow -\rho^2 \int_0^{\infty} r J_0(\rho r) u(r, z) dr + \frac{d^2}{dz^2} \int_0^{\infty} r u(r, z) J_0(\rho z) d\rho = 0$$

$$\text{or } \frac{d^2}{dz^2} \hat{u}(\rho, z) = \rho^2 \hat{u}(\rho, z)$$

$$\text{Hence } \hat{u}(\rho, z) = A e^{\rho z} + B e^{-\rho z}$$

for finiteness as $z \rightarrow \infty$ then $A = 0$

$$\hat{u}(\rho, z) = B(\rho) e^{-\rho z}$$

Hence

$$u(r, z) = \int_0^{\infty} B(p) e^{-pz} J_0(pr) p dp.$$

Boundary condition

$$u(r, 0) = f(r) = \int_0^{\infty} B(p) J_0(pr) p dp.$$

and

$$B(p) = \int_0^{\infty} f(r) J_0(pr) r dr.$$

————— 0 —————

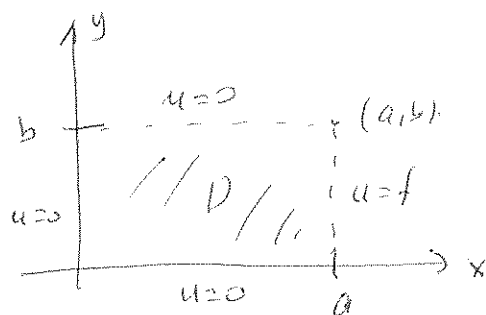
Example 3 let D be the rectangular region as shown in the figure

$$\nabla^2 u = 0 \quad \text{in } D$$

$$u(x, 0) = u(x, b) = 0 \quad 0 \leq x \leq a$$

$$u(0, y) = 0, \quad 0 \leq y \leq b$$

$$u(a, y) = f(y), \quad 0 \leq y \leq b$$



solution: using separation of variables the formal solution can be found. It reads

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh(nx) \sin\left(\frac{n\pi}{b}y\right) \\ \forall (x, y) \in D$$

at $x = a$:

$$u(a, y) = f(y) = \sum_{n=1}^{\infty} A_n \sinh(na) \sin\left(\frac{n\pi}{b}y\right)$$

$$\Rightarrow \sinh(na) A_n = \frac{2}{b} \int_0^b f(y) \sin\left(\frac{n\pi}{b}y\right) dy$$

As an exercise study the well-posedness of this problem.

The maximum - Minimum Principle for the Laplace equation

Lemma 1. Let $u \in C^2(D) \cap C(\bar{D})$ where D is a bounded ~~region~~ domain

a) If $\nabla^2 u \leq 0$ in D and $u \geq 0$ on B (boundary of D) then $u \geq 0$ in D

b) If $\nabla^2 u \geq 0$ in D and $u \leq 0$ on B then $u \leq 0$ in D

Proof: a) Suppose there are some points in D where $u < 0$ (since on the boundary $u \geq 0$) There must be a point c such that $u(c)$ is the smallest negative value of u

Let $\varepsilon > 0$ then define

$$v(x) = u(x) - \varepsilon \|x - c\|^2$$

for all $x \in B$. We can fix ε so that

$$v(x) \geq -\varepsilon \|x - c\|^2 \geq -\varepsilon d^2 \geq u(c) = v(c)$$

where d is the largest distance in D .

Hence $v(x)$ can not take its minimum

value on B . Then there is a point $p \in D$ so that v takes its minimum value

$$v(p) \leq v(x) \quad \forall x \in D.$$

on the other hand

$$\nabla^2 v = \nabla^2 u - 6\varepsilon < 0$$

which is a contradiction. Hence the assumption that $u < 0$ at some point in D is not valid, $u(x) \geq 0$ in D .

b) Just let $u \rightarrow -u$ and use (a).

Theorem (The Maximum Principle).

i) If $\nabla^2 u \leq 0$ in D
then $u(x) \geq \min_{y \in B} u(y)$
 $\forall x \in D$.

ii) If $\nabla^2 u \geq 0$ in D
then $u(x) \leq \max_{y \in B} u(y)$
 $\forall x \in D$

iii) If $\nabla^2 u = 0$ in D then
 $m \leq u(x) \leq M$

Proof: i) let $m = \min_{y \in B} u(y)$ and define

$$v(x) = u(x) - m \quad \text{then}$$

$$\nabla^2 v \leq 0 \quad \text{in } D.$$

$$v(x) \geq 0 \quad \text{on } B.$$

$$\Rightarrow v(x) \geq 0 \quad \text{in } D.$$

Hence

$$u(x) \geq m \quad \forall x \in D.$$

ii) let $M = \max_{y \in B} u(y)$ and define

$$v(x) = M - u(x) \quad \text{then}$$

$$\nabla^2 v \leq 0 \quad \text{in } D$$

$$v(x) \geq 0 \quad \text{on } B.$$

$$\Rightarrow v(x) \geq 0 \quad \text{in } D$$

Hence

$$u(x) \leq M \quad \forall x \in D$$

iii) using (i) and (ii) we get

$$m \leq u(x) \leq M \quad \forall x \in D$$

Theorem. There is at most one solution of
the boundary value problem (Dirichlet's problem)

$$\nabla^2 u = h \quad \text{in } D$$

$$u = f \quad \text{on } B.$$

proof: let us assume that there exist
solutions u_1 and u_2 for the same data
(f, h). Then define the difference function
 $w = u_1 - u_2$, then we get

$$\nabla^2 w = 0 \quad \text{in } D$$

$$w = 0 \quad \text{on } B$$

According to the third part of the last
theorem $M = m = 0 \Rightarrow 0 \leq w \leq 0$ or
 $w = 0 \quad \forall x \in D$. This means $u_1 = u_2$.
Hence there can't be two different
solutions of the Dirichlet problem corresponding
to the same data.

