

MATHS44: APPLIED MATHEMATICS II

Lecture 4

- 1) Parabolic Equations (Heat eqn).
- 2) The Maximum principle for the heat eqn and consequences
- 3) Green's function for the Heat eqn.
- 4) Heat flow in an infinite rod.

Parabolic equations

1. Initial and Boundary value problems for a finite interval

$$\left. \begin{array}{l} u_t = a u_{xx}, \quad t > 0 \quad 0 < x < L \quad (a > 0) \\ \text{(BV)} \quad u(0, t) = u(L, t) = 0, \quad t > 0 \\ \text{(IV)} \quad u(x, 0) = f(x), \quad 0 \leq x \leq L \end{array} \right\} \quad (1)$$

Formal solution:

Since the PDE and the boundary conditions are homogeneous then we can use the method of separation of variables

$$u(x, t) = X(x)T(t) \quad (2)$$

Then using this ansatz in the heat equation we get

$$\frac{T'}{T} = a \frac{X''}{X} \quad (3)$$

so that $X(0) = X(L) = 0$. From (3) we get two equations

$$T' = -a\lambda^2 T, \quad X'' + \lambda^2 X = 0 \quad (4)$$

The reason that (3) is equal to $-\lambda^2 a$ depends on the boundary conditions. You can show that this is the only choice. The eigenvalue problem

$$X'' + \lambda^2 X = 0, \quad 0 < x < L$$

(5)

$$X(0) = X(L) = 0$$

has a nontrivial solution if and only if (3) is equal to $-\lambda^2$.
The solution of the above problem is $(\lambda = k_n)$.

$$X(x) = A_n \sin(k_n x), \quad k_n = \frac{n\pi}{L}, \quad n = 1, 2, \dots \quad (6)$$

and the function $T(t)$ is easily found as

$$T(t) = e^{-ak_n^2 t} \quad (7)$$

Hence

$$u_n(x, t) = X_n(x) T_n(t) = A_n e^{-ak_n^2 t} \sin(k_n x), \quad (8)$$

$n = 1, 2, \dots$. The heat equation with the boundary conditions has infinitely many solutions $u_n(x, t)$ in (8). To satisfy the initial condition we use the method of superposition, hence

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-ak_n^2 t} \sin(k_n x), \quad t > 0 \quad (9)$$

at $t = 0$

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin(k_n x) = f(x) \quad (10)$$

$$\Rightarrow A_n = \frac{2}{L} \int_0^L f(x) \sin(k_n x) dx \quad (11)$$

(3)

Then $u(x,t)$ in (9) is the formal solution of the above problem in (1) in $D = \{ (x,t) \mid t > \tilde{t}/2, 0 \leq x \leq L \}$.

• Justification of the problem (of the solution):

Term by term differentiation of the expression in (9) requires uniform convergence

$$u(x,t) \Rightarrow \sum_{n=1}^{\infty} A_n e^{-a k_n^2 t} \sin(k_n x) \quad (12)$$

$$u_{xx}(x,t) \Rightarrow -a \sum_{n=1}^{\infty} k_n^2 A_n e^{-a k_n^2 t} \sin(k_n x). \quad (13)$$

$$u_t(x,t) \Rightarrow -a \sum_{n=1}^{\infty} k_n^2 A_n e^{-a k_n^2 t} \sin(k_n x) \quad (14)$$

Hence the uniform convergence of the series in the last expression requires:

$$\left| \sum_{n=1}^{\infty} k_n^2 A_n e^{-a k_n^2 t} \sin(k_n x) \right| \leq \sum_{n=1}^{\infty} k_n^2 |A_n| e^{-a k_n^2 t} \quad (15)$$

It is enough to have $|A_n| \leq M$ where M is any positive constant, then

$$\left| \sum_{n=1}^{\infty} k_n^2 A_n e^{-a k_n^2 t} \sin(k_n x) \right| \leq M \sum_{n=1}^{\infty} k_n^2 e^{-a k_n^2 t} \quad (16)$$

$$< M \sum_{n=1}^{\infty} k_n^2 e^{-a k_n^2 \tilde{t}} \quad (17)$$

using the Weierstrass M-test one can show that (4)

$$\sum_{n=1}^{\infty} k_n^2 e^{-a k_n^2 \tilde{t}}, \quad a > 0 \quad (18)$$

Hence.

$$\left| \sum k_n^2 A_n e^{-a k_n^2 t} \sin(k_n x) \right| < \infty \quad (19)$$

for all $x \in [0, L]$ and $t \geq \tilde{t}/2 > 0$. Hence $u(x, t)$ in (9) with $|A_n| \leq M \quad \forall n=1, 2, \dots$ is the solution of the problem (1) in $x \in [0, L]$ and $t \geq \tilde{t}/2 > 0$. The condition $|A_n| \leq M$ imposes

~~For $t \geq 0$ we have to show that the series in (10) converges also uniformly.~~

a condition on the initial value $f(x)$. from (11)

$$|A_n| \leq \frac{2}{L} \int_0^L |f(x)| dx \quad (20)$$

~~since~~ It is enough to assume that $f(x)$ is continuous in $[0, L]$. Define M so that

$$|f(x)| \leq \frac{M}{2} \quad \forall x \in [0, L] \quad (21)$$

we get

$$|A_n| \leq M \quad (22)$$

(5)

For $t=0$ we have to show that the series in (10) converges also uniformly. For this purpose we need $A_n \sim \frac{1}{n^2}$. Hence using the integration by parts we get

$$\begin{aligned} A_n &= \frac{2}{L} \int_0^L f(x) \sin(k_n x) dx \\ &= \frac{2}{L} \left[-\frac{1}{k_n} f(x) \cos(k_n x) \right]_0^L + \frac{1}{k_n} \int_0^L f'(x) \cos(k_n x) dx \\ &= \frac{2}{L k_n} \int_0^L f'(x) \cos(k_n x) dx \end{aligned} \quad (23)$$

$$f(0) = f(L) = 0 \quad (24)$$

$$\begin{aligned} A_n &= \frac{2}{L k_n} \int_0^L f'(x) \cos(k_n x) dx \\ &= \frac{2}{L k_n} \left[\frac{1}{k_n} f'(x) \sin(k_n x) \right]_0^L - \frac{1}{k_n} \int_0^L f''(x) \sin(k_n x) dx \\ &= -\frac{2}{L k_n^2} \int_0^L f''(x) \sin(k_n x) dx, \end{aligned} \quad (25)$$

$$\Rightarrow |A_n| \leq \frac{2}{L k_n^2} \int_0^L |f''(x)| dx, \quad (26)$$

(6)

let

$$M_2 = \int_0^L |f''(x)| dx \Rightarrow \quad (27)$$

$$|A_n| \leq \frac{2}{Lk_n^2} M_2 = \frac{2M_2 L}{n^2 \pi^2} = \frac{1}{n^2} \left(\frac{2M_2 L}{\pi^2} \right) \quad (28)$$

Then we have the following Theorem.

Theorem: If $f(x)$ is bounded and integrable in $(0, L)$ then the series $\sum_{n=1}^{\infty} A_n e^{-ak_n^2 t} \sin(k_n x)$, $k_n = \frac{n\pi}{L}$, $n=1, 2, \dots$

yields ~~the~~ ^a unique solution of the heat equation.

$u_t = a u_{xx}$ $(a > 0)$ with $u(0, t) = u(L, t) = 0$ for $t > 0$

and $0 \leq x \leq L$. If in addition $f(x)$ has a

continuous second derivative and $f(0) = f(L) = 0$

then $u(x, t) = \sum A_n e^{-ak_n^2 t} \sin k_n x$ is uniformly

convergent for $t > 0$ on $0 \leq x \leq L$ and also

satisfies the initial condition $u(x, 0) = f(x)$.

