

MATH 544 : Methods of Applied Mathematics II
Lecture 2 . Partial Differential Equation

Text book: "Partial Differential Equations
of Mathematical Physics" by
Ronald B Gunther and John
W Lee . (Prentice - Hall, 1988)

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Classification of Second Order Partial Differential Equations:

Let u be a twice differentiable function (having continuous second order partial derivatives) of the independent variables x and y . A second order PDE satisfied by u is in general has the following form

~~$$H(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$~~

$$H(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \quad (1)$$

where H is an arbitrary function of 8 variables $x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}$. We shall consider a special form of H . A Half Linear PDE

$$a(x, y) u_{xx} + 2b(x, y) u_{xy} + c(x, y) u_{yy} = f(x, y, u, u_x, u_y) \quad (2)$$

where a, b, c are twice continuously differentiable functions of x and y . To simplify (2) we look for a suitable coordinate system (ξ, η) . The new form is called the "canonical form" of the PDE

Let

$$x = \Phi(\xi, \eta) \quad , \quad y = \Psi(\xi, \eta)$$

where Φ and Ψ are some differentiable functions of x and y . in some open domain $U \subseteq \mathbb{R}^2$. let the Jacobian

$$J = \begin{vmatrix} \Phi_\xi & \Phi_\eta \\ \Psi_\xi & \Psi_\eta \end{vmatrix} \neq 0 \quad \forall \xi, \eta \in U$$

Then by the inverse function theorem we have

$$\xi = \phi(x, y) \quad , \quad \eta = \psi(x, y)$$

where ϕ and ψ are some functions having continuous second order partial derivatives with respect to x and y . Then we have $u(x, y) = \tau(\xi, \eta)$, and

$$u_x = \tau_\xi \phi_x + \tau_\eta \psi_x \quad ,$$

$$u_y = \tau_\xi \phi_y + \tau_\eta \psi_y \quad ,$$

$$u_{xx} = \tau_{\xi\xi} \phi_x^2 + 2\tau_{\xi\eta} \phi_x \psi_x + \tau_{\eta\eta} \psi_x^2 + \tau_\xi \phi_{xx} + \tau_\eta \psi_{xx}$$

$$u_{xy} = \tau_{\xi\xi} \phi_x \phi_y + \tau_{\xi\eta} (\phi_x \psi_y + \phi_y \psi_x) + \tau_{\eta\eta} \psi_x \psi_y + \tau_{\xi\eta} \phi_{xy} + \tau_{\eta\eta} \psi_{xy}$$

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$$u_{yy} = \zeta_{\xi\xi} \phi_y^2 + 2\zeta_{\xi\eta} \phi_y \psi_y + \zeta_{\eta\eta} \psi_y^2 + \zeta_\xi \phi_{yy} + \zeta_\eta \psi_{yy}$$

Hence the LHS of (2) becomes

$$A \zeta_{\xi\xi} + 2B \zeta_{\xi\eta} + C \zeta_{\eta\eta} = F(\xi, \eta, \zeta, \zeta_\xi, \zeta_\eta) \quad (3)$$

where

$$A = a \phi_x^2 + 2b \phi_x \phi_y + c \phi_y^2$$

$$C = a \psi_x^2 + 2b \psi_x \psi_y + c \psi_y^2$$

$$B = a \phi_x \psi_y + b(\phi_x \psi_y + \phi_y \psi_x) + c \psi_x \psi_y$$

Lemma. let $\Delta = b^2 - ac$, $\tilde{\Delta} = B^2 - AC$, then

$$\tilde{\Delta} = J^2 \Delta \quad \text{where } J \text{ is the Jacobian of the transformation and } J \neq 0 \quad \forall (x, y) \in U \subseteq \mathbb{R}^2.$$

Hence, as a consequence of this lemma is

$$\text{Sign}(\Delta) = \text{Sign}(\tilde{\Delta}).$$

An invariant of the transformation. Using this invariant we can make a classification of the PDE of second order. according to the sign of $\Delta = b^2 - ac$.

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a) Hyperbolic type PDEs: $\Delta > 0$. When this condition is satisfied we can find the roots of the following quadratic equation

$$a\lambda^2 + 2b\lambda + c = 0$$

The roots λ_1 and λ_2 are real and distinct.

Hence by choosing

$$\lambda_1 = \frac{\phi_x}{\phi_y}, \quad \lambda_2 = \frac{\psi_x}{\psi_y}$$

we can make $A = C = 0$. Hence we obtain the canonical form of the corresponding PDE

$$\zeta_\xi \eta = F/2B, \quad B \neq 0$$

The equations

$$\phi_x - \lambda_1 \phi_y = 0, \quad \psi_x - \lambda_2 \psi_y = 0$$

determine ξ and η respectively. As an illustration we solve $\phi_x - \lambda_1(x, y) \phi_y = 0$ by using the Lagrange method

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$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{d\phi}{0}$$

$$\Rightarrow v_1 = \phi = c_1 \text{ (constant)}$$

$$\text{Soln of } dy + \lambda_1 dx = dv_2 = 0$$

$$v_2 = \text{const}$$

$\Rightarrow$ simplest choice of $v_2(x, y)$ is taken as ϕ because the general soln of the above eqn is

$$f(\phi, v_2) = 0$$

where f is an arbitrary funcn.