Theorem: suppose $u \in C^2(D) \cap C(\bar{D})$ and
 satisfied $\nabla^2 u = h$ in D and $u = f$ on B .
 Suppose that h is bounded in D , i.e.,
 $|h| \leq H$ where H is a positive constant,
 and f is bounded in B , i.e., $|f| \leq F$
 where F is another positive constant. Then
 there is a constant C (depends only on the
 domain D) such that

$$|u(x)| \leq F + CH$$

Proof: let w be a function satisfying

$$\nabla^2 w \leq -H \quad \text{in } D, \quad \text{and}$$

$$w \geq \pm u(x) \quad \text{on } B$$

Define $v(x) = w(x) \pm u(x)$ then

$$\nabla^2 v \leq -H \pm h < 0 \quad \text{in } D$$

$$v(x) \geq 0 \quad \text{on } B$$

then using our previous results

$$v(x) \geq 0 \quad \text{in } D$$

or

$$\forall(x) \geq \pm u(x), \quad x \in D \quad \text{also} \quad (1)$$

let

$$w(x) = F + H (e^d - e^{d-x'}) \quad , \quad x' \in D \quad (2)$$

$$\nabla^2 w = -H e^{d-x'} \leq -H \quad \text{in } D.$$

$$w(x) \geq F \quad \text{on } B.$$

Here d is the diameter of the region D
(Largest distance between the points of D)

Then

$$w(x) \geq F \geq \pm u \quad \text{on } B$$

From (1)

$$|u(x)| \leq |w(x)| \quad x \in D$$

$$\text{but From (2)} \quad |w(x)| \leq F + H e^d$$

$$\Rightarrow |u(x)| \leq F + H e^d \quad \forall x \in D.$$

$$C = e^d.$$

Theorem: suppose that $u_i \in C^2(D) \cap C(\bar{D})$, $i=1,2$
satisfying

$$\nabla^2 u_i = h_i \quad \text{in } D$$

$$u_i = f_i \quad \text{on } B.$$

where h_i, f_i are given functions ($i=1,2$)

If

$$|h_1 - h_2| < \varepsilon, \quad \text{for } x \in D$$

$$|f_1 - f_2| < \varepsilon,$$

then

$$|u_1 - u_2| < \varepsilon(1+C)$$

where C is a constant determined by the domain D .

proof: Let $u_1 - u_2 = v$ in D .

$$\Rightarrow u_1 - u_2 = f_1 - f_2 = f \quad \text{on } B.$$

$$\text{and } \nabla^2 v = h_1 - h_2 = h \quad \text{in } D$$

with

$$|h| < \varepsilon = H \quad \text{in } D$$

$$|f| < \varepsilon = F \quad \text{on } B.$$

use the previous Theorem. That

$$|v| \leq F + Hc < \varepsilon + \varepsilon c$$

$$\Rightarrow |u_1(x) - u_2(x)| < \varepsilon(1+c)$$

where $c = e^d$, d is the diameter of D .

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Green's Functions for the Laplace equation.

let D be a closed domain and let $u, v \in C^2(D) \cap C^1(\bar{D})$, then.

$$\nabla (u \nabla v - v \nabla u) = u \nabla^2 v - v \nabla^2 u. \quad (1)$$

We have also the following

$$1) \quad \int_D (u \nabla^2 v - v \nabla^2 u) dV = \int_{B=\partial D} (u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}) dS$$

$$2) \quad \int_D \vec{\nabla} (u \vec{\nabla} v) dV = \int_B u \frac{\partial v}{\partial n} dS$$

$$3) \quad \int_D \nabla^2 u dV = \int_B \frac{\partial u}{\partial n} dS$$

Theorem: (uniqueness of the Dirichlet problem).

Let $u \in C^2(D) \cap C(\bar{D})$ with

$$\nabla^2 u = h \quad \text{in } D$$

$$u = f \quad \text{on } B$$

This D.P. has at most one solution in D .

Proof: let us suppose that there exist two different solutions u_1 and u_2 for the same data (f, h) . Define the difference function $w = u_1 - u_2$. Then we have

$$\nabla^2 w = 0 \quad \text{in } D$$

$$w = 0 \quad \text{on } B$$

use property (2) by letting $u = v = w$, then

$$\int_D \nabla(w \cdot \nabla w) dV = \int_B w \frac{\partial w}{\partial n} dS = 0$$

$$= \int_D (w \nabla^2 w + \nabla w \cdot \nabla w) dV = 0$$

$$= \int_D \|\nabla w\|^2 dV = 0 \Rightarrow \nabla w = 0 \text{ in } D$$

Hence w is constant in D . It is equal to zero on the boundary B . Since w is continuous on $\bar{D} \Rightarrow w(x) = 0$ in D . This means that $u_1 = u_2 \quad \forall x \in D$.

Theorem: (Uniqueness of the Robin problem)

Let $u \in C^2(D) \cap C(\bar{D})$ with

$$\nabla^2 u = h \quad \text{in } D$$

$$\frac{\partial u}{\partial n} + \alpha u = \beta \quad \text{on } B \quad \text{where } \alpha > 0$$

Then this Robin problem has at most one solution in D .

proof: let us suppose that there exist two different solutions u_1 and u_2 for the same data (h, α, β) . Define the difference function $w = u_1 - u_2$, then we have

$$\nabla^2 w = 0 \quad \text{in } D$$

$$\frac{\partial w}{\partial n} + \alpha w = 0 \quad \text{on } B, \quad \alpha > 0$$

using the property (2) by letting $u = v = w$ we get

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$$\begin{aligned}\int_D \nabla (w \sigma w) dV &= \int_B w \frac{\partial w}{\partial n} ds \\ &= - \int_B \alpha w^2 ds\end{aligned}$$

or

$$\int_D \nabla (w \sigma w) dV + \int_B \alpha w^2 ds = 0$$

$$\int_D \|\nabla w\|^2 dV + \int_B \alpha w^2 ds = 0$$

Each term is positive in the LHS hence

$$\vec{\nabla} w = 0 \quad \text{in } D$$

$$w = 0 \quad \text{on } B$$

$\Rightarrow w = 0$ in D which leads to
 $u_1 = u_2$ in D .