2. Heat equation with source (inhomogeneous heat eqn). (7)

Suppose now that heat sources or sinks are present in the rod. Then the initial and boundary value problem for the rod becomes

$$\left. \begin{aligned} u_t &= a u_{xx} + F(x,t), \quad t > 0, \quad x \in (0,L) \quad (27) \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0 \\ u(x,0) &= f(x), \quad 0 \leq x \leq L \end{aligned} \right\} \quad (29)$$

where $F(x,t)$ is a given function of x and t (source density). We assume that, from our experience of the previous problem.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) e^{-a k_n^2 t} \sin(k_n x) \quad (30)$$

where $k_n = \frac{n\pi}{L}$, $n=1,2,\dots$. To have a solution the source function $F(x,t)$ must have a similar series expansion

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t) e^{-a k_n^2 t} \sin(k_n x) \quad (31)$$

Then

$$F_n(t) e^{-a k_n^2 t} = \frac{2}{L} \int_0^L F(x,t) \sin(k_n x) dx \quad (32)$$

for all $n=1,2,\dots$

substituting (31) into the heat equation in (29) we get ⁽⁸⁾

$$u_n'(t) = F_n(t), \quad n=1,2,\dots \quad (33)$$

with the solution

$$u_n(t) = u_n(0) + \int_0^t F_n(\tau) d\tau. \quad (34)$$

for all $n=1,2,\dots$ with, from (30)

$$\sum_{n=1}^{\infty} u_n(0) \sin(k_n x) = f(x) \quad (35)$$

or

$$A_n = u_n(0) = \frac{2}{L} \int_0^L f(x) \sin(k_n x) dx, \quad n=1,2,\dots \quad (36)$$

Hence the formal solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-a k_n^2 t} \sin(k_n x) + \sum_{n=1}^{\infty} \left[\int_0^t F_n(\tau) d\tau \right] e^{-a k_n^2 t} \sin(k_n x) \quad (37)$$

If $f(x)$ is bounded and integrable in $(0,L)$ then the first ~~term~~ series in (37) is uniformly convergent.

Furthermore, the Fourier series in (35) is uniformly convergent if $f(0)=f(L)=0$ and $f(x)$ has continuous second derivatives in $0 \leq x \leq L$.

The second term in (37): $F(0,t) = F(L,t) = 0$ from (9)
 its property in (31). The Fourier series in (32)
 is uniformly convergent if $F(x,t)$ has continuous partial
 derivative wrt x in $(0,L)$ then

$$F_n(t) e^{-ak_n^2 t} = -\frac{2}{Lk_n^2} \int_0^L F''(x,t) \sin(k_n x) dx, \quad n=1,2,\dots \quad (38)$$

Then

$$|F_n(t) e^{-ak_n^2 t}| \leq \frac{2}{Lk_n^2} \int_0^L |F''(x,t)| dx, \quad n=1,2,\dots \quad (39)$$

Since $F''(x,t) = F_{xx}(x,t)$ is continuous in $(0,L)$, $t \geq 0$
 we have $|F_{xx}(x,t)| < M \quad \forall \quad x \in [0,L] \Rightarrow$

$$|F_n(t) e^{-ak_n^2 t}| \leq \frac{2M}{k_n^2} = \frac{1}{n^2} \left(\frac{2ML^2}{\pi^2} \right) \quad n=1,2,\dots \quad (40)$$

Then

$$\sum_{n=1}^{\infty} F_n(t) e^{-ak_n^2 t} < \infty \quad (41)$$

consequently Weierstrass M-test guarantees the
 uniform convergence of the series in the RHS of (31)

(10)

• Term by term differentiation, for instance u_t :

First term in (32) is alright (we have studied it in the first problem). We shall study the second term in (32), $(u_t) = \text{first} + \text{second}$.

$$\text{second} = \sum \left[F_n(t) e^{-ak_n^2 t} \sin k_n x - \left(\int_0^t F_n(\tau) d\tau \right) a k_n^2 e^{-ak_n^2 t} \sin(k_n x) \right] \quad (42)$$

$$\begin{aligned} |\text{second}| &\leq \sum \left[|F_n(t) e^{-ak_n^2 t}| + a k_n^2 \int_0^t |F_n(\tau)| d\tau e^{-ak_n^2 t} \right] \\ &\leq \frac{2M L^2}{\pi^2} \sum \frac{1}{n^2} + \sum_{n=1} a k_n^2 \left(\int_0^t |F_n(\tau)| e^{-ak_n^2 \tau} d\tau \right) e^{ak_n^2 (t-\tau)} \\ &\leq \frac{2M L^2}{6} + \sum a k_n^2 \frac{2M}{k_n^2} \int_0^t e^{ak_n^2 (\tau-t)} d\tau. \end{aligned}$$

Here we used (40).

$$\begin{aligned} |\text{second}| &\leq \frac{2M L^2}{6} + 2Ma \sum \int_0^t e^{ak_n^2 (\tau-t)} d\tau \\ &\leq \frac{2M L^2}{6} + 2Ma \sum \frac{1}{a k_n^2} (1 - e^{-ak_n^2 t}) \\ &\leq \frac{2M L^2}{6} + \frac{2M L^2}{\pi^2} \sum \frac{1}{n^2} = \frac{2M L^2}{3} < \infty \end{aligned}$$

(11)

Weierstrass M-test implies the uniform convergence of the series (42) and hence RHS of (37). Then we have the following Theorem.

Theorem: Suppose $f(x)$ is bounded and integrable in $0 \leq x \leq L$ and $F(x, t)$ has continuous second order partial derivatives with respect to x , with $F(0, t) = F(L, t) = 0$, $t \geq 0$. Then the series (30) with $u_n(t)$ given in (34) is a solution of the inhomogeneous heat equation and the boundary conditions in (29). If in addition $f(x)$ has continuous second derivative in $(0, L)$ and $f(0) = f(L) = 0$ then (30) also satisfies the initial condition in (29).

3) Heat equation with Inhomogeneous boundary conditions

(12)

Suppose that the previous initial and boundary value problem becomes

$$u_t = a u_{xx} + F(x, t), \quad 0 < x < L, \quad t > 0 \quad (42)$$

$$\left. \begin{aligned} u(0, t) &= g(t), \quad u(L, t) = h(t), \quad t > 0 \\ u(x, 0) &= f(x) \end{aligned} \right\} (43)$$

where $g(t)$ and $h(t)$ are some differentiable functions of t . We can transform the above problem to the initial value problem with homogeneous boundary conditions by letting

$$u(x, t) = v(x, t) + \frac{L-x}{L} g(t) + \frac{x}{L} h(t) \quad (44)$$

which leads to

$$\left. \begin{aligned} v_t &= a v_{xx} + \bar{F}(x, t) \quad 0 < x < L, \quad t > 0 \\ v(0, t) &= v(L, t) = 0, \quad t > 0 \\ v(x, 0) &= \bar{f}(x) \quad 0 \leq x \leq L \end{aligned} \right\} (45)$$

where

$$\left. \begin{aligned} \bar{F}(x, t) &= F(x, t) - \frac{L-x}{L} g'(t) - \frac{x}{L} h'(t) \\ \bar{f}(x) &= f(x) - \frac{L-x}{L} g(0) - \frac{x}{L} h(0) \end{aligned} \right\} (46)$$

Hence $v(x, t)$ satisfies our previous problem.