Example: $u_{xx} - \kappa^2 u_{yy} = 0$

$$a=1, b=0, c=-\kappa^2 \Rightarrow \Delta = b^2 - ac = \kappa^2$$

$$\text{Hence taking } U = \{ (x, y) \in \mathbb{R}^2 \mid \kappa \neq 0 \}$$

Then the type of PDE is hyperbolic. Then

$$\lambda^2 - \kappa^2 = 0 \Rightarrow \lambda_1 = \kappa, \lambda_2 = -\kappa$$

$$\frac{\phi_x}{\phi_y} = \kappa \Rightarrow \phi_x - \kappa \phi_y = 0$$

$$\frac{dx}{1} = -\frac{dy}{\kappa} = \frac{d\phi}{0} \Rightarrow \underline{v_1 = y + \frac{1}{2}\kappa^2 x^2 = \phi = \xi}$$

$$\frac{\psi_x}{\psi_y} = -\kappa \Rightarrow \psi_x + \kappa \psi_y = 0$$

$$\Rightarrow \underline{\psi = y - \frac{1}{2}\kappa^2 x^2}$$

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$$\xi = y + \frac{1}{2}x^2, \quad \eta = y - \frac{1}{2}x^2$$

$$u_x = \eta_{\xi} x - \zeta_{\eta} x.$$

$$\begin{aligned} u_{xx} &= x [\zeta_{\xi\xi} x - \zeta_{\xi\eta} x] - x [\zeta_{\eta\xi} x - \zeta_{\eta\eta} x] \\ &= x^2 \zeta_{\xi\xi} + x^2 \zeta_{\eta\eta} - 2x^2 \zeta_{\xi\eta} + \zeta_{\xi} - \zeta_{\eta}. \end{aligned}$$

$$u_y = \zeta_{\xi} + \zeta_{\eta}.$$

$$u_{yy} = \zeta_{\xi\xi} + \zeta_{\eta\eta} + 2\zeta_{\xi\eta}$$

$$\begin{aligned} u_{xx} - x^2 u_{yy} &= x^2 [\zeta_{\xi\xi} + \zeta_{\eta\eta} - 2\zeta_{\xi\eta} - \zeta_{\xi\xi} - \zeta_{\eta\eta} - 2\zeta_{\xi\eta}] \\ &\quad + \zeta_{\xi} - \zeta_{\eta} \\ &= -4x^2 \zeta_{\xi\eta} + \zeta_{\xi} - \zeta_{\eta} = 0 \end{aligned}$$

$$\Rightarrow \zeta_{\xi\eta} = \frac{\zeta_{\eta} - \zeta_{\xi}}{4x^2} = \frac{\zeta_{\eta} - \zeta_{\xi}}{4(\xi - \eta)}$$

"Canonical form of the given PDE"

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b) Parabolic case: $\Delta = b^2 - ac = 0$

In this case the roots of the quadratic equation

$$a\lambda^2 + 2b\lambda + c = 0$$

are equal, $\lambda_1 = \lambda_2 = -\frac{b}{a}$.

Hence we can choose one of the coefficients A or C to be zero. Let us choose

$$\lambda = -\frac{b}{a} = \frac{\phi_x}{\phi_y} \quad (3)$$

 $\Rightarrow A = 0$, since $\tilde{\Delta} = 0$ and $A = 0 \Rightarrow$ $B = 0$ in parabolic type. Then thecanonical form is

$$\zeta_{\eta\eta} = F/\zeta, \quad \zeta \neq 0$$

From the above eqn. (3) we find ξ . To find η we can make a guess so that $\zeta \neq 0$.Example: $u_{xx} + 2u_{xy} + u_{yy} = 0$

$$a=1, b=1, c=1 \Rightarrow \lambda^2 + 2\lambda + 1 = 0$$

$$\Rightarrow \lambda = -1. \text{ let}$$

$$\frac{\phi_x}{\phi_y} = -1 \Rightarrow \phi_x + \phi_y = 0$$

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This equation has a simple solution

$$\xi = x - y$$

choose

$$\eta = x + y$$

Then

$$u_x = \zeta_\xi + \zeta_\eta$$

$$u_{xx} = \zeta_{\xi\xi} + \zeta_{\eta\eta} + 2\zeta_{\xi\eta}$$

$$u_y = -\zeta_\xi + \zeta_\eta$$

$$u_{yy} = \zeta_{\xi\xi} - 2\zeta_{\xi\eta} + \zeta_{\eta\eta}$$

$$u_{xy} = -\zeta_{\xi\xi} - \zeta_{\xi\eta} + \zeta_{\xi\eta} + \zeta_{\eta\eta} = -\zeta_{\xi\xi} + \zeta_{\eta\eta}$$

$$\begin{aligned} u_{xx} + 2u_{xy} + u_{yy} &= \zeta_{\xi\xi} + \zeta_{\eta\eta} + 2\zeta_{\xi\eta} \\ &\quad - 2\zeta_{\xi\xi} + 2\zeta_{\eta\eta} \\ &\quad + \zeta_{\xi\xi} - 2\zeta_{\xi\eta} + \zeta_{\eta\eta} = 4\zeta_{\eta\eta} = 0 \end{aligned}$$