Theorem (uniqueness of the Neuman Problem).

let $u \in C^2(D) \cap C(\bar{D})$ with

$$\nabla^2 u = h \quad \text{in } D$$

$$\frac{\partial u}{\partial n} = f \quad \text{on } B$$

Then the Neumann problem has at most one solution in D provided that any two solutions differ by constant.

proof: Proof of this thm is similar to the proof of the Dirichlet problem up to the point

$$\int_D \|\nabla w\|^2 dV = 0 \Rightarrow \nabla w = 0 \text{ in } D$$

hence $w = \text{constant}$.

$$\text{Since } \frac{\partial w}{\partial n} = 0 \text{ on } B \Rightarrow$$

$$u_1 - u_2 = \text{constant in } D.$$

Hence Neuman problem has a unique solution up to a constant term.

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Green's function for the Laplace equation.

Remember the first identity in Page 21

let $u, v \in C^2(D) \cap C(\bar{D}) \Rightarrow$

$$\int_D (v \nabla^2 u - u \nabla^2 v) dV = \int_B (v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}) ds$$

let v be the Green's function, i.e. $v = G(x, y)$.
satisfying

$$\nabla_y^2 G(x, y) = -4\pi \delta(x - y), \quad x, y \in D$$

and $\nabla^2 u = h$ in D . Then the above
Green's identity turns out to be

$$\int_D [v h - 4\pi \delta(x - y) u(y)] dV_y = \int_B (G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n}) ds_y$$

$$\int_D G(x, y) h(y) dV_y + 4\pi u(x) = \int_B \left[G(x, y) \frac{\partial u}{\partial n} - u(y) \frac{\partial G}{\partial n} \right] ds_y$$

let $G(x, y)|_B = 0 \Rightarrow$

$$4\pi u(x) = - \left(\int_D G(x, y) h(y) dV_y + \int_B u(y) \frac{\partial G}{\partial n} ds_y \right)$$

$$\text{if } u|_B = f \Rightarrow$$

$$u(x) = \frac{-1}{4\pi} \int_D G(x,y) h(y) dV_y + \frac{1}{4\pi} \int_B f(y) \frac{\partial G}{\partial x} dS_y$$

The RHS contains the GF satisfying

$$\nabla^2 G(x,y) = 4\pi \delta(x-y) \quad \text{in } D.$$

$$G|_B = 0.$$

Hence the solution of the Dirichlet problem is reduced to the construction of the GF satisfying above equations.

Lemma 1 . Let $\psi(\vec{r})$ be a solution of the Laplace equation in $D \subseteq \mathbb{R}^3$ then

$\frac{a}{r} \psi\left(\frac{a^2}{r^2} \vec{r}\right)$ solves also the Laplace equation

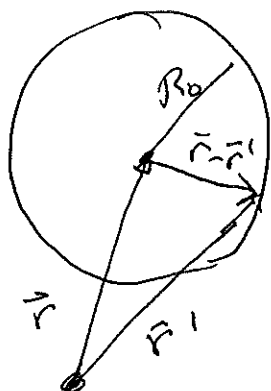
This is known as the "Kelvin Theorem"

Lemma 2 . Let $\vec{R} = (x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}$
 $= \vec{r} - \vec{r}'$ a difference vector in $\mathbb{R}^3$

$$R = \|\vec{R}\| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\text{Then } \nabla^2 \frac{1}{R} = 0 \quad \vec{r} \neq \vec{r}'$$

Consider the sphere of radius R_0 at the tip of $\vec{r}$



$$\|\vec{r} - \vec{r}'\| \leq R_0$$

Integrate $\nabla^2 \frac{1}{R}$ over the sphere

$$\begin{aligned} \int_D \nabla'^2 \frac{1}{R} dV &= \int_S (\vec{N}' \cdot \nabla' \frac{1}{R}) dS' \\ &= \int_S \frac{\vec{R}'}{R'} \left(-\frac{\vec{R}'}{R'^3} \right) R'^2 d\Omega \\ &\rightarrow \\ &= - \int_S d\Omega = -4\pi \end{aligned}$$

Lemma 3 $\nabla'^2 \frac{1}{R} = -4\pi \delta(\vec{r} - \vec{r}')$

also $\nabla^2 \frac{1}{R} = -4\pi \delta(\vec{r} - \vec{r}')$

Lemma 4 in $\mathbb{R}^2$ $\nabla^2 \frac{1}{R} = -2\pi \delta(\vec{r} - \vec{r}')$

Remember the expression for $u(x)$ satisfying the Dirichlet problem with $h=0$

$$u(x) = \frac{1}{4\pi} \int_B f(y) \frac{\partial G}{\partial n} ds_y$$

Using the Lemma 1 and Lemma 2 in page 27 (and Lemma 3 in page 28) we can find the Green's function as follows

I) In spherical regions.

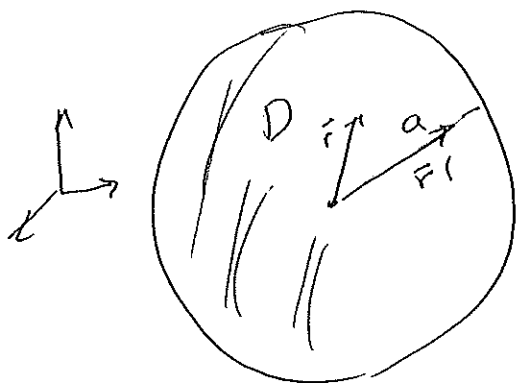
a) $\mathbb{R}^3$

$$G(\vec{r}, \vec{r}') = \frac{1}{\|\vec{R}\|} - \frac{a/r}{\|\vec{R}^*\|}$$

$$\vec{R} = \vec{r} - \vec{r}', \quad \vec{R}^* = \frac{a^2}{r^2} \vec{r} - \vec{r}'$$

