

# Hyperbolic Type of Equations

## Lecture 3

Spring 2012

1. Cauchy Problem of hyperbolic equations with constant coefficients: Existence and Uniqueness of the solutions
2. The Riemann Problem: To solve the Cauchy Problem of hyperbolic equations one of the best methods is the "Riemann Problem" which is equivalent to the method of "Green's function".
3. Higher Dimensional hyperbolic equations and some examples.

## Hyperbolic Equations.

1. Cauchy Problem for hyperbolic equations with constant coefficients

$$u_{tt} - c^2 u_{xx} + b u_t + c u_x + d u = F(x, t), \quad (1)$$

where  $a, b, d$  are constants and initial condition

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad (2)$$

Transforming to the canonical form

$$\xi = x + ct, \quad \eta = x - ct \quad (3)$$

$\Rightarrow$

$$u_{\xi\eta} + \alpha u_{\xi} + \beta u_{\eta} + \gamma u = G(\xi, \eta). \quad (4)$$

where  $\alpha, \beta, \gamma$  are also constants. Using the transformation

$$u(\xi, \eta) = v(\xi, \eta) e^{\lambda \xi + \mu \eta} \quad (5)$$

we obtain

$$v_{\xi\eta} + k v = g(\xi, \eta) \quad (6)$$

proof:

$$u_{\xi} = V_{\xi} e^{\lambda \xi + \mu \eta} + \lambda V e^{\lambda \xi + \mu \eta}.$$

$$u_{\eta} = V_{\eta} e^{\lambda \xi + \mu \eta} + \mu V e^{\lambda \xi + \mu \eta}.$$

$$u_{\xi \eta} = V_{\xi \eta} e^{\lambda \xi + \mu \eta} + (\mu V_{\xi} + \lambda V_{\eta}) e^{\lambda \xi + \mu \eta} \\ + \lambda \mu V e^{\lambda \xi + \mu \eta}.$$

Hence

$$V_{\xi \eta} + \alpha (V_{\xi} + \lambda V) + \beta (V_{\eta} + \mu V) \\ + \mu V_{\xi} + \lambda V_{\eta} + \lambda \mu V + \gamma V = G(\xi, \eta) e^{-\lambda \xi - \mu \eta}$$

Choose  $\mu = -\alpha, \lambda = -\beta \Rightarrow$

$$V_{\xi \eta} + (-\alpha\beta - \alpha\beta + \alpha\beta \gamma) V = G(\xi, \eta) e^{-\lambda \xi - \mu \eta}$$

$$\Rightarrow k = \gamma - \alpha\beta, \quad g(\xi, \eta) = G(\xi, \eta) \quad \text{we get}$$

$$V_{\xi \eta} + k V = g(\xi, \eta)$$

Initial conditions at  $\xi = \eta$ . ( $t=0$ ).

$$u(\xi, \xi) = \bar{f}(\xi).$$

$$v_{\xi}(\xi, \xi) = \bar{g}(\xi).$$

$$v_{\eta}(\xi, \xi) = \bar{h}(\xi)$$

but

$$\bar{f}_{\xi} = \bar{g}(\xi) + \bar{h}(\xi)$$

only two of the are independent

- Existence and uniqueness of the initial value problem

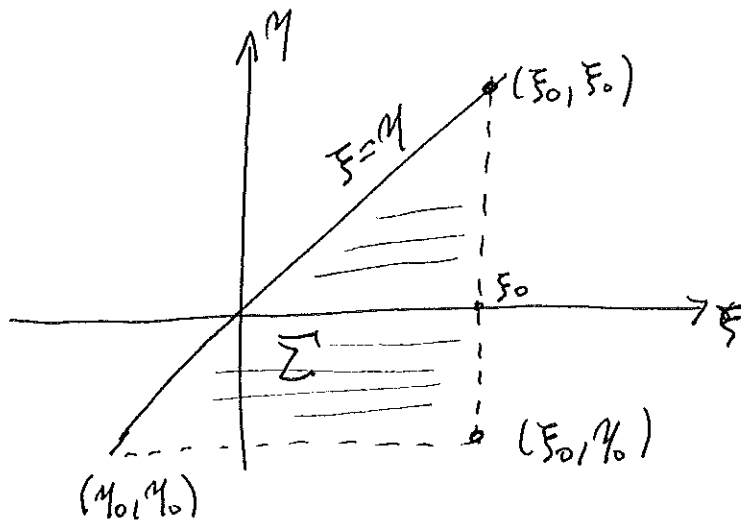
$$v_{\xi\eta} + k v = g(\xi, \eta) \quad (7)$$

$$v(\xi, \xi) = \bar{f}(\xi) \quad (8)$$

$$v_{\xi}(\xi, \xi) = \bar{g}(\xi) \quad (9)$$

$$v_{\eta}(\xi, \xi) = \bar{h}(\xi) \quad (10)$$

Define the region  $\bar{Z}$



Integrating  $V_{xy}$   
on  $\bar{Z}$

$$\iint_{\bar{Z}} V_{xy} dxdy$$

$$\iint_{\bar{Z}} V_{xy} dxdy = \int_{y_0}^{x_0} \int_y^{x_0} V_{xy} dxdy = \int_{y_0}^{x_0} (V_y(x_0, y) - V_y(y, y)) dy$$

$$= V(x_0, y) \Big|_{y_0}^{x_0} - \int_{y_0}^{x_0} \bar{h}(y) dy$$

$$= V(x_0, x_0) - V(x_0, y_0) - \int_{y_0}^{x_0} \bar{h}(y) dy$$

$$\iint_{\bar{Z}} V_{xy} dxdy = -V(x_0, y_0) + f(x_0) - \int_{y_0}^{x_0} \bar{h}(y) dy \quad (11)$$

Integrating now first wrt  $y$  we get

$$\begin{aligned} \iint_{\bar{Z}} V_{xy} dy dx &= \int_{x_0}^{x_0} \int_{y_0}^x V_{xy} dy dx \\ &= \int_{x_0}^{x_0} [V_x(x, x) - V_x(x, y_0)] dx \end{aligned}$$

$$\begin{aligned} \iint_{\bar{Z}} V_{\xi\eta} d\eta d\xi &= \int_{\eta_0}^{\xi_0} \bar{g}(\xi) d\xi - V(\xi_0, \eta_0) + V(\eta_0, \eta_0) \\ &= -V(\xi_0, \eta_0) + \bar{f}(\eta_0) + \int_{\eta_0}^{\xi_0} \bar{g}(\xi) d\xi. \end{aligned} \quad (12)$$