Discussions

a) It is clear from the solution of the homogeneous heat equation

$$u(x,t) = \sum A_n e^{-ak_n^2 t} \sin(k_n x)$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) \sin k_n x$$

with

$$|A_n| \leq 2 \max_{x \in [0,L]} |f(x)|$$

and

$$|u(x,t)| \leq \max |f(x)| \gamma$$

where $\gamma = 2 \sum e^{-ak_n^2 t}$. Indeed we shall show later that $|u(x,t)| \leq \max_{x \in [0,L]} |f(x)|$.

b) Solution is unique: This is clear from the above expression. Given data f we have a unique solution

(14)

c) continuous dependence on the data (stability of the solution)

$$|u_1 - u_2| \leq \sum |A_n^1 - A_n^2| e^{-ak_n^2 t} \leq |A_n^1 - A_n^2| e^{-ak_n^2 \tilde{t}/2} \\ \leq \sum |A_n^1 - A_n^2|, \quad t \geq \tilde{t}/2 > 0$$

but

$$A_n^1 - A_n^2 = \frac{2}{Lk_n^2} \int_0^L (f_1'' - f_2'') \sin k_n x \, dx$$

$$|A_n^1 - A_n^2| \leq \frac{2}{Lk_n^2} \max |f_1'' - f_2''|$$

$$\Rightarrow |u_1 - u_2| \leq \frac{2L^2}{\pi^2} \max |f_1'' - f_2''| \frac{\pi^2}{6} \\ \leq \frac{L^2}{3} \max |f_1'' - f_2''|$$

This proves the stability of the initial and boundary value problem

d) Internal energy

$$E(t) = \rho c \int_0^L u(x,t) dx$$

$$\text{as } t \rightarrow \infty \quad u(x,t) \rightarrow 0 \rightarrow E(t) \rightarrow 0$$

$$\frac{dE}{dt} = \rho c \int_0^L \frac{\partial u}{\partial t} dx = \rho c \int_0^L u_{xx} dx$$

$$= \rho c \left. u_x \right|_0^L = \rho c (u_x(L,t) - u_x(0,t))$$

$$u_x = \sum' A_n e^{-ak_n^2 t} k_n \cos k_n x$$

$$\begin{aligned} u_x(L,t) - u_x(0,t) &= \sum' A_n e^{-ak_n^2 t} k_n ((-1)^n - 1) \\ &= -2 \sum_{n=\text{odd}} A_n k_n e^{-ak_n^2 t} \end{aligned}$$

$$\frac{dE}{dt} < 0 \quad t > \tilde{t}_{1/2} > 0$$

e) If $f(x) \geq 0$ the initial temperature then
 $u(x,t) \geq 0$ for all $t \geq 0, 0 \leq x \leq L$

$$\text{and } \lim_{t \rightarrow \infty} u(x,t) = 0$$

let $a > 0$

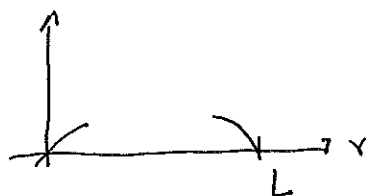
$$u_t - a u_{xx} = 0, \quad 0 < x < L, \quad t > 0$$

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L$$

i) $0 \leq u(x, t) \leq \max_{x \in [0, L]} f(x)$

ii) since $u(0, t) = u(L, t) = 0$ then



$$u_x(L, t) \leq 0$$

$$u_x(0, t) \geq 0$$

iii) $E(t) = \int_0^L u(x, t) dx > 0, \quad E(0) = 0$
if $u(x, 0) = 0$

$$\frac{dE}{dt} = a \int_0^L u_{xx}(x, t) dx = a [u_x(L, t) - u_x(0, t)] \leq 0$$

$$= -a \gamma(t), \quad \gamma(t) = -u_x(0, t) + u_x(L, t) \geq 0$$

$$E(t) = -a \int_0^t \gamma(t) dt \leq 0 \quad \text{contradiction}$$

$$E(0) = \int_0^L f(x) dx \quad \text{where } u(x, 0) = f(x)$$

$$\Rightarrow E(t) = E(0) - a \int_0^t \gamma(t) dt > 0$$

f) If $F \neq 0$ (a non-zero source).

Assume

$$\lim_{t \rightarrow \infty} F(x, t) = F(x).$$

Then what is the limit

$$\lim_{t \rightarrow \infty} u(x, t) = ?$$

stationary state $u_t = 0$

$$a u_{xx} + F = 0 \quad u(0) = u(L) = 0$$

$$u(x, t) = \sum_{n=1} A_n e^{-a k_n^2 t} \sin k_n x + \sum \left[\int_0^t F_n(\tau) d\tau \right] e^{-a k_n^2 t} \sin k_n x$$

where

$$F_n(t) e^{-a k_n^2 t} = \frac{2}{L} \int_0^L F(x, t) \sin k_n x dx$$

$$\lim_{t \rightarrow \infty} u(x, t) = \sum \lim_{t \rightarrow \infty} \left(\int_0^t F_n(\tau) d\tau / e^{a k_n^2 t} \right) \sin k_n x$$

$$= \sum \frac{F_n(t)}{a k_n^2 e^{a k_n^2 t}} \sin k_n x$$

(14)

$$\lim_{t \rightarrow \infty} u(x, t) = \sum \frac{1}{ak_n^2} F_n u_1 e^{-ak_n^2 t} \sin k_n x$$

$$= \sum \frac{1}{ak_n^2} \left(\lim_{t \rightarrow \infty} \left(\frac{2}{L} \int_0^L F(x', t) \sin k_n x' dx' \right) \right) \sin k_n x$$

$$= \sum \frac{2}{ak_n^2 L} \lim_{t \rightarrow \infty} \left(\int_0^L F(x', t) \sin k_n x' dx' \right) \sin k_n x$$

$$= \sum \frac{2}{ak_n^2 L} \int_0^L \left(\lim_{t \rightarrow \infty} F(x', t) \right) \sin k_n x' dx' \sin k_n x$$

$$= \sum_{n=1} \frac{2}{ak_n^2 L} \left(\int_0^L F(x') \sin k_n x' dx' \right) \sin k_n x$$

$$u u_{xx} + F = 0$$

$$u(x).$$

$$= - \sum_{n=1} \frac{2}{ak_n^2 L} a \int_0^L u_{xx}' \sin k_n x' dx' \sin k_n x$$

$$= - \sum_{n=1} \frac{2}{k_n^2 L} \left[\cancel{u_x' \sin k_n x'} \Big|_0^L - k_n \int_0^L u_x' \cos k_n x' dx' \right] \sin k_n x$$

$$= \sum \frac{2}{k_n L} \left(\int_0^L u_x' \cos k_n x' dx' \right) \sin k_n x$$

$$= \sum \frac{2}{k_n L} \left[-\cancel{u \sin k_n x'} \Big|_0^L + k_n \int_0^L u \sin k_n x' dx' \right] \sin k_n x$$

$$= + \sum \frac{2}{L} \left(\int_0^L u \sin k_n x' dx' \right) \sin k_n x$$

$$= \sum_{n=1} a_n \sin k_n x = u(x) \quad \checkmark$$

The Maximum Principle of the heat equation
and its consequences.

Let

$$u_t - a u_{xx} = F(x, t) \leq 0$$

$$u(0, t) = g(t) \leq 0$$

$$u(L, t) = h(t) \leq 0$$

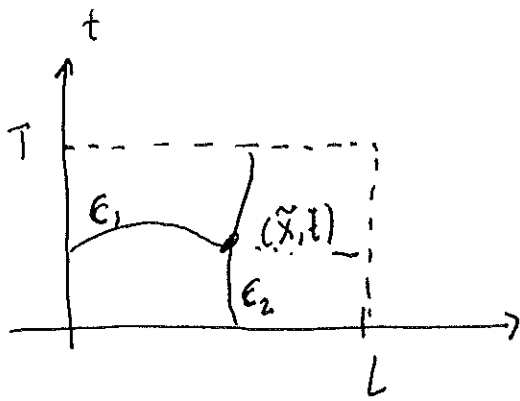
$$u(x, 0) = f(x) \leq 0$$

we expect that $u(x, t) \leq 0 \quad \forall 0 \leq x \leq L, t \geq 0$

We shall prove this: First we prove a proposition

proposition: Suppose that $w(x, t)$ satisfies the
differential inequality $w_t - a w_{xx} < 0$ in $0 < x < L$,
 $0 < t \leq T$, where $a > 0$. Then $w(x, t)$ can
not assume a local maximum value at any
point in the rectangle $0 < x < L, 0 < t \leq T$

proof. Assume the contrary: Let $w(\tilde{x}, \tilde{t})$ be a
local maximum value for w at $(\tilde{x}, \tilde{t})$,
($0 < \tilde{x} < L, 0 < \tilde{t} \leq T$)



a) Along the curve C_1 :
 $w(x, \tilde{t})$, $(0 < x < L)$ has a
 local maximum at $x = \tilde{x}$, hence
 $w_{xx} < 0$ at $(\tilde{x}, \tilde{t})$

b) Along the curve C_2 : $w(\tilde{x}, t)$, $(0 < t \leq T)$ has
 a local maximum value at $t = \tilde{t}$, hence
 $w_t \geq 0$ at $(\tilde{x}, \tilde{t})$ (strict inequality
 occurs at $\tilde{t} = T$)

From (a) and (b) we have $w_t - a w_{xx} \geq 0$
 at $(\tilde{x}, \tilde{t})$ which is a contradiction. Hence there
 exist no point in the rectangle $0 < x < L$, $0 < t \leq T$,
 so that $w(x, t)$ assume a local maximum value.