$$\begin{aligned} \nabla^2 G(\vec{r}, \vec{r}') &= -4\pi \delta(\vec{r} - \vec{r}') \\ &\quad + 4\pi \delta\left(\frac{a^2}{r^2} \vec{r} - \vec{r}'\right) \end{aligned}$$

in D



$$= -4\pi \delta(\vec{r} - \vec{r}')$$

and $G|_B = 0$

b) $\mathbb{R}^2$

$$G(\vec{r}, \vec{r}') = \ln R - \ln\left(\frac{a}{r} R^*\right)$$

$$\nabla^2 G = -2\pi \delta(\vec{r} - \vec{r}')$$

$$G|_B = 0$$

II. Planar regions

a) $\mathbb{R}^3$: $G(x, y, z) = \frac{1}{R} - \frac{1}{R^*}$

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

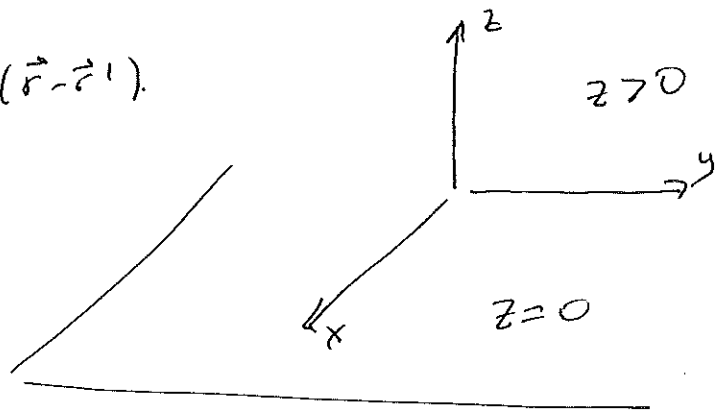
$$R^* = \sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}$$

$$\nabla^2 G = -4\pi \delta(\vec{r} - \vec{r}')$$

$$G|_B = 0$$

$$B: z=0$$

$$D: z>0$$

b) $\mathbb{R}^2$

$$G(\vec{r}, \vec{r}') = \ln R/R^*$$

$$\nabla^2 G = -2\pi \delta(\vec{r} - \vec{r}')$$

Green's functions

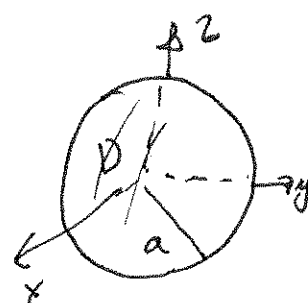
(31)

In $\mathbb{R}^3$

• Spherical regions:

$$G(\vec{r}, \vec{r}') = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\Theta}} - \frac{a}{\sqrt{a^4 + r^2 r'^2 - 2a^2 rr'\cos\Theta}}$$

$$\left. \frac{\partial G}{\partial r'} \right|_{r'=a} = \frac{r^2 - a^2}{a(r^2 + a^2 - 2ar\cos\Theta)^{3/2}}$$



Solution of the Dirichlet problem

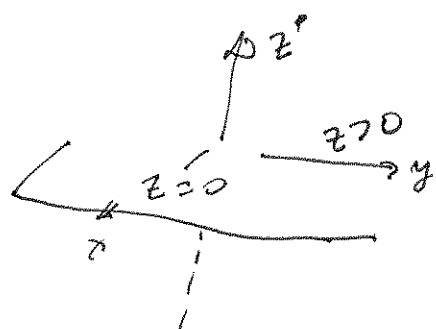
$$u(r, \theta, \phi) = \frac{a(r^2 - a^2)}{4\pi} \int_0^{2\pi} \int_0^\pi \frac{f(\theta', \phi') d\theta' d\phi'}{(a^2 + r^2 - 2ar\cos\Theta)^{3/2}}$$

where

$$\cos\Theta = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi')$$

• Cartesian (rectangular) regions:

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} - \frac{1}{|\vec{r}^* - \vec{r}'|}$$



$$= \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}}$$

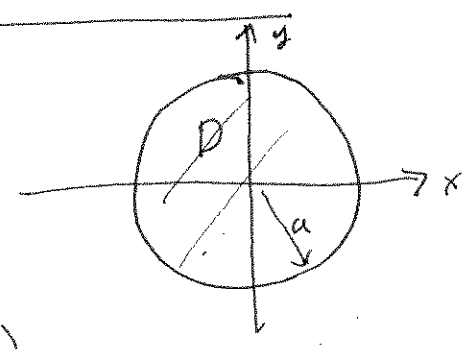
$$\left. \frac{\partial G}{\partial z'} \right|_{z'=0} = - \frac{2z}{(z^2 + (x-x')^2 + (y-y')^2)^{3/2}}$$

Solution of the Dirichlet problem

$$u(x, y, z) = \frac{z}{2\pi} \iint_{-\infty}^{\infty} \frac{f(x', y') dx' dy'}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}}$$

in $\mathbb{R}^2$:

Circular regions:



$$G(\vec{r}, \vec{r}') = \ln \left(\frac{r}{a} \left| \frac{a^2}{r^2} \vec{r} - \vec{r}' \right| \right)$$

$$= \frac{1}{2} \ln \left(\frac{a^2 + \frac{r^2 r'^2}{a^2} - 2arr' \cos(\theta - \theta')}{r^2 + r'^2 - 2rr' \cos(\theta - \theta')} \right)$$

$$\left. \frac{\partial G}{\partial r'} \right|_{r'=a} = - \frac{(a^2 - r^2)}{a(a^2 - 2ar \cos(\theta - \theta') + r^2)}$$