Adding (11) and (12) and dividing by 2 we get

$$\begin{aligned} \iint_{\bar{Z}} V_{\xi\eta} d\xi d\eta &= -V(\xi_0, \eta_0) + \frac{1}{2} [\bar{f}(\xi_0) + \bar{f}(\eta_0)] \\ &\quad - \frac{1}{2} \int_{\eta_0}^{\xi_0} \bar{h}(\eta) d\eta + \frac{1}{2} \int_{\eta_0}^{\xi_0} \bar{g}(\xi) d\xi \\ &= -V(\xi_0, \eta_0) + \frac{1}{2} [\bar{f}(\xi_0) + \bar{f}(\eta_0)] \\ &\quad + \frac{1}{2} \int_{\eta_0}^{\xi_0} (\bar{g}(\xi) - \bar{h}(\xi)) d\xi. \end{aligned} \quad (13)$$

using (7) we get

$$\iint_{\bar{Z}} V_{\xi\eta} d\xi d\eta = -k \iint_{\bar{Z}} V(\xi, \eta) d\xi d\eta + \iint_{\bar{Z}} g(\xi, \eta) d\xi d\eta \quad (14)$$

comparing (13) with (14) we obtain

$$V(\xi_0, \eta_0) = k \iint_{\bar{Z}(\xi_0, \eta_0)} V(\xi, \eta) d\xi d\eta + \Phi(\xi_0, \eta_0) \quad (15)$$

where

$$\begin{aligned} \Phi(\xi_0, \eta_0) = & \frac{1}{2} [\bar{f}(\xi_0) + \bar{f}(\eta_0)] + \frac{1}{2} \int_{\eta_0}^{\xi_0} (\bar{h}(\xi) - \bar{g}(\xi)) d\xi \\ & - \iint_{\bar{Z}(\xi_0, \eta_0)} g(\xi, \eta) d\xi d\eta. \end{aligned} \quad (16)$$

is a given function.

Theorem: The initial value problem 7-10 has at most one solution.

proof: Let  $V_1$  and  $V_2$  ~~have~~ be the solution of this initial value problem with the same data  $\Phi$ , then  $\Psi = V_1 - V_2$  we have

$$\Psi(\xi_0, \eta_0) = k \iint_{\bar{Z}(\xi_0, \eta_0)} \Psi(\xi, \eta) d\xi d\eta. \quad (17)$$

Let  $M = \max |\Psi(\xi, \eta)|$  for  $(\xi, \eta) \in \bar{Z}(\xi_0, \eta_0)$

$$\begin{aligned}
\Rightarrow |\Psi(\xi_0, \eta_0)| &\leq k \int_{\eta_0}^{\xi_0} \int_{\eta_0}^{\xi_0} M \, d\xi \, d\eta \\
&\leq k M \int_{\eta_0}^{\xi_0} (\xi_0 - \eta) \, d\eta \\
&\leq k M \left( \xi_0 \eta - \frac{1}{2} \eta^2 \right) \Big|_{\eta_0}^{\xi_0} \\
&\leq k M \left( \xi_0^2 - \frac{1}{2} \xi_0^2 - \xi_0 \eta_0 + \frac{1}{2} \eta_0^2 \right) \\
&\leq k M \frac{1}{2} (\xi_0 - \eta_0)^2.
\end{aligned}$$

$$|\Psi(\xi_0, \eta_0)| \leq \frac{1}{2} k M |\xi_0 - \eta_0|^2$$

for all  $(\xi_0, \eta_0) \in \mathbb{R}^2 \Rightarrow$

$$|\Psi(\xi, \eta)| \leq \frac{1}{2} k M |\xi - \eta|^2 \quad (18)$$

using this in

$$\begin{aligned}
|\Psi(\xi_0, \eta_0)| &\leq k \iint |\Psi(\xi, \eta)| \, d\xi \, d\eta \\
&\leq \frac{1}{2} k^2 M \iint (\xi - \eta)^2 \, d\xi \, d\eta.
\end{aligned}$$

$$\int_{\eta_0}^{\xi_0} \int_{\eta}^{\xi_0} (\xi - \eta)^2 d\xi d\eta = \int_{\eta_0}^{\xi_0} \int_0^{\xi_0 - \eta} u^2 du d\eta.$$

$$= \int_{\eta_0}^{\xi_0} \frac{1}{2} (\xi_0 - \eta)^3 d\eta = \frac{1}{3} \int_{\xi_0 - \eta_0}^0 u^3 (-du).$$

$$= \frac{1}{12} (\xi_0 - \eta_0)^4 \Rightarrow.$$

$$|\Psi(\xi_0, \eta_0)| \leq \frac{1}{24} h^2 M |\xi_0 - \eta_0|^4.$$

Algorithmically we proceed and find that

$$|\Psi(\xi_0, \eta_0)| \leq \frac{h^n}{(2n)!} M |\xi_0 - \eta_0|^{2n}.$$

letting  $n \rightarrow \infty$  we get  $\Psi(\xi_0, \eta_0) \rightarrow 0$

Hence

$$\Psi(x_0, y_0) = 0 \Rightarrow v_1 = v_2$$

Hence  $v$  is unique

## Existence

$$V(\xi, \eta) = \iint_{\bar{Z}(\xi, \eta)} V(r, s) dr ds + \Phi(\xi, \eta)$$

Define the sequence

$$V_n(\xi, \eta) = \iint_{\bar{Z}(\xi, \eta)} V_{n-1}(r, s) dr ds + \Phi(\xi, \eta), \quad n=1, 2, \dots$$

$$V_1(\xi, \eta) = \iint_{\bar{Z}(\xi, \eta)} V_0(r, s) dr ds + \Phi(\xi, \eta)$$

Sequence  $\{V_n\}$ , we expect that it converges to the solution  $V$  as  $n \rightarrow \infty$ , or

$$\lim_{n \rightarrow \infty} V_n = V$$

Assuming convergence, we start with  $V_0 = 0$

$$V_1(\xi, \eta) = \Phi(\xi, \eta)$$

$$V_2(\xi, \eta) = k \iint_{\bar{Z}(\xi, \eta)} \Phi(r, s) dr ds + \Phi(\xi, \eta)$$

$$V_3(\xi, \eta) = k \iint_{\bar{Z}(\xi, \eta)} V_2(r, s) dr ds + \Phi(\xi, \eta)$$

...

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$$\begin{aligned}
 V_3(\xi, \eta) &= \Phi(\xi, \eta) + k \iint_{\bar{Z}(\xi, \eta)} \left[ \Phi(r, s) + k \iint_{\bar{Z}(r, s)} \Phi(r', s') dr' ds' \right] dr ds \\
 &= \Phi(\xi, \eta) + k \iint_{\bar{Z}(\xi, \eta)} \Phi(r, s) dr ds \\
 &\quad + k^2 \iint_{\bar{Z}(\xi, \eta)} \left( \iint_{\bar{Z}(r, s)} \Phi(r', s') dr' ds' \right) dr ds.
 \end{aligned}
 \tag{11}$$