Theorem: Let $a \geq 0$ and suppose that $u(x, t)$ is
 continuous in $0 \leq x \leq L$, $0 \leq t \leq T$ (rectangle
 with its boundaries, $R \cup \partial R = \bar{R}$ and satisfies

$$u_t - a u_{xx} \leq 0 \quad 0 < x < L, \quad 0 < t \leq T$$

$$u(0, t) \leq 0, \quad u(L, t) \leq 0, \quad 0 \leq t \leq T$$

$$u(x, 0) \leq 0, \quad 0 \leq x \leq L$$

Then $u(x, t) \leq 0$ for $(x, t) \in R \cup \partial R$

proof: By contradiction: Assume to the contrary that u is positive at some point, say $u(\tilde{x}, \tilde{t}) > 0$. Define

$$w(x, t) = u(x, t) - \varepsilon t$$

where $\varepsilon > 0$ and $0 \leq x \leq L$, $0 \leq t \leq T$. Then for any choice of ε we find.

$$w_t - a w_{xx} = u_t - a u_{xx} - \varepsilon < 0$$

$$w(0, t) = u(0, t) - \varepsilon t \leq 0 \quad 0 \leq t \leq T$$

$$w(L, t) = u(L, t) - \varepsilon t \leq 0$$

$$w(x, 0) = u(x, 0) \leq 0 \quad 0 \leq x \leq L$$

previous lemma and the above boundary conditions show that for every $\varepsilon > 0$ $w(x, t)$ can not have a (positive) maximum in the closed interval $0 \leq x \leq L$, $0 \leq t \leq T$. On the other hand, for small enough $\varepsilon > 0$ we have

$$w(\tilde{x}, \tilde{t}) = u(\tilde{x}, \tilde{t}) - \varepsilon \tilde{t} > 0$$

but this is a contradiction. Hence there exists no point $(\tilde{x}, \tilde{t})$ in R so that $u(\tilde{x}, \tilde{t}) > 0$.

Corollary 1. let $a > 0$ and suppose that $u(x,t)$ is continuous in $0 \leq x \leq L$, $0 \leq t \leq T$ and satisfies

$$u_t - a u_{xx} \geq 0, \quad \text{for } 0 < x < L, \quad 0 < t \leq T$$

$$u(0,t) \geq 0, \quad u(L,t) \geq 0 \quad \text{in } 0 \leq t \leq T$$

$$u(x,0) \geq 0 \quad \text{in } 0 \leq x \leq L$$

Then $u(x,t) \geq 0$ for $0 \leq x \leq L$, $0 \leq t \leq T$

proof: Take $v = -u$. Use ~~theorem~~ the previous thm. for v .

Corollary 2 let $a > 0$, suppose $u(x,t)$ is defined and continuous in $0 \leq x \leq L$, $0 \leq t \leq T$ and satisfies

$$u_t = a u_{xx} \quad 0 < x < L, \quad 0 < t \leq T$$

$$\text{let } M = \max \left\{ \max_{x \in [0,L]} u(x,0), \max_{0 < t \leq T} u(0,t), \max_{0 < t \leq T} u(L,t) \right\}$$

$$m = \min \left\{ \min_{x \in [0,L]} u(x,0), \min_{0 < t \leq T} u(0,t), \min_{0 < t \leq T} u(L,t) \right\}$$

Then $m \leq u(x,t) \leq M$ for all $0 \leq x \leq L$, $0 \leq t \leq T$.

proof. i) let $V(x,t) = u(x,t) - M$. Then.

$$V_t - V_{xx} = 0 \quad 0 < x < L, \quad 0 < t \leq T$$

$$V(0,t) \leq 0, \quad V(L,t) \leq 0 \quad 0 < t \leq T.$$

$$V(x,0) \leq 0 \quad 0 \leq x \leq L$$

$$\Rightarrow V(x,t) \leq 0 \Rightarrow u(x,t) \leq M.$$

ii) let $W(x,t) = m - u(x,t)$. Then

$$W_t - W_{xx} = 0.$$

$$W(0,t) \leq 0, \quad W(L,t) \leq 0 \quad 0 \leq t \leq T$$

$$W(x,0) \leq 0 \quad 0 \leq x \leq L$$

$$\Rightarrow W(x,t) \leq 0 \Rightarrow u(x,t) \geq m.$$

Hence we obtain $m \leq u(x,t) \leq M$

for all $0 \leq x \leq L, \quad 0 \leq t \leq T$.

Corollary 3. Let $a > 0$, suppose that $u(x,t)$ defined and continuous in $0 \leq x \leq L$, $0 \leq t \leq T$ and satisfies

$$u_t - au_{xx} = F(x,t) \quad 0 < x < L, \quad 0 < t \leq T$$

and $F(x,t)$ is bounded by a constant N that

$$|F(x,t)| \leq N \quad \text{for all } 0 \leq x \leq L, \quad 0 \leq t \leq T.$$

Let

$$M = \max \left\{ \max_{x \in [0,L]} |u(x,0)|, \max_{0 \leq t \leq T} |u(0,t)|, \max_{0 \leq t \leq T} |u(L,t)| \right\}$$

Then

$$|u(x,t)| \leq M + tN \quad \text{for all } 0 \leq x \leq L, \quad 0 \leq t \leq T$$

Proof: i) Let $v(x,t) = u(x,t) - (M + tN)$

$$v_t - av_{xx} = u_t - au_{xx} - N = F - N \leq 0$$

$$v(0,t) \leq 0, \quad v(L,t) \leq 0 \quad 0 \leq t \leq T$$

$$v(x,0) \leq 0 \quad 0 \leq x \leq L$$

$$\Rightarrow v(x,t) \leq 0 \quad \Rightarrow u(x,t) \leq M + tN$$

$$\text{ii) Let } w(x,t) = -u(x,t) - (M + tN)$$

Then

$$w_t - a w_{xx} = -F - N \leq 0$$

$$w(0, t) \leq 0, \quad w(L, t) \leq 0 \quad 0 \leq t \leq T$$

$$w(x, 0) \leq 0$$

$$\Rightarrow w(x, t) \leq 0$$

$$\Rightarrow -(M + tN) \leq u(x, t)$$

As a result

$$|u(x, t)| \leq M + tN \leq M + TN$$

Theorem: There exist at most one function $u(x, t)$ which is continuous in $0 \leq x \leq L$, $0 \leq t$ and satisfies

$$u_t - a u_{xx} = F(x, t) \quad 0 < x < L, \quad t > 0$$

$$u(0, t) = g(t), \quad u(L, t) = h(t), \quad t \geq 0$$

$$u(x, 0) = f(x) \quad 0 \leq x \leq L$$

where f, g, h are given continuous functions (data)

proof: let $u_1 - u_2 = w$. Then w where

u_1 and u_2 are the solutions of the above initial and boundary value problem with the same data. Hence it satisfies

$$w_t - a w_{xx} = 0 \quad 0 < x < L, \quad t > 0$$

$$w(0,t) = w(L,t) = 0 \quad t > 0$$

$$w(x,0) = 0 \quad 0 \leq x \leq L$$

This means that $m = M = 0$ (corollaries).

($m \leq w \leq M$) Hence $w(x,t) = 0$ which leads

to $u_1 = u_2$.