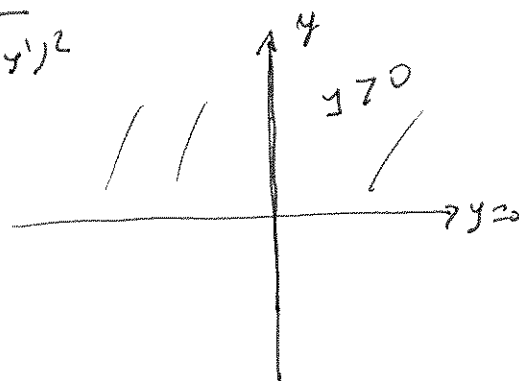
Solution of the Dirichlet problem

$$u(r, \theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\theta') d\theta'}{a^2 - 2ar \cos(\theta - \theta') + r^2}$$

rectangular regions:

$$G(\bar{r}, \bar{r}') = \frac{1}{2} \ln \frac{(x-x')^2 + (y+y')^2}{(x-x')^2 + (y-y')^2}$$

$$\left. \frac{\partial G}{\partial n'} \right|_{y'=0} = - \frac{2y}{(x-x')^2 + y^2}$$



solution of the Dirichlet problem

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x') dx'}{(x-x')^2 + y^2}$$



ASSIGNED EXERCISES OF MATH544: PDE set 7 April, 2001

THE LAPLACE EQUATION

Ronald B Guenther and John W Lee, Partial Differential Equations of Mathematical Physics and Integral Equations

In all types of equations we studied the following:

1. Exact Solutions
2. Uniqueness of solutions
3. Maximum-Minimum Principles

Here, in the elliptic case, I will include into our study the subject **Green's Function Technique** for the Laplace operator.

1. **Green's function for the Laplace operator:** Let D be a domain and B its boundary. Let the closure of D be $\bar{D} = D \cup B$. Let $u, v \in C^2(D) \cap C^1(\bar{D})$. Then the following is an identity

$$\int_D [v \nabla^2 u - u \nabla^2 v] dV = \int_B [v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}] dS \quad (1)$$

where $\frac{\partial u}{\partial n} = \langle N, \text{grad } u \rangle$. Here N is the unit normal vector field defined at each point of the surface B . Now consider the following Dirichlet problem

$$\nabla^2 u = h, \quad \text{in } D, \quad (2)$$

$$u = f, \quad \text{on } B. \quad (3)$$

where h and f some continuous functions in D and on B respectively. Let $v = G(x, y)$ with

$$\nabla_y^2 G(x, y) = \delta(x - y), \quad \text{in } D \quad (4)$$

$$G(x, y)|_{y \in B} = 0. \quad (5)$$

Then we find that

$$u(x) = \int_D G(x, y) h(y) dv_y + \int_B f(y) \frac{\partial G(x, y)}{\partial n_y} dS_y \quad (6)$$

Hence the solution of a Dirichlet problem reduces to the determination of Green's function $G(x, y)$ satisfying (4)-(5).

2. Find the Green function when D is a ball and B is the sphere with radius a . Since $\frac{1}{|x-y|}$ satisfies

$$\nabla^2 \frac{1}{|x-y|} = 4\pi\delta(x-y)$$

then the following form for G is more suitable

$$G(x, y) = \frac{1}{4\pi} \left[\frac{1}{|x-y|} - \frac{q}{|x^*-y|} \right]$$

where q and x^* are to be determined through the boundary condition. Let $y \in B$ then

$$G|_{y \in B} = 0$$

or

$$\frac{1}{|x-y|} = \frac{q}{|x^*-y|}$$

This gives

$$q = \frac{a}{r}, \quad x^* = q^2 x. \quad (7)$$

where $r^2 = |x|^2$. We find also

$$\frac{\partial G(x, y)}{\partial n_y} = \left\langle \frac{y}{a}, \text{grad}_y G(x, y) \right\rangle |_{y \in B} = \frac{r^2 - a^2}{4\pi |x-y|^3}$$

Hence the solution of the Dirichlet problem when $h = 0$ is

$$u(x) = \int_B \frac{a^2 - r^2}{4\pi |x-y|^3} f(y) dS_y = \frac{1}{4\pi} \iint \frac{(a^2 - r^2) f(\varphi, \phi) \sin \theta d\theta d\phi}{|r^2 + a^2 - 2ar \cos \theta|}$$

2

$$|x-y| = \sqrt{x^2 + a^2 - 2ar \cos \theta}$$

$$\cos \theta = \sin \theta \sin \theta' \cos(\phi - \phi') + \cos \theta \cos \theta'$$

3. Using the Green's function technique solve the following Dirichlet problem.

$$\nabla^2 u = 0, \quad -\infty < x, y < \infty, \quad z > 0, \quad (8)$$

$$u(x, y, 0) = f(x, y), \quad -\infty < x, y < \infty. \quad (9)$$

where $f(x, y)$ is a continuous function in R^2 .

4. Solve the above problem by use of separation of variables

5. Prove the following theorem:

Theorem. Let D be a bounded normal domain, and suppose that the Green's function $G(x, y)$ for the Dirichlet problem exists. Then

$$G(x, y) = G(y, x)$$

for $x, y \in D$. Consequently, in addition to $\nabla_y^2 G(x, y) = 0$ for $y \neq x$, we also have $\nabla_x^2 G(x, y) = 0$ for $x \neq y$.

Solution. *First way.* Let $u = G(z, x)$ and $v = G(z, y)$ in (1). Using (4) and (5) and using the property

$$\int_D f(y) \delta(y - x) d^3y = f(x)$$

of the delta-distribution one obtains $G(x, y) = G(y, x)$. *Second way.* Let $K_a(c)$ represent a ball with radius a centered at c and $S_a(c)$ is the sphere bounding this ball.

$$K_a(c) = \{x \in R^3 \mid |x - c| < a\}, \quad S_a(c) = \{x \in R^3 \mid |x - c| = a\}$$

Fix $x, x' \in D$. Fix $\varepsilon > 0$ so small that $K_\varepsilon(x), K_\varepsilon(x') \subset D$ and $|x - x'| > 2\varepsilon$. Apply (1) to the domain

$$D_\varepsilon = D - (K_\varepsilon(x) \cup K_\varepsilon(x'))$$

for $u(y) = G(x, y)$, and $v(y) = G(x', y)$

6. In two dimensions let D be a disc with radius a and B is the circle bounding D with radius a . Consider

$$\nabla^2 u = 0, \quad r < a, \quad (10)$$

$$u = f \quad \text{on } B. \quad (11)$$

- a. Find the formal solution.
 - b. Find reasonable restrictions on f so that the formal solution is a solution.
7. Let $D = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$ and $B = \{(x, y, z) | z = 0\}$. Solve the following Dirichlet problem

$$\nabla^2 u = 0, \quad \text{in } D, u = f(x, y) \quad \text{on } B. \quad (12)$$

Remark 1. The last two problems were solved in class. They respectively represent discrete and continuous spectrum in the method of separation of variables.