The last term can be simplified by integration by parts

$$\begin{aligned}
 &\iint_{\bar{Z}(\xi, \eta)} \left( \iint_{\bar{Z}(r, s)} \Phi(r', s') dr' ds' \right) dr ds \\
 &= \int_{\eta}^{\xi} \int_s^{\xi} \left( \int_s^r \int_{s'}^r \Phi(r', s') dr' ds' \right) dr ds. \\
 &= \int_{\eta}^{\xi} \int_{\eta}^r \left( \int_s^r \int_{s'}^r \Phi(r', s') dr' ds' \right) ds dr. \\
 &= \int_{\eta}^{\xi} \left[ s \left( \int_s^r \int_{s'}^r \Phi(r', s') dr' ds' \right) \right]_{s=\eta}^{s=r} \\
 &\quad + \int_{\eta}^{\xi} s \left( \int_s^r \Phi(r', s') dr' \right) ds dr
 \end{aligned}$$

$$= \int_{\eta}^{\xi} \int_{\eta}^r (s'-\eta) \int_s^r \Phi(r', s') dr' ds' dr.$$

$$= r \int_{\eta}^r (s'-\eta) \int_s^r \Phi(r', s') dr' ds' \Big|_{r=\eta}^{r=\xi}.$$

$$= \int_{\eta}^{\xi} (s'-\eta) r \Phi(r, s') ds' dr.$$

$$= \int_{\eta}^{\xi} \int_s^{\xi} (\xi-r)(s'-\eta) \Phi(r, s') ds' dr.$$

$$= \iint_{Z(\xi, \eta)} (\xi-r)(s-\eta) \Phi(r, s) dr ds.$$

Hence

$$V_3(\xi, \eta) = \Phi(\xi, \eta) + k \iint_{Z(\xi, \eta)} \Phi(r, s) dr ds + k^2 \iint_{Z(\xi, \eta)} (\xi-r)(s-\eta) \Phi(r, s) dr ds$$

$$= \Phi(\xi, \eta) + k \iint_{Z(\xi, \eta)} \sum_{j=0}^{\infty} \frac{k^j (\xi-r)^j (s-\eta)^j}{(j!)^2} \Phi(r, s) dr ds$$

Continuing this way we obtain

$$V_n(z, y) = \Phi(z, y) + k \iint_{\bar{Z}(z, y)} \sum_{j=0}^{n-2} k^j \frac{(z-r)^j (y-s)^j}{(j!)^2} \Phi(r, s) dr ds$$

as  $n \rightarrow \infty$ , since

$$\sum_{j=0}^{\infty} \frac{z^j}{(j!)^2} = \sum_{j=0}^{\infty} \frac{(2\sqrt{z})^{2j}}{2^{2j} (j!)^2} = I_0(2\sqrt{z})$$

where  $I_0$  is the modified Bessel function, then

$$V(z, y) = \Phi(z, y) + k \iint_{\bar{Z}(z, y)} I_0(2\sqrt{k(z-r)(y-s)}) \Phi(r, s) dr ds$$

Hence we have the following theorem

Theorem: Let  $g(z, y)$ ,  $\bar{f}(z)$ ,  $\bar{g}(z)$ , and  $\bar{h}(z)$  be continuous for  $(z, y) \in \bar{Z}(z, y)$ . Then the Cauchy problem (7-10) has a unique solution.

## 2. The Riemann Method

This is a method to solve the initial value problem of hyperbolic type of equations. I will explain this method by solving a specific problem.

1. Solve the following initial value problem

$$u_{xy} - y u_x = 1, \quad y > -x, \quad -\infty < x < \infty \quad (1)$$

$$\left. \begin{aligned} u(x, -x) &= x \\ u_x(x, -x) &= x^2 \\ u_y(x, -x) &= x^2 - 1 \end{aligned} \right\} \quad -\infty < x < \infty. \quad (2)$$

initial curve is  $\Gamma = \{(x, y) \in \mathbb{R}^2 \mid y = -x\}$   
 compatibility condition on the initial conditions

$$u_t = u_x \dot{x} + u_y \dot{y}, \quad t = x. \quad (\text{on } \Gamma)$$

$$u_x = u_x - u_y = x^2 - (x^2 - 1) = 1. \quad \checkmark$$

~~Step 1~~ ~~Write the differential equation as~~  
~~a divergence of a vector field  $(B, A)$~~   
 ~~$A_y + B_x = C$~~

(15)

Step 1. Define a new function  $v(x, y)$  which satisfied the adjoint equation. In this example

$$v_{xy} + (yv)_x = 0 \quad (3)$$

Homogeneous adjoint equation. Using (1) and (3) we get (multiply (1) by  $v$  and multiplying (3) by  $u$  and subtract)

$$A_y + B_x = 0 \quad (4)$$

where

$$A = \frac{1}{2} (v u_x - u v_x), \quad B = \frac{1}{2} (v u_y - u v_y) - y u v$$

This (Eq. (4)) is nothing but the Lagrange identity that we studied in MATH 543

$$v L u - u L^t v = \frac{dQ}{dt}$$

for the ODEs. In this case

$$L = \frac{\partial^2}{\partial x \partial y} - y \frac{\partial}{\partial x}, \quad L^t = \frac{\partial^2}{\partial x \partial y} + y \frac{\partial}{\partial x}$$

so that, in this case

$$L u = 1, \quad L^t v = 0 \quad (5)$$

$$\begin{aligned}
 v L u - u L^+ v &= v \frac{\partial^2 u}{\partial x \partial y} - y \frac{\partial^2 u}{\partial x^2} - u \left( \frac{\partial^2 v}{\partial x \partial y} + y \frac{\partial^2 v}{\partial x^2} \right) \\
 &= v u_{xy} - u v_{xy} - y v u_x - y u v_x \\
 &= (v u_x - u v_x)_y - (y u v)_x
 \end{aligned}$$

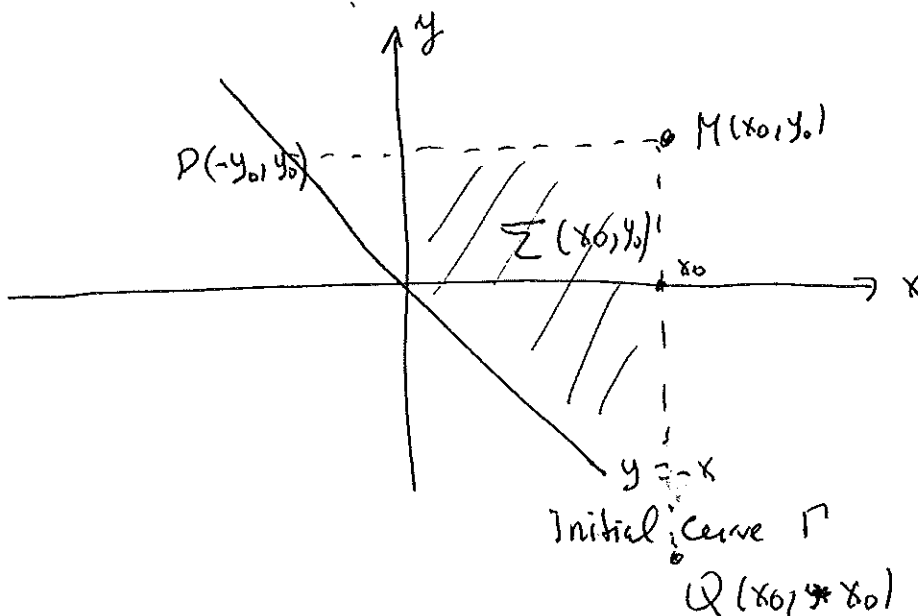
or

$$\begin{aligned}
 v L u - u L^+ v &= \frac{1}{2} [v u_x - u v_x]_y + \frac{1}{2} [v u_y - u v_y]_x - (y u v)_x \\
 &= \text{div}
 \end{aligned}$$

$$A = \frac{1}{2} [v u_x - u v_x], \quad B = \frac{1}{2} [v u_y - u v_y] - y u v \quad (6)$$

which is nothing but (4).