Theorem. Let $a > 0$, suppose that $u(x,t)$ is continuous on $0 \leq x \leq L$, $0 \leq t \leq T$ and satisfies the initial and boundary value problem.

$$\frac{\partial}{\partial t} u - a \frac{\partial^2}{\partial x^2} u = F \quad 0 < x < L, \quad 0 < t \leq T$$

$$u(0,t) = g(t), \quad u(L,t) = h(t), \quad 0 \leq t \leq T$$

$$u(x,0) = f(x), \quad 0 \leq x \leq L$$

Then this initial and boundary value problem is stable, i.e., the solutions continuously depend on the data.

proof: let the data $(F_1, f_1, g_1, h_1) \rightarrow u_1(x,t)$
and $(F_2, f_2, g_2, h_2) \rightarrow u_2(x,t)$, i.e.

$$\frac{\partial}{\partial t} u_i - \alpha \frac{\partial^2}{\partial x^2} u_i = F_i(x, t) \quad , \quad 0 < x < L, \quad 0 < t \leq \tau$$

$$u_i(0, t) = g_i(t), \quad u_i(L, t) = h_i(t) \quad , \quad 0 \leq t \leq \tau$$

$$u_i(x, 0) = f_i(x) \quad , \quad 0 \leq x \leq L$$

for $i=1, 2$. Suppose further that.

$$|f_1 - f_2| < \varepsilon, \quad |g_1 - g_2| < \varepsilon$$

$$|h_1 - h_2| < \varepsilon, \quad |F_1 - F_2| < \varepsilon$$

for all $0 \leq x \leq L, \quad 0 \leq t \leq \tau$. Then.

using the Lemma Corollary 3 we get

$$|u_1 - u_2| \leq (1+T)\varepsilon. \quad (N=M=\varepsilon)$$

— • —

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Another proof of the uniqueness theorem in the previous theorem.

proof: let $w(x,t) = u_1 - u_2$ where u_1 and u_2 are two different solutions of the initial and boundary value problem. ~~Take~~ In the same data, then

$$w_t - a w_{xx} = 0 \quad 0 < x < L, t > 0$$

$$w(0,t) = w(L,t) = 0 \quad t > 0$$

$$w(x,0) = 0, \quad 0 \leq x \leq L$$

Define now the energy functional

$$E(t) = \int_0^L w^2 dx \geq 0$$

with $E(0) = 0$, and

$$\frac{dE}{dt} = \int_0^L 2w w_t dx = 2a \int_0^L w w_{xx} dx$$

$$= 2a \left[w w_x \Big|_0^L - \int_0^L w_x^2 dx \right] = -2a \int_0^L w_x^2 dx \leq 0$$

hence $E' \leq 0$ for all $t > 0$. Hence we

have

$$E(t) \geq 0 \quad \forall t \geq 0$$

$$E(0) = 0$$

$$E'(t) \leq 0 \quad \forall t \geq 0$$

Hence the only possibility is that $E(t) = 0$
 $\forall t \geq 0$. This means that

$$w(x, t) = 0 \quad \forall x \in (0, L), t \geq 0$$

$$\text{or } u_1(x, t) = u_2(x, t) \quad \forall x \in [0, L], t \geq 0.$$

This proves the uniqueness.

The Green's Function (of the heat equation)

We have studied that the initial value problem

$$u_t = a u_{xx} \quad 0 \leq x \leq L, \quad t > 0$$

$$\begin{aligned} u(x, 0) &= f(x) & 0 \leq x \leq L \\ u(0, t) &= u(L, t) = 0 & t > 0 \end{aligned} \quad (1)$$

has the solution

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-a \lambda_n^2 t} \sin(\lambda_n x), \quad \lambda_n = \frac{n\pi}{L} \quad (1)$$

where

$$c_n = \frac{2}{L} \int_0^L f(y) \sin(\lambda_n y) dy \quad (2)$$

we assumed that $f(0) = f(L) = 0$ and $f(x)$ is twice differentiable function. For $t > 0$ we insert (2) in (1)

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} \frac{2}{L} \int_0^L f(y) \sin(\lambda_n y) dy e^{-a \lambda_n^2 t} \sin(\lambda_n x) \\ &= \int_0^L \left[\frac{2}{L} \sum_{n=1}^{\infty} e^{-a \lambda_n^2 t} \sin \lambda_n y \sin \lambda_n x \right] f(y) dy \end{aligned}$$

$\uparrow$ converges uniformly and all derivatives are uniformly conv. on $0 \leq x \leq L, \quad t > t_0$ for any $t_0 > 0$

hence interchanging sum and integral is legal

Define

$$G(x, y, t) = \frac{2}{L} \sum_{n=1}^{\infty} e^{-a\lambda_n^2 t} \sin \lambda_n y \sin \lambda_n x$$

for $0 \leq x, y \leq L$ and $t > 0$.

$$\Rightarrow u(x, t) = \int_0^L G(x, y, t) f(y) dy.$$

at $t=0$ $G(x, y, 0) = \delta(x-y).$

$G(x, y, t)$ is the G-F for the heat equation.

Properties:

i) $G_t(x, y, t) = a G_{xx}(x, y, t) = a G_{yy}(x, y, t)$

for $0 \leq x, y \leq L, t > 0$

ii) $G(0, y, t) = G(L, y, t) = 0$

for $0 \leq y \leq L, t > 0$

iii) $G(x, y, t) = G(y, x, t)$

for $0 \leq x, y \leq L, t > 0$

iv) $\lim_{(x, t) \rightarrow (x_0, 0)} \left(\int_0^L G(x, y, t) dy \right) = f(x_0).$

$$0 \leq x_0 \leq L$$

$$u(x,t) = \int_0^L G(x,y,t) f(y) dy$$

$u(x,t)$ satisfies (term by term differentiation of $G(x,y,t)$) the heat equation and satisfies the boundary condition. $f(x)$ has a continuous second derivative

sequence of
Given $\forall$ twice differentiable function $\{f_n(x)\}$ with $f_n(0) = f_n(L) = 0$. Then given $\varepsilon > 0$ there is an index $N(\varepsilon)$ such that

$$n, m > N(\varepsilon) \Rightarrow |f_n(x) - f_m(x)| < \varepsilon \quad \text{for } 0 \leq x \leq L$$

Let $u_n(x,t)$ be the solution to the heat equation with zero boundary condition and initial temperature $f_n(x)$. By the maximum principle

$$n, m > N(\varepsilon) \text{ implies that } |u_n(x,t) - u_m(x,t)| < \varepsilon$$

for $0 \leq x \leq L$, $t > 0$, and consequently the sequence $\{u_n(x,t)\}$ converges uniformly to a function $u(x,t)$ on $0 \leq x \leq L$, $t > 0$.

Evidently $u(x,t)$ is ~~contin~~ continuous. Moreover for each (x,t) with $t > 0$

$$\begin{aligned} u(x,t) &= \lim_{n \rightarrow \infty} u_n(x,t) = \lim_{n \rightarrow \infty} \int_0^L G(x,y,t) f_n(y) dy \\ &= \int_0^L G(x,y,t) f(y) dy \end{aligned}$$

Since $u(x,t)$ is continuous on $0 \leq x \leq L$, $t > 0$ and $\Rightarrow$

$$u(x_0, 0) = f(x_0)$$

Then we have

Theorem: Let the initial temperature $f(x)$ be continuous on $0 \leq x \leq L$ and satisfy $f(0) = f(L) = 0$. Then initial and boundary value problem (1-3) has a unique solution, continuous solution defined on $0 \leq x \leq L$, $t \geq 0$. This solution is given by

$$u(x, t) = \int_0^L G(x, y, t) f(y) dy$$

for $t > 0$

Heat flow in infinite and semi-infinite Rod

The temperature $u(x, t)$ in an infinite rod, taken to be the entire x -axis is given by

$$\begin{aligned} u_t &= a u_{xx} & -\infty < x < \infty, \quad t > 0 \\ u(x, 0) &= f(x) & -\infty < x < \infty \end{aligned}$$

where $f(x)$ is the initial temperature distribution.