8. In two dimensions consider

$$\nabla^2 u = 0, \quad 0 < x < a, \quad 0 < y < b, \quad (13)$$

$$u(x, 0) = f(x), \quad u(x, b) = 0, \quad 0 \leq x \leq a, \quad (14)$$

$$u(0, y) = u(a, y) = 0, \quad 0 \leq y \leq b. \quad (15)$$

- a. Find the formal solution,
- b. Find reasonable restrictions on $f(x)$ so that the formal solution is a solution.

9. Consider

$$\nabla^2 u(r, \theta) = 0, \quad 0 \leq r < a, \quad (16)$$

$$\frac{\partial u(a, \theta)}{\partial r} + \alpha u(a, \theta) = f(\theta), \quad (17)$$

with f is 2π periodic continuous function and $\alpha > 0$ a constant.

- a. Solve formally by separation of variables.
- b. Find reasonable conditions on f so that the formal solution is a solution.

10. Suppose that $k > 0$ is a constant and $\nabla^2 u - k^2 u = 0$. Formulate Dirichlet, Neumann, and Robin problems of this operator $L = \nabla^2 - k^2$ and prove that each problem has at most one solution.

11. Prove the following theorems.

Theorem 1. Let $u \in C^2(D) \cap C^1(\bar{D})$, where D is a bounded domain.

- (i) If $\nabla^2 u \leq 0$ in D and $u \geq 0$ on B , then $u \geq 0$ in D ,
- ii) If $\nabla^2 u \geq 0$ in D and $u \leq 0$ on B , then $u \leq 0$ in D .

Theorem 2. Let $u \in C^2(D) \cap C^1(\bar{D})$, where D is a bounded domain.

- (i) If $\nabla^2 u \geq 0$ in D , then $u(x) \leq M = \max_{y \in B} u(y)$ for all $x \in D$.
- (ii) If $\nabla^2 u \leq 0$ in D , then $u(x) \geq m = \min_{y \in B} u(y)$ for all $x \in D$.
- (iii) If $\nabla^2 u = 0$ in D , then $m \leq u(x) \leq M$ for all $x \in D$.

Theorem 3. Let D be a bounded domain and B its boundary. Let $u \in C^2(D) \cap C^1(\bar{D})$. Then there is at most one solution of the problem

$$\nabla^2 u = h, \quad \text{in } D, \quad (18)$$

$$u = f, \quad \text{on } B. \quad (19)$$

Theorem 4. Suppose that $u \in C^2(D) \cap C^1(\bar{D})$ and satisfies the Dirichlet problem as in the previous theorem (Thm. 3). Suppose that $h(x)$ is bounded

by a constant H in D and f is bounded by F on B . Then there is a constant C depending only on the domain D such that $|u(x)| \leq F + CH$

Proof. Find a function w such that

$$\nabla^2 w \leq -H \text{ in } D \text{ and } w \geq \pm u \text{ on } B. \quad (20)$$

Given such a function we have

$$\nabla^2 (w \pm u) \leq -H \pm h \leq 0 \text{ in } D \text{ and } w \pm u \leq 0 \text{ on } B$$

Using Thm. 1 above we must have

$$w \pm u \geq 0$$

or $w \geq |u|$ in D . Hence we have to find an w satisfying (20). Let

$$w(x) = F + H[e^d - ex_1 - c_e], \quad x \in \bar{D} \quad (21)$$

where D lies in between the planes $x_1 = c_1$ and $x_1 = c_1 + d$ and $C = e^d$. Prove that this w satisfies all requirements of (20).

Theorem 5. Suppose that the functions $u_i \in C^2(D) \cap C^1(\bar{D})$, $i = 1, 2$, satisfies $\nabla^2 u_i = h_i$ in D and $u_i = f_i$ on B , where h_i and f_i are given. If

$$|h_1(x) - h_2(x)| \leq \varepsilon$$

for all $x \in D$ and

$$|f_1(x) - f_2(x)| \leq \varepsilon$$

for all $x \in B$. then

$$|u_1(x) - u_2(x)| \leq (1 + C)\varepsilon$$

for all $x \in D$, where C is a constant determined by the domain D .

Remark 2. First three theorems were proved in the class.

12. Prove the following theorem.

Theorem 6. Let $u \in C^2(D) \cap C^1(\bar{D})$ for some bounded domain D .

- (1) If $\nabla^2 u < 0$ in D and $u \geq 0$ on B , then $u \geq 0$ in D .
- (ii) If $\nabla^2 u > 0$ in D and $u \leq 0$ on B , then $u \leq 0$ in D .

We shall now discuss how the Poisson integrals are well-defined. Which functions f provide Poisson's integrals or when the Poisson integrals are defined.