Step 2. Integrate the divergence equation (4) in a triangular region  $\Sigma(x_0, y_0)$



$$\iint_{\bar{Z}(x_0, y_0)} (A_y + B_x) dx dy = \iint_{\bar{Z}(x_0, y_0)} V(x, y) dx dy. \quad (7)$$

Assuming the differentiability of  $A$  and  $B$  in  $\Sigma$   
we use the Green's thm in  $\Sigma$

$$\iint_{\bar{Z}(x_0, y_0)} (A_y + B_x) dx dy = \oint_C (-A dx + B dy).$$

Where the closed curve  $C$  is the union of  
the curves  $QM$ ,  $MP$ , and  $PQ$ ., i.e.

$$\oint_C (-A dx + B dy) = \int_{\substack{QM \\ (x=x_0)}} B dy - \int_{\substack{MP \\ (y=y_0)}} A dx + \int_{\substack{PQ \\ y=-x}} (-A dx + B dy)$$

$$\begin{aligned} \text{i) } \int_{QM} B dy &= \frac{1}{2} \int_{QM} (V u_y - u v_y) dy - \int_{QM} y u v dy \\ &= \frac{1}{2} \int_{-x_0}^{y_0} V(x_0, y) u_y(x_0, y) - u(x_0, y) v_y(x_0, y) dy \\ &\quad - \int_{-x_0}^{y_0} y u(x_0, y) V(x_0, y) dy \end{aligned}$$

$$\int_{QM} B \, dy = \frac{1}{2} V(x_0, y) U(x_0, y) \Big|_{-y_0}^{y_0} - \frac{1}{2} \int_{-x_0}^{y_0} U(x_0, y) V_y(x_0, y) \, dy$$

$$- \int_{-x_0}^{y_0} y U(x_0, y) V(x_0, y) \, dy.$$

$$= \frac{1}{2} V(x_0, y) U(x_0, y) \Big|_{-y_0}^{y_0} - \int_{-x_0}^{y_0} [V_y(x_0, y) + y V(x_0, y)] U(x_0, y) \, dy$$

$$ii) \int_{MP} A \, dx = \frac{1}{2} \int_{x_0}^{-y_0} (V(x, y_0) U_x(x, y_0) - U(x, y_0) V_x(x, y_0)) \, dx$$

$$= \frac{1}{2} \int_{x_0}^{-y_0} (V(x, y_0) U(x, y_0)) \, dx$$

$$- \int_{x_0}^{-y_0} U(x, y_0) V_x(x, y_0) \, dx$$

$$= \frac{1}{2} V(x, y_0) U(x, y_0) \Big|_{x_0}^{-y_0} - \int_{x_0}^{-y_0} U(x, y_0) V_x(x, y_0) \, dx$$

(9)

(19)

$$\begin{aligned}
\int_{QM} B dy - \int_{MP} A dx &= \frac{1}{2} V(x_0, y) u(x_0, y) \Big|_{-x_0}^{y_0} - \frac{1}{2} V(x, y_0) u(x, y_0) \Big|_{x_0}^{-y_0} \\
&\quad - \int_{-x_0}^{y_0} [V_y(x_0, y) + y V(x_0, y)] u(x_0, y) dy \\
&\quad + \int_{x_0}^{-y_0} u(x, y_0) V_x(x, y_0) dx
\end{aligned}$$

$$\begin{aligned}
&= V(x_0, y_0) u(x_0, y_0) - \frac{1}{2} V(x_0, -x_0) u(x_0, -x_0) \\
&\quad - \frac{1}{2} V(-y_0, y_0) u(-y_0, y_0)
\end{aligned}$$

we let

$$V_y(x_0, y) + y V(x_0, y) = 0 \quad y \leq y_0$$

$$V_x(x, y_0) = 0$$

$$x \neq x_0$$

$$V(x_0, y_0) = 1$$

(10)

$$\Rightarrow u(x_0, y_0) - \frac{1}{2} x_0 V(x_0, -x_0) + \frac{1}{2} y_0 V(-y_0, y_0)$$

$$+ \int_{PQ} (-A dx + B dy) = \iint_{Z(x_0, y_0)} V(x, y) dx dy$$

(11)

(20)

$$u(x_0, y_0) = \frac{1}{2} x_0 v(x_0, -x_0) + \frac{1}{2} y_0 v(-y_0, y_0).$$

$$+ \iint_{\Sigma(x_0, y_0)} v(x, y) dx dy - \int_{PQ} (-A dx + B dy)$$

(12)

The function  $v(x, y)$  is indeed depend on four variables

$$v(x, y) = R(x, y, x_0, y_0). \quad (14)$$

satisfying the differential equation.

$$v_{xy} + y v_x = 0 \quad \text{in } \Sigma. \quad (15)$$

and the conditions (10)

$$\begin{aligned} & v_y(x_0, y) + y v(x_0, y) = 0 \quad y \leq y_0 \\ & v_x(x, y_0) = 0 \quad x \neq x_0 \\ & v(x_0, y_0) = 0 \end{aligned} \quad (16)$$

So the problem reduced to determining the auxiliary function  $v(x, y)$  called the Riemann function (a kind of Green's function that we studied in MATH 543). This problem is the "Goursat problem"

Step 3. Solving the Goursat problem.

solution of (16) is easy

$$V(x, y) = h(x) e^{-\frac{1}{2}y^2} + g(y).$$

Here  $h(x)$  and  $g(y)$  may depend on  $x_0$  and  $y_0$

From the first eqn. of (16) we get  $g(y) = 0$   
and second equation gives  $h = h(x_0, y_0)$

Hence

$$V(x, y) = h(x_0, y_0) e^{-\frac{1}{2}y^2} \quad (17)$$

using the last eqn in (16) we get

$$h(x_0, y_0) e^{-\frac{1}{2}y_0^2} = 1.$$

$$\Rightarrow V(x, y) = R(x, y; x_0, y_0) = e^{\frac{1}{2}(y_0^2 - y^2)} \quad (18)$$

We have the following properties.

$$V(x_0, -x_0) = e^{\frac{1}{2}(y_0^2 - x_0^2)}. \quad (19)$$

$$V(-y_0, y_0) = 1. \quad (20)$$

~~1.  $V(x, y) = 1$~~

~~2.  $V(-y_0, y_0) = 1$~~

~~3.  $V(x_0, -x_0) = e^{\frac{1}{2}(y_0^2 - x_0^2)}$~~

~~4.  $V(x, y) = e^{\frac{1}{2}(y_0^2 - y^2)}$~~

~~5.  $V(x, y) = e^{\frac{1}{2}(y_0^2 - x_0^2)}$~~

(23)

$$\int_{PQ} = -\frac{3}{2} \int_{-y_0}^{x_0} x^2 V(x, -x) dx - \frac{1}{2} \int_{y_0}^{-x_0} V(-y, y) dy$$