We use Fourier transform

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} g(x) dx$$

inverse transform

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{g}(\omega) d\omega$$

(GFS)

multiply the heat equation by $e^{i\omega x}$ and integrate

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$$\int_{-\infty}^{\infty} (u_t - a u_{xx}) e^{i\omega x} dx = \hat{u}_t - a \int_{-\infty}^{\infty} u_{xx} e^{i\omega x} dx$$

$$= \hat{u}_t + i a \omega \int_{-\infty}^{\infty} u_x e^{i\omega x} dx$$

$$u_x \Big|_{|x| \rightarrow \infty} = 0 \quad \Rightarrow \quad \hat{u}_t + a \omega^2 \hat{u} = 0$$

$$u \Big|_{|x| \rightarrow \infty} = 0 \quad \Rightarrow \quad \hat{u}(\omega, t) = A(\omega) e^{-a\omega^2 t}$$

$$= \int_{-\infty}^{\infty} e^{i\omega x} u(x, t) dx$$

$$\hat{u}(\omega, 0) = A(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x) dx$$

$\Rightarrow$

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{u}(\omega, t) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) e^{-a\omega^2 t - i\omega x} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega y} f(y) dy e^{-a\omega^2 t - i\omega x} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dy f(y) G(x, y, t)$$

$$u(x,t) = \int_{-\infty}^{\infty} k(x-y,t) f(y) dy$$

where ~~$k(x,y,t)$~~

$$u(x,t) = \int_{-\infty}^{\infty} dy f(y) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-a\omega^2 t - i\omega(x-y)} d\omega$$

use $\int_{-\infty}^{\infty} e^{-i\omega\beta - \omega^2/4\alpha} d\omega = \sqrt{4\pi\alpha} e^{-\alpha\beta^2}, \alpha > 0$

with $\alpha = 1/4at$, $\beta = x-y$ for $t > 0$

$$u(x,t) = \frac{1}{\sqrt{4\pi at}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4at} f(y) dy$$

$$= \int_{-\infty}^{\infty} k(x-y,t) f(y) dy$$

with $k(x,y,t) = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}$

for $-\infty < x < \infty$, $t > 0$

function $k(x,y,t)$ is called the "fundamental solution" to the heat equation. It plays the same role for the ~~rod~~ infinite rod that GF does for finite length rod.

We found the formal solution: By changing variable we get

$$u(x,t+1) = \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-\xi^2} f(x - \sqrt{4at} \xi) d\xi.$$

change variable $\xi = \frac{x-y}{\sqrt{4at}}$

The solution $u(x,t+1) = \int_{-\infty}^{\infty} k(x-y, t+1) f(y) dy$

Cannot give a solution to the IVP for all continuous function $f(y)$. There should be restriction on f . This integral diverges if $f(x) = e^{xy}$. This suggests that f should be bounded on the real line.

It is easy to show that

$$k_x(x,t+1) = -\frac{2x}{4t} k(x,t+1)$$

$$k_t = k_{xx} = \frac{x^2 - 2t}{4t^2} k(x,t+1)$$

We have the following lemma

Lemma: let $0 < t_0 < T$, $R > 0$ and $\gamma > 1$ be given. Then there is a constant N depending on t_0, T, R and γ such that

$$0 \leq k(x-y, t+1) \leq N e^{-y^2/4\gamma t}$$

$$|k_x(x-y, t)| \leq N e^{-y^2/4\delta a T}$$

$$|k_t(x-y, t)| = |k_{xx}(x-y, t)| \leq N e^{-y^2/4\delta a T}$$

for all $|x| \leq R$, $t_0 \leq t \leq T$

proof:

$$k(x-y, at) = \frac{1}{\sqrt{4\pi at}} e^{-\frac{(x-y)^2}{4at}} \leq \frac{1}{\sqrt{4\pi at_0}} e^{-\frac{(x-y)^2}{4\pi a T}}$$

$$\leq \frac{1}{\sqrt{4\pi at_0}} e^{-\frac{-x^2 + 2xy - y^2}{4aT}} \leq \frac{1}{\sqrt{4\pi at_0}} e^{\frac{2R|y|}{4aT}} e^{-\frac{y^2}{4aT}}$$

$$\leq \frac{1}{\sqrt{4\pi at_0}} e^{\frac{2R|y|}{4aT}} e^{-\frac{y^2}{4aT} [1 - \frac{1}{R}]} e^{-\frac{y^2}{4\delta a T}}$$

bounded and continuous

has a max. value = N.

$$\leq N e^{-y^2/4\delta a T}$$

all other functions k_t, k_x, k_{xx} are bounded
the bound N is the largest for all these cases

Since $\int_{-\infty}^{\infty} e^{-\frac{y^2}{4\delta a T}} dy = \sqrt{4\pi\delta a T} < \infty$

hence for any bounded, integrable function $f(y)$
the integral

$$\int_{-\infty}^{\infty} k(x-y, at) f(y) dy$$

Can be differentiated under the integral sign once wrt t and twice wrt x for $x \in \mathbb{R}$, $0 < t_0 \leq t \leq T$. Let $t_0 \rightarrow 0$ and $R, T \rightarrow \infty$ to see that here derivative can be taken under the integral sign for $-\infty < x < \infty$ and $t > 0$

and since $h_t(x, t) = h_{xx}(x, t) \Rightarrow$ the solution to the heat eqn. hence we have

Theorem: Suppose that the initial temperature distribution is continuous and bounded on $-\infty < x < \infty$. Then the diffusion problem has a solution which is continuous on $-\infty < x < \infty$, $t > 0$. For $t > 0$ the soln. is

$$u(x, t) = \int_{-\infty}^{\infty} h(x-y, t) f(y) dy$$

In fact we have

$$u(x, t) = \begin{cases} \int_{-\infty}^{\infty} h(x-y, t) f(y) dy & -\infty < x < \infty \quad t > 0 \\ f(x) & -\infty < x < \infty \quad t = 0 \end{cases}$$

we showed this part

We have to show that the continuity of this solution. we proved the continuity for all $t > 0$ for $t = 0$ we must show

$$\lim_{(x, t) \rightarrow (x_0, 0)} \int_{-\infty}^{\infty} h(x-y, t) f(y) dy = f(x_0)$$

proof:

$$i) \int_{-\infty}^{\infty} k(x-y, at) dy = 1.$$

like GF

$$k(x-y, at) = \frac{1}{\sqrt{4\pi at}} e^{-\frac{(x-y)^2}{4at}}$$

$$\frac{1}{\sqrt{4\pi at}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4at}} dy$$

$$\xi = \frac{x-y}{\sqrt{4at}}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2} d\xi = 1. \quad \checkmark$$

$$ii) \int_{-\infty}^{\infty} k(x-y, at) f(y) dy = f(x_0)$$

$$= \int_{-\infty}^{\infty} k(x-y, at) [f(y) - f(x_0)] dy$$

$$= \int_{|y-x_0| < \delta} k(x-y, at) [f(y) - f(x_0)] dy \quad S_1$$

$$+ \int_{|y-x_0| > \delta} k(x-y, at) [f(y) - f(x_0)] dy \quad S_2$$

Given $\epsilon > 0$ choose $\delta > 0$ so that $|f(y) - f(x_0)| < \epsilon$
for $|y - x_0| < \delta \Rightarrow$

$$\Rightarrow |S_1| < \varepsilon$$

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in $|y-x_0| > \delta$ we have $f(x)$ is bounded

$$|f(x)| < M$$

$$|S_2| < \int_{|y-x_0| > \delta} h(x-y, t) |f(y) - f(x_0)| dy$$

$$< 2M \int_{-\infty}^{x_0-\delta} h(x-y, t) dy + 2M \int_{x_0+\delta}^{\infty} h(x-y, t) dy$$

$$\leq \frac{2M}{\sqrt{\pi}} \left[\int_{-\infty}^{(x_0-\delta-x)/\sqrt{4\pi t}} e^{-\xi^2} d\xi + \int_{(x_0+\delta-x)/\sqrt{4\pi t}}^{\infty} e^{-\xi^2} d\xi \right]$$

$$\text{with } \xi = \frac{y-x}{\sqrt{4\pi t}}$$

For $|x-x_0| < \delta/2$ we have

$$|S_2| \leq \frac{2M}{\sqrt{\pi}} \left[\int_{-\infty}^{-\frac{\delta}{2\sqrt{4\pi t}}} e^{-\xi^2} d\xi + \int_{\frac{\delta}{2\sqrt{4\pi t}}}^{\infty} e^{-\xi^2} d\xi \right] < \varepsilon$$

provided that $t > 0$ is sufficiently close to ∞
completing the proof.