Theorem (spherical case in $\mathbb{R}^2$) Suppose that f is continuous on the boundary $\partial D = \mathbb{C}$ of D (Circular region). (∂D is a circle of radius a and D is the ball $K_a(0)$). Then the function

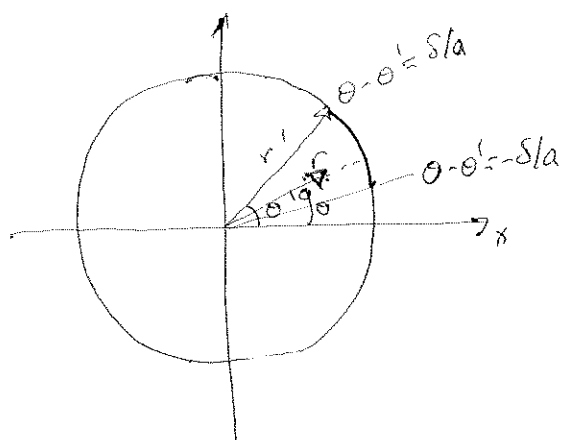
$$u(r, \theta) = \begin{cases} \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\theta') d\theta'}{r^2 + a^2 - 2ar(\cos(\theta - \theta'))} & , \quad \bar{r} \in K_a(0) \text{ (} r \neq a \text{)} \\ f(\theta) & , \quad r = a. \end{cases}$$

which satisfies the Laplace equation for $r < a$ and is continuous in $r \leq a$.

Proof: For $r < a$, it is straightforward to check that repeated differentiation of the function under the integral sign is permissible and $\nabla^2 u = 0$ (Indeed we found this integral represented also from the method separation of variables). It is also clear that it satisfies the boundary condition at $r = a$. Since we know that $u \in C^2(K_a(0))$ we must only show that $u(r, \theta)$ is continuous at $r = a$, i.e.,

$$\lim_{r \rightarrow a} u(r, \theta) = f(\theta) \quad , \quad \forall 0 \leq \theta \leq 2\pi$$

or for a given $\varepsilon > 0$ there exist a $\delta' > 0$ such that for all $|r-a| < \delta' \Rightarrow |u(r, \theta) - f(\theta)| < \varepsilon$
 $\forall \theta \in [0, 2\pi]$.



We know that $f(\theta)$ is continuous in $[0, 2\pi]$, i.e. Given $\varepsilon > 0$ there exist a $\delta > 0$ such that $\forall \theta, \theta' \in [0, 2\pi]$
 $|\theta - \theta'| < \frac{\delta}{a} \Rightarrow |f(\theta) - f(\theta')| < a \frac{\varepsilon}{2}$

Writing

$$u(r, \theta) - f(\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\theta') d\theta'}{r^2 + a^2 - 2ar \cos(\theta - \theta')} - f(\theta)$$

If the function $f=1 \Rightarrow u=1 \Rightarrow$

$$\frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{d\theta'}{r^2 + a^2 - 2ar \cos(\theta - \theta')} = 1$$

then

$$\begin{aligned} u(r, \theta) - f(\theta) &= \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{[f(\theta') - f(\theta)] d\theta'}{r^2 + a^2 - 2ar \cos(\theta - \theta')} \\ &= \frac{a^2 - r^2}{2\pi} \int_{\theta - \delta/a}^{\theta + \delta/a} \frac{[f(\theta') - f(\theta)] d\theta'}{r^2 + a^2 - 2ar \cos(\theta - \theta')} \\ &\quad + \frac{a^2 - r^2}{2\pi} \left(\int_0^{\theta - \delta/a} + \int_{\theta + \delta/a}^{2\pi} \right) \end{aligned}$$

Hence

$$|u(r, \theta) - f(\theta)| \leq \frac{|a^2 - r^2|}{2\pi} \int_{\theta - \delta/a}^{\theta + \delta/a} \frac{|f(\theta') - f(\theta)|}{r^2 + a^2 - 2ar \cos(\theta - \theta')} d\theta' \\ + \frac{|a^2 - r^2|}{2\pi} \left(\int_0^{\theta - \delta/a} \frac{|f(\theta') - f(\theta)|}{R^2} d\theta' + \int_{\theta + \delta/a}^{2\pi} \frac{|f(\theta') - f(\theta)|}{R^2} d\theta' \right)$$

where $R^2 = r^2 + a^2 - 2ar \cos(\theta - \theta') = \|\vec{r} - \vec{r}'\|^2$

~~Let~~ in $f(\theta)$ satisfies $|f(\theta) - f(\theta')| < \frac{\varepsilon}{2}$ in $(\theta - \delta/2, \theta + \delta/2)$ and bounded outside of this region (interval) ($|f| \leq M$ in $(0, 2\pi)$)

$$|u(r, \theta) - f(\theta)| \leq \frac{a^2 - r^2}{2\pi} \int_{\theta - \delta/2}^{\theta + \delta/2} \frac{\varepsilon/2}{r^2 + a^2 - 2ar \cos(\theta - \theta')} d\theta' \\ + \frac{|a^2 - r^2|}{2\pi} 2M \left(\int_0^{\theta - \delta/2} \frac{d\theta'}{R^2} + \int_{\theta + \delta/2}^{2\pi} \frac{d\theta'}{R^2} \right)$$

Since

$$\frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{d\theta'}{R^2} = \frac{a^2 - r^2}{2\pi} \int_{\theta - \delta/2}^{\theta + \delta/2} \frac{d\theta'}{R^2} + \frac{a^2 - r^2}{2\pi} \left(\int_0^{\theta - \delta/2} \frac{d\theta'}{R^2} + \int_{\theta + \delta/2}^{2\pi} \frac{d\theta'}{R^2} \right)$$