$\Rightarrow$  From (12) we get

$$u(x_0, y_0) = \frac{1}{2} x_0 e^{\frac{1}{2}(y_0^2 - x_0^2)} - \frac{1}{2} y_0$$

$$+ \frac{3}{2} \int_{-y_0}^{x_0} x^2 e^{\frac{1}{2}(y_0^2 - x^2)} dx$$

$$+ \frac{1}{2} \int_{y_0}^{-x_0} e^{\frac{1}{2}(y_0^2 - y^2)} dy + \iint_{\Sigma(x_0, y_0)} V(x, y) dx dy$$

or

$$u(x, y) = \frac{1}{2} x e^{\frac{1}{2}(y^2 - x^2)} - \frac{1}{2} y + \frac{3}{2} \int_{-y}^x \xi^2 e^{\frac{1}{2}(y^2 - \xi^2)} d\xi$$

$$+ \frac{1}{2} \int_y^{-x} e^{\frac{1}{2}(y^2 - \eta^2)} d\eta + \int_{-x}^x \int_{-y}^y V(\xi, \eta) d\xi d\eta$$

$\nearrow e^{\frac{1}{2}(y^2 - \eta^2)}$

$$\iint_{\Sigma} v(x, y) dx dy = \int_{-x}^y (x+y) e^{\frac{1}{2}(y^2-y^2)} dy.$$

$\Rightarrow$

$$u(x, y) = \frac{1}{2} x e^{\frac{1}{2}(y^2-x^2)} - \frac{1}{2} y + \frac{3}{2} \int_{-y}^x \xi^2 e^{\frac{1}{2}(y^2-\xi^2)} d\xi \\ - \frac{1}{2} \int_{-y}^x e^{\frac{1}{2}(y^2-\eta^2)} d\eta + \int_{-y}^x (x-\eta) e^{\frac{1}{2}(y^2-\eta^2)} d\eta$$

$$u(x, y) = \frac{1}{2} x e^{\frac{1}{2}(y^2-x^2)} - \frac{1}{2} y + \int_{-y}^x \left( \frac{3}{2} \eta^2 - \frac{1}{2} + x - \eta \right) e^{\frac{1}{2}(y^2-\eta^2)} d\eta$$

Solution

$$u(x, y) = \frac{1}{2} x e^{\frac{1}{2}(y^2-x^2)} - \frac{1}{2} y + \int_{-y}^x \left( \frac{3}{2} \eta^2 - \frac{1}{2} + x - \eta \right) e^{\frac{1}{2}(y^2-\eta^2)} d\eta$$


---

Exercise: Solve

$$u_{xy} - \frac{2}{(x+y)^2} u = 0$$

$$u(x, x) = 0, \quad u_x(x, x) = +x^2 \Rightarrow u_y(x, x) = -x^2$$

Solution:

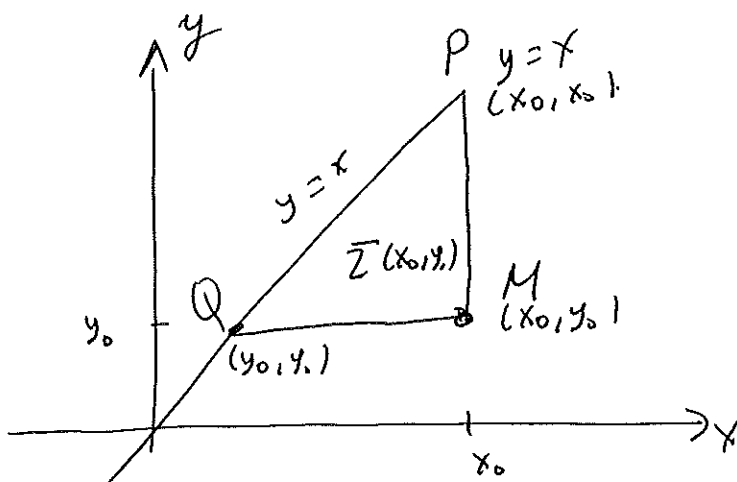
$$u_{xy} - \frac{2}{(x+y)^2} u = 0$$

$$v_{xy} - \frac{2}{(x+y)^2} v = 0$$

$$u_{xy} v - u v_{xy} = 0$$

or

$$(v u_x)_y - (u v_y)_x = 0$$



Integrate the last equation in  $\Sigma$  and use the GT

$$\iint_{\bar{Z}} [ (v u_x)_y - (u v_y)_x ] dx dy$$

$$= \oint_C [ v u_x dx - u v_y dy ] = 0$$

$$= \int_{QM.} v u_x dx - \int_{MP} u v_y dy + \int_{\Gamma} (v u_x dx - u v_y dy)$$

$y=y_0$                        $x=x_0$

$$= v(x, y_0) u(x, y_0) \Big|_{y_0}^{x_0} - \int_{y_0}^{x_0} u(x, y_0) v_x(x, y_0) dx$$

$$- \int u(x_0, y) v_y(x_0, y) dy + \int_{\Gamma} (v u_x dx - u v_y dy) = 0$$

Let

$$v_x(x, y_0) = 0$$

$$x \leq x_0$$

$$v_y(x_0, y) = 0$$

$$y \leq y_0$$

$$v(x_0, y_0) = 1$$

$$u(x_0, y_0) - v(y_0, y_1) u(y_0, y_0)$$

$$= \int_{y_0}^{y_0} - \int_{x_0}^x (v u_x dx - u v_y dy)$$

$$u(x_0, y_0) = - \int_{x_0}^{y_0} v(x, x) u_x(x, x) dx + \int_{x_0}^{y_0} u(x, x) v_y dy$$

$$= - \int_{x_0}^{y_0} x^2 v(x, x) dx$$

solution of the Goursat problem

$$v_{xy} - \frac{2}{(x+y)^2} v = 0$$

$$v_x(x, y_0) = 0$$

$$v_y(x_0, y) = 0$$

$$v(x_0, y_0) = 0$$

$$\Rightarrow v(x, y, x_0, y_0) = \frac{(x_0 + y)(x + y_0) + (x_0 - x)(y_0 - y)}{(x + y)(x_0 + y_0)}$$

$$v(x, x) = \frac{x_0 y_0 + x^2}{x(x_0 + y_0)}$$

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$$u(x_0, y_0) = - \frac{1}{x_0 + y_0} \int_{x_0}^{y_0} x (x_0 y_0 + x^2) dx$$

$$= - \frac{1}{x_0 + y_0} \left[ \frac{1}{2} (y_0^2 - x_0^2) x_0 y_0 + \frac{1}{4} (y_0^4 - x_0^4) \right]$$

$$= - \frac{1}{4(x_0 + y_0)} (y_0^2 - x_0^2) [2x_0 y_0 + (x_0^2 + y_0^2)]$$

$$= - \frac{1}{4} (y_0 - x_0) (x_0 + y_0)^2$$

~~$$u(x, y) = - \frac{1}{4} (y - x) (x + y)^2$$~~

$$\Rightarrow u(x, y) = - \frac{1}{4} (y - x) (x + y)^2$$

Example : Using the Riemann method solve

$$u_{xy} + k u = g(x, y).$$

Initial curve  $\Gamma : x = y$

$$u(x, x) = h(x)$$

$$u_x(x, x) = f(x).$$

Solution : let

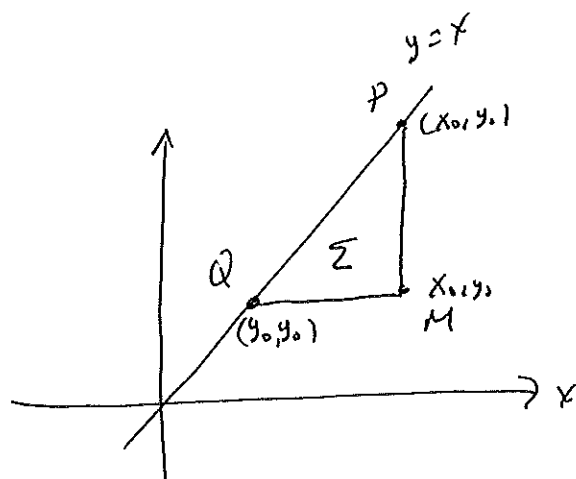
$$v_{xy} + k v = 0$$

$$v u_{xy} + k v u = v g$$

$$u v_{xy} + k u v = 0$$

$$v u_{xy} - u v_{xy} = g v.$$

$$(v u_x)_y - (u v_y)_x = g v.$$



Integrate the last equation in  $\bar{\Sigma}$

$$\iint_{\bar{\Sigma}} [(v u_x)_y - (u v_y)_x] dx dy = \iint_{\Sigma} g(x, y) v(x, y) dx dy$$