$$\Rightarrow \lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} h(x-y, t) f(y) dy = f(x)$$

Remark: let $\delta_n(x)$ be a mass point
Assume initially that the entire mass
of a substance is localized at the point x_0
and the total mass is φ . Let $\delta_n(x)$ be
a sequence of continuous functions which
vanish for $|x - x_0| > \frac{1}{n}$ and satisfy

$$\int_{-\infty}^{\infty} \delta_n(x) dx = 1$$

Then $\varphi \delta_n(x)$ is the concentration of a
mass distributed ^{nearby} localized at x_0 and with
total mass φ

let $u_n(x, t)$ be the solution with the initial
data $\varphi \delta_n(x)$ then for $t > 0$

$$\begin{aligned} u_n(x, t) &= \int_{-\infty}^{\infty} k(x-y, t) \varphi \delta_n(y) dy \\ &= \varphi \int_{x_0 - 1/n}^{x_0 + 1/n} k(x-y, t) \delta_n(y) dy \end{aligned}$$

According to the MVT for integrals

$$\begin{aligned} u_n(x, t) &= k(x-y_n, t) \int_{x_0 - 1/n}^{x_0 + 1/n} \varphi \delta_n(y) dy \\ &= \varphi k(x-y_n, t) \end{aligned}$$

$$n \rightarrow \infty \quad \lim_{n \rightarrow \infty} u_n(x, t) = \varphi k(x-x_0, t)$$

$\Rightarrow k(x-x_0, t)$ represents the concentration of unit mass
concentrated at x_0 when $t=0$

Semi-infinite rod.

GF12

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$$u_t(x,t) - a u_{xx} = 0$$

$$x > 0, t > 0$$

$$u(x,0) = f(x)$$

$$x > 0$$

$$u(0,t) = 0$$

$$t > 0$$

define

$$u^+(x,t) = \int_0^{\infty} h(x-y, at) f(y) dy$$

$$u^-(x,t) = \int_{-\infty}^0 h(x-y, at) f(-y) dy$$

$$\equiv \int_0^{\infty} h(x+y, at) f(y) dy$$

$$u(x,t) = u^+ - u^- = \int_0^{\infty} G(x,y, at) f(y) dy$$

$$G(x,y, at) = h(x-y, at) - h(x+y, at)$$

GF for semi-infinite interval

$$0 < x < \infty$$

ASSIGNED EXERCISES OF MATH544: PDE set 6 April, 2001

THE HEAT EQUATION

Ronald B Guenther and John W Lee, Partial Differential Equations of Mathematical Physics and Integral Equations

1. Solve the following initial value problem

$$u_t = au_{xx}, \quad (a > 0), t > 0, \quad x \in [0, L], \quad (1)$$

$$u(0, t) = u(L, t) = 0, \quad t \geq 0, \quad (2)$$

$$u(x, 0) = f(x), \quad x \in [0, L] \quad (3)$$

step 1. Formal Solution: Using the separation of variables we first let $u(x, t) = T(t)X(x)$. Then find that

$$T' = a k T, \quad X'' = k X$$

where k is any constant. It must be a negative number $k = -\lambda^2$. we find that

$$T = e^{-a\lambda_n^2 t}, X = \sin(\lambda_n x) \quad , n = 1, 2, \dots$$

where $\lambda_n = \frac{n\pi}{L}$. Hence

$$u(x, t) = \sum_{n=1}^{\infty} \alpha_n e^{-a\lambda_n^2 t} \sin(\lambda_n x) \quad (4)$$

Here α_n are constants , will be determined by the initial condition (assuming the t goes to zero exists which requires the uniform convergence of the sine series)

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin(\lambda_n x), \quad (5)$$

with

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin(\lambda_n x) dx, \quad n = 1, 2, \dots \quad (6)$$

step 2. Is the formal solution a solution of the initial and boundary value problem we started with?. For this the given function must satisfy certain conditions. Term-by term differentiation of the series representing $u(x, t)$ in (4) can be justified at each point $(\tilde{x}, \tilde{t})$ with $\tilde{t} > 0$ and $0 \leq \tilde{x} \leq L$ by showing that the series in (4) and the series representing u_t , u_x and u_{xx} that result from term-by-term differentiation, converge uniformly in a region D which contains the point $(\tilde{x}, \tilde{t})$. Such a region is

$$D = \{(x, t) | 0 \leq x \leq L, t \geq \frac{\tilde{t}}{2}\}.$$

We first assume that the function $f(x)$ is bounded in $[0, L]$

$$|f(x)| \leq M, \quad \text{for all } x \in [0, L]$$

Then from (6) we have

$$|\alpha_n| \leq 2M$$

and series (4) satisfies

$$\left| \sum_{n=1}^{\infty} \alpha_n e^{-a\lambda_n^2 t} \sin(\lambda_n x) \right| \leq 2M \sum_{n=1}^{\infty} \lambda_n^2 e^{-a\lambda_n^2 \tilde{t}/2}.$$

The series at the right hand side converges uniformly in D . Similar reasoning applies also for the series representing u_t , u_x , and u_{xx} . Consequently (4) satisfies the equations (1) and (2).

To guarantee that (4) will also satisfy the initial condition (5), we need further assumptions on $f(x)$. For the uniform convergence of the series in (5) we need $\sum_{n=1}^{\infty} |\alpha_n| < \infty$. If $f(0) = f(L) = 0$ then

$$\alpha_n = -\frac{2}{\lambda_n^2 L} \int_0^L f'' \sin(\lambda_n x) dx.$$

Hence, if f'' is continuous on $0 \leq x \leq L$, then it is bounded say by M'' , and

$$|\alpha_n| \leq \frac{2M''L^2}{\pi^2 n^2}$$

This estimate implies that the series (4) for $u(x, t)$ is uniformly convergent for $t \geq 0$ and $0 \leq x \leq L$. Since $u(x, t)$ is the uniform limit of continuous functions, it is continuous. Hence we have the following:

Theorem. *If $f(x)$ is bounded and integrable, the series in (4) yields a solution to (1) and (2) for $t > 0$ and $0 \leq x \leq L$. If in addition f has continuous second derivative and $f(0) = f(L) = 0$, then $u(x, t)$ defined in (4) is uniformly convergent and continuous for $t \geq 0$ and $0 \leq x \leq L$ and also satisfies the initial condition (3).*

2. Solve the following initial and boundary value problem:

$$u_t - au_{xx} = F(x, t), \quad t > 0, \quad 0 < x < L, \quad (7)$$

$$u(0, t) = u(L, t) = 0, \quad t \geq 0, \quad (8)$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L \quad (9)$$

Complete the method of solution given in class. The method of solution is similar to previous problem.

3. Solve the problem:

$$u_t - au_{xx} = F(x, t), \quad 0 < x < L, \quad t > 0, \quad (10)$$

$$u_x(0, t) = u_x(L, t) = 0, \quad t \geq 0, \quad (11)$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L. \quad (12)$$

Explain also how you can prove the uniqueness of the solution of this problem.