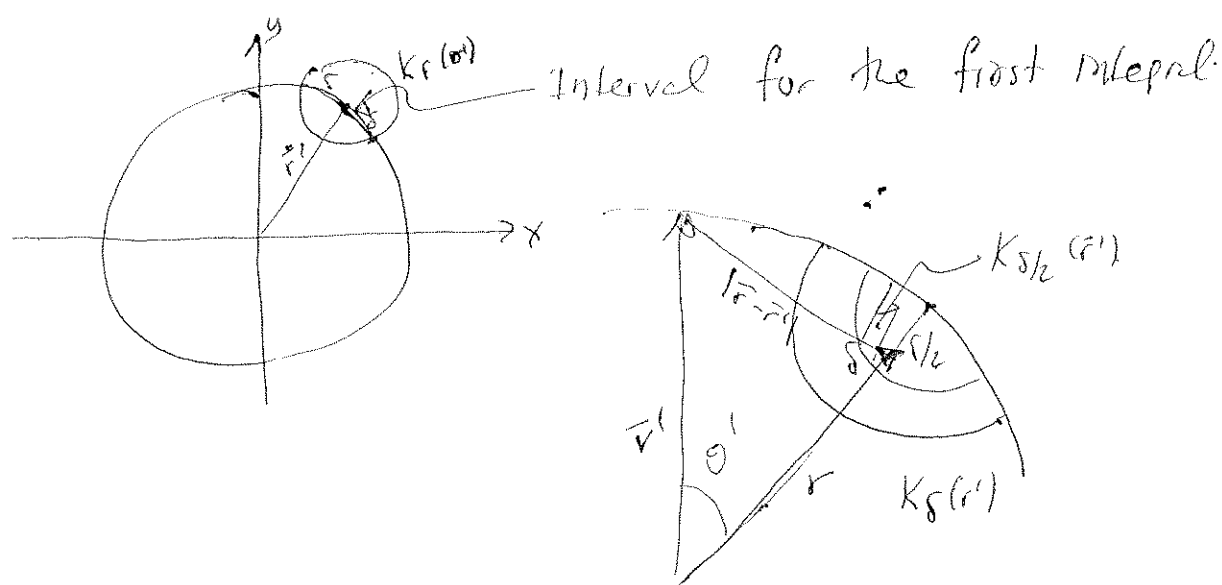
$$= 1$$

$$\Rightarrow \frac{a^2 - r^2}{2\pi} \int_{\theta - \delta/2}^{\theta + \delta/2} \frac{d\theta'}{R^2} \leq 1$$

Then

$$|u(r, \theta) - f(\theta)| \leq \frac{\varepsilon}{2} + \frac{M(a^2 - r^2)}{2\pi} \left(\int_0^{\theta - \delta/2} \frac{d\theta'}{R^2} + \int_{\theta + \delta/2}^{2\pi} \frac{d\theta'}{R^2} \right)$$

Now let r be in the region $K_{\delta/2}(r')$



Let $\vec{r} \in K_{\delta/2}(r')$

In the second and third integral θ' is out of the region $(\theta - \delta/2, \theta + \delta/2)$
 hence $\|\vec{r} - \vec{r}'\| \geq \delta/2$ or

$$\frac{1}{R} = \frac{1}{\|\vec{r}' - \vec{r}\|} \leq \frac{2}{\delta}$$

$\Rightarrow$

$$\begin{aligned} |u(r, \theta) - f(\theta)| &\leq \frac{\varepsilon}{2} + \frac{M(a^2 - r^2)}{\pi} \left(\frac{2}{\delta} \right)^2 \left(\theta - \frac{\delta}{2} + 2\pi - \theta - \frac{\delta}{2} \right) \\ &\leq \frac{\varepsilon}{2} + \frac{M(a^2 - r^2)}{\pi} (2\pi - \delta) \leq \frac{\varepsilon}{2} + \frac{4M}{\pi} a(\pi - \frac{\delta}{2})(a - r) \end{aligned}$$

$$|u(r,0) - f(0)| \leq \frac{\varepsilon}{2} + g(r)$$

$$g(r) = \frac{4\pi}{\pi} a \left(\pi - \frac{\pi}{2} \right) (a-r)$$

clearly $g(r)$ is continuous at $r=a$, i.e.

given $\varepsilon > 0$ there exists $\delta' > 0$ so that $\forall r$

$$|r-a| < \delta' \Rightarrow |g(r) - g(a)| < \varepsilon/2 \Rightarrow$$

$$|u(r,0) - f(0)| < \varepsilon$$

————— 0 —————

As an exercise prove the continuity of the Poisson integral

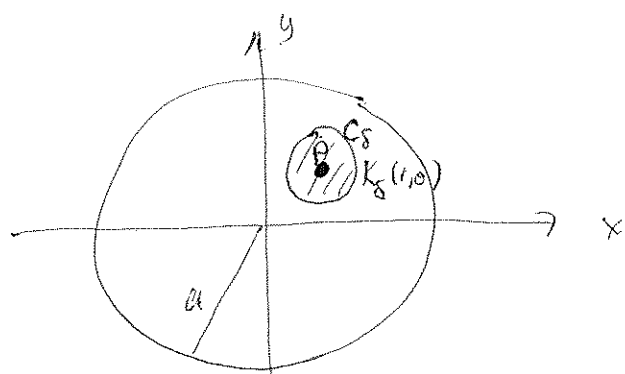
$$u(x,y) = \begin{cases} \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x') dx'}{y^2 + (x-x')^2}, & y \neq 0 \\ f(x) & y = 0 \end{cases},$$

Mean value property of the harmonic functions

$$u(0, \theta) = \frac{a^2}{2\pi} \int_0^{2\pi} \frac{f(\theta') d\theta'}{a^2} = \frac{1}{2\pi} \int_0^{2\pi} f(\theta') d\theta'$$

$$u(0, \theta) = \text{Ar}(f|_{\theta})$$

This is not only valid at the origin of the circle $\text{refin } K_a(0)$ but at any point (r, θ)



$$u_P(r, \theta) = \text{Ar } f|_{\theta}$$

$$\theta \in S_\delta(r, \theta)$$

$$f|_{\theta} = u|_{C_\delta}$$

This property leads to max-min theorem very easily. Harmonic functions take their maximum and minimum values at the boundaries.

Proof: Assume that there exists a point $P \in D$ so that u takes its maximum value. Now draw a circle centered at P with radius δ

Since

$$u_P = \text{av}(f|_{C_\delta}), \quad f = u|_{C_\delta}$$

u_P is the average value of u on the circle C_δ at some points u must exceed the value u_P which is a contradiction.