using the GT we get

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$$\oint_C v u_x dx - u v_y dy = \iint_{\bar{Z}} g v \cdot dx dy$$

Similar to what we did in the previous example  
we get

$$\begin{aligned} & -\cancel{u(x_0, y_0)} + \cancel{v(y_0, y_0)} \cancel{h(x)} \\ & v(x_0, y_0) u(x_0, y_0) - v(y_0, y_0) h(y_0) - \int_{y_0}^{x_0} u(x, y_0) v_x(x, y_0) dx \\ & - \int_{y_0}^{x_0} u(x_0, y) v_y(x_0, y) dy + \int_{\Gamma} = \iint_{\bar{Z}} g v \cdot dx dy \end{aligned}$$

we let

$$v_x(x, y_0) = 0$$

$$x \leq x_0$$

$$v_y(x_0, y) = 0$$

$$y \leq y_0$$

$$v(x_0, y_0) = 1$$

$$\text{and } \nabla u + h v = 0$$

$\Rightarrow$

$$u(x_0, y_0) = v(y_0, y_0) h(y_0) + \int_{y_0}^{x_0} + \iint_{\bar{I}} g v \, dx \, dy$$

$$\int_{\bar{I}} = \int_{y_0}^{x_0} v u_x \, dx - \int_{y_0}^{x_0} u v_y \, dy.$$

$$= \int_{y_0}^{x_0} v(x, x) u_x(x, x) \, dx - \int_{y_0}^{x_0} u(y, y) v_y(y, y) \, dy.$$

$$= \int_{y_0}^{x_0} p(x) v(x, x) \, dx.$$

$$u(x_0, y_0) = v(y_0, y_0) h(y_0) - \int_{y_0}^{x_0} p(x) v(x, x) \, dx + \iint_{\bar{I}} g v \, dx \, dy$$

Solution of the Goursat problem.

let

$$\xi = (x - x_0)(y - y_0).$$

then  $v$  satisfies

$$-\xi v_{\xi\xi} + v_{\xi} + h v = 0 \Rightarrow v = I_0(2/\sqrt{k} \xi)$$

This result we found in page 13

# ASSIGNED EXERCISES OF MATH544: PDE set 5 *March, 2001*

## THE METHOD OF RIEMANN

Ronald B Guenther and John W Lee, Partial Differential Equations of Mathematical Physics and Integral Equations

1. Solve the following initial value problem

$$u_{xy} - yu_x = 1, \quad y > -x, -\infty \leq x \leq \infty, \quad (1)$$

$$u(x, -x) = x, \quad u_x(x, -x) = x^2, \quad u_y(x, -x) = x^2 - 1, \quad (2)$$

$$\Gamma = \{x, y \in \mathbb{R}^2 | y = -x\} \quad (3)$$

**step 1.** Define the Riemann function  $v(x, y)$  which satisfies the homogeneous adjoint equation

$$v_{xy} + (yv)_x = 0$$

Using the equations for  $u$  and  $v$  we obtain

$$A_y + B_x = v$$

where

$$A = \frac{1}{2}(vu_x - uv_x), \quad B = \frac{1}{2}(vu_y - uv_y) - yuv$$

**step 2.** Integrate the above divergence form in a triangular region with the vertices  $M(x_0, y_0)$ ,  $P(-y_0, y_0)$ , and  $Q(x_0, -x_0)$  and using Green's theorem we get

$$\int_C [-Adx + Bdy] = \int \int_{T(x_0, y_0)} v(x, y) dx dy$$

where  $C = L_{QM}uL_{MP}uL_{PQ}$ . Evaluating each integral

$$\begin{aligned} \int_{QM} &= \int_{QM} Bdy = \frac{1}{2} \int_{QM} (vu_y - uv_y)|_{x=x_0} dy - \int_{QM} yuv|_{x=x_0} dy \\ &= \frac{1}{2}(uv)|_Q^M - \int_{QM} u(x_0, y) (v_y + yv) dy \end{aligned}$$

Since the last term contains an unknown value  $u(x_0, y)$  of the function  $u(x, y)$  we then let its coefficient to vanish.

$$v_y(x_0, y) + yv(x_0, y) = 0, \quad y \leq y_0 \quad (4)$$

Hence

$$\int_{QM} = \frac{1}{2}u(M)v(M) - \frac{1}{2}u(Q)v(Q)$$

Similarly we find that

$$\int_{MP} = - \int_{MP} A|_{y=y_0} dx = \frac{1}{2}u(M)v(M) - \frac{1}{2}u(P)v(P)$$

with

$$v_x(x, y_0) = 0, \quad x \leq x_0 \quad (5)$$

We then find that

$$u(M)v(M) = \frac{1}{2}u(P)v(P) + \frac{1}{2}u(Q)v(Q) - \int_{\Gamma} [-Adx + Bdy] + \int_{T(x_0, y_0)} v(x, y) dx dy \quad (6)$$

Here

$$\begin{aligned} u(M) &= u(x_0, y_0), \quad v(M) = v(x_0, y_0), \\ u(P) &= u(-y_0, y_0) = y_0, \quad v(P) = v(-y_0, y_0) \\ u(Q) &= u(x_0, -x_0) = x_0, \quad v(Q) = v(x_0, -x_0), \end{aligned}$$

**step 3.** Determination of the Riemann (Green) function: Let us collect all the conditions on the (Riemann) function  $v(x, y)$ . We usually use the notation  $v(x, y) = R(x_0, y_0, x, y)$

$$v_{xy}(x, y) + yv(x, y) = 0, \quad y > -x, x \in R, \quad (7)$$

$$v_y(x_0, y) + yv(x_0, y) = 0, \quad y \leq y_0, \quad (8)$$

$$v_x(x, y_0) = 0, \quad x \leq x_0, \quad (9)$$

$$v(x_0, y_0) = 1. \quad (10)$$

The complete solution of the above problem (known as the Goursat-Problem) is easy and given by

$$v(x, y) = e^{\frac{1}{2}(-y^2+y_0^2)}$$

**step 4.** Solution: Inserting the above result in the equation (6) we obtain that

$$u(x_0, y_0) = -\frac{1}{2}y_0 + \frac{1}{2}x_0 e^{(y_0^2-x_0^2)/2} + \frac{3}{2} \int_{-y_0}^{x_0} x^2 e^{(y_0^2-x^2)/2} dx \quad (11)$$

$$-\frac{1}{2} \int_{-x_0}^{y_0} e^{(y_0^2-y^2)/2} dy + \int_{T(x_0, y_0)} e^{(y_0^2-y^2)/2} dx dy, \quad (12)$$

$$= -\frac{1}{2}y_0 + \frac{1}{2}x_0 e^{(y_0^2-x_0^2)/2} + \int_{-y_0}^{x_0} \left[ \frac{3}{2}\xi^2 - \frac{1}{2} + x_0 - \xi \right] e^{(y_0^2-\xi^2)/2} d\xi \quad (13)$$