4. Let $u(x, t)$ be a solution to the problem

$$u_t - au_{xx} = 0, \quad t > 0, \quad 0 \leq x \leq L, \quad (13)$$

$$u(0, t) = u(L, t) = 0, \quad t \geq 0 \quad (14)$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L \quad (15)$$

let $\tilde{u}$ be a solution of the above problem with f replaced by $\tilde{f}$. Denote the respective Fourier coefficients by α_n and $\tilde{\alpha}_n$. Show $|\alpha_n - \tilde{\alpha}_n| \leq 2\|f'' - \tilde{f}''\|/\lambda_n^2$

and hence

$$\|u - \tilde{u}\| \leq 2\|f'' - \tilde{f}''\| \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \right)$$

where $\|h\| = \max|h(x)|$ for $0 \leq x \leq L$. Deduce continuous dependence and hence the well-posedness of the problem.

5. For $T > 0$ let

$$Q_T = \{(x, t) | 0 < x < L, 0 < t \leq T\}$$

(this is exactly the region D defined in the lecture), be a rectangle with its bottom and two vertical sides removed. Let

$$\bar{Q}_T = \{(x, t) | 0 \leq x \leq L, 0 \leq t \leq T\}.$$

Then $B_T = \bar{Q}_T - Q_T$ consists of the three missing sides of the original rectangle Q_T . Suppose that $w(x, t)$ satisfies the differential inequality

$$w_t - a(x, t)w_{xx} - b(x, t)w_x - c(x, t)w < 0 \quad (x, t) \in Q_T$$

where $a(x, t) \geq 0$ and $c(x, t) \leq 0$ in Q_T . Then $w(x, t)$ can not achieve a positive local maximum in Q_T

6. Let a, b and c be as in the previous problem. Suppose that $u(x, t)$ is continuous on $\bar{Q}_T$ and satisfies

$$u_t - a(x, t)u_{xx} - b(x, t)u_x - c(x, t)u \leq 0, \quad (x, t) \in Q_T, \quad (16)$$

$$u(x, t) \leq 0, \quad (x, t) \in B_T \quad (17)$$

Then $u(x, t) \leq 0$ on $\bar{Q}_T$.

7. Let a, b and c be as in Prob.4. Suppose that $u(x, t)$ is continuous on $\bar{Q}_T$ and satisfies

$$u_t - a(x, t)u_{xx} - b(x, t)u_x - c(x, t)u = 0$$

in Q_T . If u has a positive maximum, it is assumed on the boundary B_T and if u has a negative minimum, it is assumed on B_T .

8. Let a, b and c be as in Prob. 4. Suppose that $F(x, t)$ is defined in Q_T and bounded by the constant N there. Suppose that $u(x, t)$ is continuous on $\bar{Q}_T$, $M = \max_{B_T} |u(x, t)|$, and $u(x, t)$ satisfies

$$u_t - a(x, t)u_{xx} - b(x, t)u_x - c(x, t)u = F(x, t)$$

in Q_T . The $|u(x, t)| \leq M + TN$ in Q_T .

9. Let $a(x, t) > 0$ and $c(x, t) \leq 0$ for all $0 < x < L$ and $t > 0$. Show that the initial boundary value problem

$$u_t - a(x, t)u_{xx} - b(x, t)u_x - c(x, t)u = F(x, t), \quad 0 < x < L, \quad t > 0, \quad (18)$$

$$u(0, t) = g(t), \quad u(L, t) = h(t), \quad t \leq 0, \quad (19)$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L \quad (20)$$

has at most one solution which is continuous on $0 \leq x \leq L$ and $t \geq 0$ [Notice that if there is, in fact a continuous solution, then the boundary and initial data must be compatible. That is, $g(0) = f(0)$, $h(0) = f(L)$].

HOMEWORK III (MATH 544): *March 7, 2012*
(For March 21, 2012)

1. An initial value problem of the wave equation:

$$\begin{aligned}u_{tt} - c^2 u_{xx} &= 0, \quad t > 0, \quad 0 < x < L, \\u(0, t) &= 0, \quad u_x(L, t) = 0, \quad t \geq 0, \\u(x, 0) &= f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq L\end{aligned}$$

where $f(x)$ and $g(x)$ are given functions (data) for all $x \in (0, L)$. Find the conditions on the data so that the above initial value problem is well posed.

2. Solve the following initial value problems.

$$x^2 u_{xx} + 3xy u_{xy} + 2y^2 u_{yy} = 0, \quad (x, y) \in D \subset \mathbb{R}^2 \tag{1}$$

with the following initial data:

(a) $u(x, x\sqrt{x}) = f(x)$ and $u_n(x, x\sqrt{x}) = g(x)$ where $f(x)$ and $g(x)$ are given functions of x and u_n is the directional derivative of u along the normal vector of the initial curve.

(b) $u(x, x) = f(x)$ and $u_n(x, x) = g(x)$ where $f(x)$ and $g(x)$ are given functions of x and u_n is the directional derivative of u along the normal vector of the initial curve.

Solution to the second Question

$$2) \quad x^2 u_{xx} + 3xy u_{xy} + 2y^2 u_{yy} = 0$$

$$\xi = \frac{x}{y}, \quad \eta = \frac{x^2}{y}$$

DE changes to $u(x, y) = \zeta(\xi, \eta)$

$$-\eta \zeta_{\xi\eta} + \zeta_{\xi} = 0$$

$$\Rightarrow \zeta(\xi, \eta) = \eta F(\xi) + G(\eta)$$

where F and G are arbitrary functions. Then

$$u(x, y) = \frac{x^2}{y} F(x/y) + G(x^2/y)$$

a) Curve is not a characteristic $x = t, y = t^{3/2}$

$$\sqrt{t} F\left(\frac{1}{\sqrt{t}}\right) + G(\sqrt{t}) = f(t)$$

$$-\left(3 + \frac{1}{t}\right) F\left(\frac{1}{\sqrt{t}}\right) + 2t\left(\frac{3}{2} + \frac{1}{t}\right) F_t$$

$$-2\sqrt{t}\left(3 + \frac{1}{t}\right) G_t = \sqrt{t + \frac{9}{4}t} g(t)$$

$$\hat{t} = \frac{1}{\sqrt{1 + \frac{9}{4}t}} \left(1, \frac{3}{2}\sqrt{t}\right)$$

$$\hat{n} = \frac{1}{\sqrt{1 + \frac{9}{4}t}} \left(-\frac{3}{2}\sqrt{t}, 1\right)$$

$$\Rightarrow F_t = \frac{\sqrt{g + 2\sqrt{t}\left(3 + \frac{1}{t}\right)} f'}{2t\left(\frac{9}{4} + \frac{1}{t}\right)}$$

" $F(1/\sqrt{t})$ "

" $G(\sqrt{t})$ "

$$G_t = f' - \frac{1}{2} \frac{1}{\sqrt{t}} F - \sqrt{t} F_t$$

then we obtain $F(t)$ and $G(t)$ in terms of data

b) Curve is a characteristic $x=t, y=t$

$$\text{let } F(t) + G(t) = f(t)$$

$$\frac{1}{\sqrt{2}} \left[-\frac{2x}{y} F - \frac{x^2}{y^2} F - \frac{x^2}{y^2} G' \right]$$

$$t = \frac{1}{\sqrt{2}} (1, 1)$$

$$\eta = \frac{1}{\sqrt{2}} (-1, 1)$$

$$\frac{1}{\sqrt{2}} \left[-\frac{2x}{y} F - \frac{x^2}{y^2} F - \frac{2x}{y} G_\eta - \frac{x^2}{y^2} F - \frac{x^3}{y^3} F \right] = g(t) \quad \text{on } \delta$$

$$\frac{1}{\sqrt{2}} \left[-3F(1) - 2F_\xi(1) - 2G_t(t) \right] = g(t)$$

$$-3F(1) - 2F_\xi(1) - 2G_t(t) = \sqrt{2} g(t)$$

$$\text{but } G(t) = -tF(1) + f(t)$$

$$G_t = -F(1) + f'$$

$$\Rightarrow -3F(1) - 2F_\xi(1) + 2F(1) - 2f' = 2g(t)$$

$$-F(1) - 2F_\xi(1) - 2f' = 2g(t)$$

$$\text{Differentiate: } f'' = -g'$$

If data satisfied this relation we get infinitely many solutions otherwise there exist no solution