2. Solve the above problem without using Riemann's method.

3. Solve the initial value problem by the use of Riemann's method:

$$u_{xy} = F(x, y), \quad u(x, x) = f(x), \quad u_x(x, x) = g(x) \quad (14)$$

4. Solve the following initial value problem:

$$u_{xy} + xu_x = 1, \quad y > x, \quad x \in R, \quad (15)$$

$$u(x, -x) = u_x(x, -x) = 0. \quad (16)$$

5. Solve

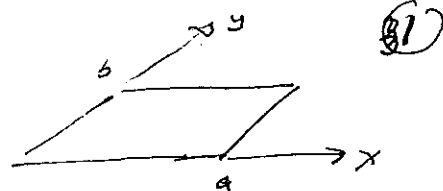
$$u_{xy} - u_x = xy, \quad (17)$$

$$u(x, x^3) = \sin x, \quad u_x(x, x^3) = \cos x, \quad u_y(x, x^3) = 0, \quad (18)$$

in the region above the curve  $\Gamma: y = x^3$ .

### 3. Hyperbolic Equations.

in two and higher dimensions.  
(A membrane with fixed edges)



$$u_{tt} - c^2 \nabla^2 u = 0 \quad 0 < x < a, \quad 0 < y < b, \quad t > 0$$

$$u(x, y, 0) = f(x, y), \quad 0 \leq x \leq a, \quad 0 \leq y \leq b.$$

$$u_t(x, y, 0) = g(x, y), \quad 0 \leq x \leq a, \quad 0 \leq y \leq b$$

$$u(x, 0, t) = 0, \quad u(x, b, t) = 0 \quad 0 \leq x \leq a, \quad t \geq 0$$

$$u(0, y, t) = 0, \quad u(a, y, t) = 0 \quad 0 \leq y \leq b, \quad t \geq 0$$

Separation of variables

$$u(x, y, t) = v(x, y) T(t).$$

We obtain

$$T'' + c^2 \lambda^2 T = 0 \quad t > 0$$

and

$$\nabla^2 v + \lambda^2 v = 0 \quad 0 < x < a, \quad 0 < y < b$$

$$v = 0 \quad \text{when} \quad x = 0, a, \quad y = 0, b$$

$\Rightarrow$  separation of variables wrt  $x$  and  $y$

$$\frac{x''}{x} = - \left( \frac{y''}{y} + \lambda^2 \right) = -\mu^2$$

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then

 $\mu > 0$ 

$$X'' + \mu^2 X = 0$$

$$X(0) = X(a) = 0$$

$$Y'' + \nu^2 Y = 0$$

$$Y(0) = Y(b) = 0$$

$$\mu = \mu_n = \frac{n\pi}{a} \Rightarrow X_n(x) = \sin(\mu_n x) \quad n=1,2,\dots$$

$$\nu = \nu_m = \frac{m\pi}{b} \Rightarrow Y_m(y) = \sin(\nu_m y), \quad m=1,2,\dots$$

$$\lambda^2 = \lambda_{mn}^2 = \mu_n^2 + \nu_m^2 = \pi^2 \left( \frac{n^2}{a^2} + \frac{m^2}{b^2} \right)$$

$$U = U_{nm} = \sin(\mu_n x) \sin(\nu_m y)$$

$$T = T_{mn}(t) = A_{mn} \cos(\lambda_{mn} t) + B_{mn} \sin(\lambda_{mn} t)$$

$$u(x,y,t) = \sum_{m,n=1}^{\infty} \left[ A_{mn} \cos(\lambda_{mn} t) + B_{mn} \sin(\lambda_{mn} t) \right] \sin \mu_n x \sin \nu_m y$$

$$u(x,y,0) = f(x,y) = \sum A_{mn} \sin \mu_n x \sin \nu_m y$$

$$u_t(x,y,0) = g(x,y) = \sum \lambda_{mn} B_{mn} \sin \mu_n x \sin \nu_m y$$

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x,y) \sin \mu_n x \sin \nu_m y \, dx \, dy$$

$$B_{mn} = \frac{4}{ab \lambda_{mn}} \int_0^a \int_0^b g(x,y) \sin \mu_n x \sin \nu_m y \, dx \, dy$$

Exercise: Under which conditions this formal solution is a solution of the IVP

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$$|\mu_n^2 A_{nn}| \leq \frac{A_1}{m^2 n^2}$$

$$|\nu_m^2 A_{mm}|^2 \leq \frac{A_2}{m^2 n^2}$$

$$|\lambda_{mn}^2 A_{mn}|^2 \leq \frac{B_1}{m^2 n^2}$$

$$f(0,0)=0$$

$$f(a,b)=f(a,y)=0$$

consider the following problem

$$u_{tt} = c^2 \nabla^2 u = c^2 \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right]$$

$$0 < r < a, \quad t > 0$$

$$u(r, \theta, 0) = 0, \quad u_t(r, \theta, 0) = f(r, \theta), \quad u(a, \theta, t) = 0$$

$(r, \theta)$  and  $(r, \theta + 2\pi)$  mark the same point.

$2\pi$  periodicity in  $-\infty < \theta < \infty$

$$u(r, \theta, t) = \varphi(r, \theta) T(t)$$

$$T'' + c^2 \lambda^2 T = 0, \quad T|_{t=0} = 0$$

$$\nabla^2 \varphi + \lambda^2 \varphi = 0 \quad \text{in } 0 \leq r < a$$

$$\varphi = 0 \quad \text{on } r = a, \quad \lambda > 0$$

$$\psi(r, \theta) = R(r) \Theta(\theta)$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial R}{\partial r} \right) + \lambda^2 + \frac{\Theta'}{r^2 \Theta} = 0$$

$$k = n^2 \quad n > \text{integer}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial R}{\partial r} \right) + \lambda^2 r^2 = - \frac{\Theta'}{\Theta} = k = n^2$$

$$R'' + \frac{1}{r} R' + \left( \lambda^2 - \frac{n^2}{r^2} \right) R = 0, \quad R(a) = 0$$

$$\Theta = \left( \frac{1}{2}, \cos n\theta, \sin n\theta \right).$$

$$R_n = A J_n(x) \quad r = \frac{x}{\lambda}, \quad R = y(x)$$

$$y'' + \frac{1}{x} y' + \left( 1 - \frac{n^2}{x^2} \right) y = 0$$

$$y = A J_n(x) + B N_n(x)$$

$n$ -th order  
Bessel function

$$N_n \rightarrow \infty \quad \text{as} \quad x \rightarrow 0$$

$$R(r) = A J_n(\lambda r) + B N_n(\lambda r)$$

$$r = a$$

$$0 = A J_n(\lambda a)$$

$$B = 0$$

$$\lambda a = \mu_{ni}$$

$$\mu_{n1} < \mu_{n2} < \dots < \mu_{ng} < \dots$$

$$\frac{1}{2} J_0(\lambda_0 r), J_n(\lambda_{nj} r) \cos n\theta, J_n(\lambda_{nj} r) \sin n\theta \quad (35)$$

$$T = \sin(c \lambda_{nj} t)$$

$$\Rightarrow u(r, \theta, t) = \sum_{j=1}^{\infty} \frac{1}{2} A_{0j} J_0(\lambda_{0j} r) \sin(c \lambda_{0j} t) \\ + \sum_{n,j=1}^{\infty} \left[ A_{nj} J_n(\lambda_{nj} r) \cos n\theta + B_{nj} J_n(\lambda_{nj} r) \sin n\theta \right] \sin(c \lambda_{nj} t)$$

$$u(r, \theta, t) = 0 \quad \checkmark \quad u(r, \theta, 0) = 0 \quad \checkmark$$

$$u_t(r, \theta, 0) = g(r, \theta)$$

$$\sum \frac{1}{2} A_{0j} J_0(\lambda_{0j} r) c \lambda_{0j}$$

$$+ \sum_{n,j=1}^{\infty} c \lambda_{nj} \left[ A_{nj} \underbrace{J_n(\lambda_{nj} r) \cos n\theta} + B_{nj} \underbrace{J_n(\lambda_{nj} r) \sin n\theta} \right]$$

$$c A_{nj} = \frac{1}{N_1} \iint_D g(r, \theta) J_n(\lambda_{nj} r) \cos n\theta \, r \, dr \, d\theta$$

$$c B_{nj} = \frac{1}{N_2} \iint_D g(r, \theta) J_n(\lambda_{nj} r) \sin n\theta \, r \, dr \, d\theta$$

$$\nabla^2 \psi + \lambda^2 \psi = 0 \quad x \in D,$$

$$\psi = 0 \quad x \in \partial$$

i)  $\lambda_1$  and  $\lambda_2$  two distinct eigen values

$$\nabla^2 \psi_1 + \lambda_1^2 \psi_1 = 0$$

$$\nabla^2 \psi_2 + \lambda_2^2 \psi_2 = 0$$

$$\int_D (\psi_2 \nabla^2 \psi_1 - \psi_1 \nabla^2 \psi_2) + (\lambda_1^2 - \lambda_2^2) \int_D \psi_1 \psi_2 dV = 0$$

"

$$\lambda_1 \neq \lambda_2 \quad \langle \psi_1, \psi_2 \rangle = 0$$

ii) Each eigen value  $\lambda_{nj}$   $n \neq 0$  has two eigen functions

$$\psi^1 = J_n(\lambda_{nj} r) \cos(n\theta), \quad \psi^2 = J_n(\lambda_{nj} r) \sin(n\theta)$$

prove that  $\langle \psi^1, \psi^2 \rangle$

any pair of distinct eigen functions are orthogonal.

## Uniqueness theory

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$$u_{tt} = c^2 \nabla^2 u + F(x, t) \quad (x, t) \in Q_T$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad x \in D.$$

$$u(x, t) = \phi(x, t) \quad (x, t) \in S_T$$

$D$  is a bounded domain in  $\mathbb{R}^3$   
and  $B$  is its boundary.

$$Q_T = \{ (x, t); x \in D, 0 < t \leq T \}.$$

its lateral surface

$$S_T = \{ (x, t); x \in B, 0 < t \leq T \}.$$

so called the parabolic boundary is the set

$$B_T = S_T \cup (\bar{D} \times \{0\}).$$

Consider

$$E(t) = \frac{1}{2} \int_D \left[ (w_t)^2 + c^2 (\nabla w)^2 \right] d^3x$$

$$w_{tt} = c^2 \nabla^2 w.$$

$$\begin{aligned} E' &= \int_D \left[ w_t c^2 (\nabla^2 w) + c^2 \nabla w_t \cdot \nabla w \right] d^3x \\ &= c^2 \int_D \nabla \cdot (w_t \nabla w) = 0 \end{aligned}$$

$$E(t) = E(0) \quad \forall t \quad \text{but } E(0) = 0$$

Theorem: The initial value problem has at most  
one solution in  $C^{2,2}(\bar{Q}_T) \cap C^{1,1}(\bar{Q}_T)$

$u_{tt} - c^2 \nabla^2 u = 0$  hyperbolic equation. of stationary

a) plane waves: let  $\xi = x \cdot \omega - ct$

let  $u(\bar{x}, t) = u(\xi)$

$$c^2 u'' - c^2 \bar{\omega} \cdot \bar{\omega} u'' = 0$$

i)  $\bar{\omega} \cdot \bar{\omega} = 1$  here  $c$  is the speed of propagation

b) spherical (or circular)

$$u_{tt} = c^2 \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right)$$

$$= c^2 \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r \left( \frac{\partial}{\partial r} (ru) \right) - u \right]$$

$$(ru)_{tt} = c^2 \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial}{\partial r} (u) - u \right] \quad u = ru$$

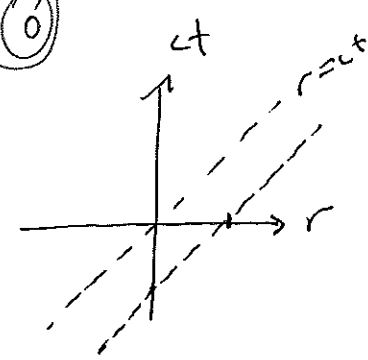
$$u_{tt} = c^2 u'' \Rightarrow v = f(r+ct) + g(r-ct)$$

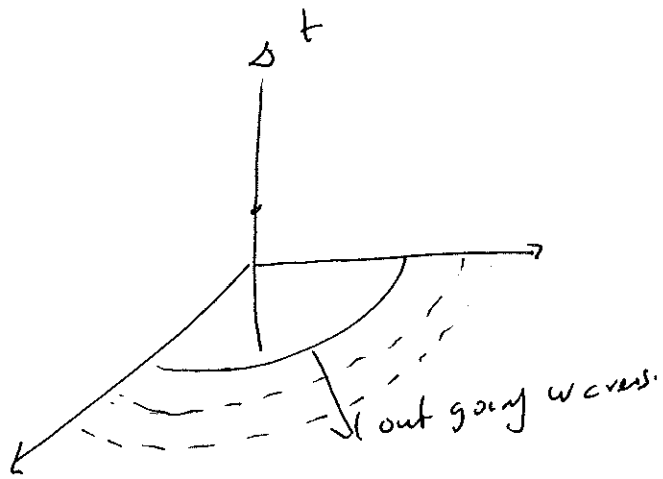
$$\Rightarrow u(r, t) = \frac{f(r+ct)}{r} + \frac{g(r-ct)}{r} \rightarrow \text{represent outgoing}$$

if  $g(r) = 1$  for  $r \leq 1$   
 $= 0$   $r > 1$

$$r-ct \leq 1$$

(3)





$$r - ct = 1$$

$$(r-1) = ct$$

$$cft =$$