Hence $\zeta(\xi, \eta) = f(\xi)\eta + g(\xi)$ and

$$u(x, y) = f(x-y)(x+y) + g(x-y)$$

where f and g are arbitrary functions.

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c) $\Delta < 0$: The quadratic equation

$$a\lambda^2 + 2b\lambda + c = 0$$

has complex roots ξ and $\eta = \xi^*$

letting $\alpha = \frac{1}{2}(\xi + \eta)$, $\beta = \frac{i}{2}(\eta - \xi)$

or $\xi = \alpha + i\beta$, $\eta = \xi^* = \alpha - i\beta$.

The canonical form for hyperbolic type of equation $\zeta_{\xi\eta}$ becomes.

$$\frac{\partial^2 \zeta}{\partial \xi \partial \eta} = \frac{1}{4} \left(\frac{\partial^2 \zeta}{\partial \alpha^2} + \frac{\partial^2 \zeta}{\partial \beta^2} \right)$$

The R.H.S is the canonical form of the "elliptic type" of PDEs

$$u_{\alpha\alpha} + u_{\beta\beta} = F(\alpha, \beta, u, u_\alpha, u_\beta)$$

Example: $u_{xx} + x^2 u_{yy} = 0$

$$a=1, b=0, c=x^2 \Rightarrow \lambda^2 + x^2 = 0$$

$$\lambda_1 = ix, \quad \lambda_2 = -ix$$

$$\Rightarrow \xi = iy + \frac{x^2}{2}, \quad \eta = iy - \frac{x^2}{2}$$

$$\alpha = \frac{1}{2}x^2, \quad \beta = y$$

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$$u_x = \zeta_\alpha x + \zeta_\beta(0) = \zeta_\alpha x$$

$$u_{xx} = x^2 \zeta_{\alpha\alpha} + \zeta_\alpha$$

$$u_y = \zeta_\beta, \quad u_{yy} = \zeta_{\beta\beta}$$

$\Rightarrow$

$$u_{xx} + x^2 u_{yy} = x^2 \zeta_{\alpha\alpha} + \zeta_\alpha + x^2 \zeta_{\beta\beta} = 0$$

$$\zeta_{\alpha\alpha} + \zeta_{\beta\beta} = -\frac{\zeta_\alpha}{x^2} = -\frac{\zeta_\alpha}{2\alpha}$$

"Canonical form of the PDE"

Other examples (i) $x^2 u_{xy} - y u_{yy} + u_x - 4u = 0$

$$a=0, \quad c=-y, \quad b=\frac{1}{2}x^2$$

Since $a=0$, we write A explicitly

$$a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = x^2\phi_x\phi_y - y\phi_y^2 = 0$$

$$\Rightarrow \phi_y (x^2\phi_x - y\phi_y) = 0$$

Two solutions

$$i) \quad \phi_y = 0 \Rightarrow \phi(x, y) = \phi(x)$$

Take $y = x$

$$ii) \quad x^2\phi_x - y\phi_y = 0$$

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$$\frac{dx}{x^2} = \frac{dy}{-y} = \frac{d\phi}{0}$$

$$-\frac{1}{x} + \ln y = \text{const}$$

$$\Rightarrow \xi = \frac{1}{x} - \ln y$$

$$\eta = x$$

$$u_x = \zeta_{\xi} \left(-\frac{1}{x^2}\right) + \zeta_{\eta}$$

$$u_{xx} = \frac{2}{x^3} \zeta_{\xi} + \frac{1}{x^2} \left[-\zeta_{\xi\xi} \frac{1}{x^2} + \zeta_{\xi\eta} \right] \\ + \left[-\zeta_{\eta\xi} \frac{1}{x^2} + \zeta_{\eta\eta} \right]$$

$$u_{xy} = -\frac{1}{x^2} \left[\zeta_{\xi\xi} \left(-\frac{1}{y}\right) + \zeta_{\xi\eta} (0) \right] + \zeta_{\eta\xi} \left(-\frac{1}{y}\right) \\ = \frac{1}{x^2 y} \zeta_{\xi\xi} - \frac{1}{y} \zeta_{\xi\eta}$$

$$u_{xy} = \zeta_{\xi} \left(-\frac{1}{y}\right), \quad u_{yy} = \frac{1}{y^2} \zeta_{\xi\xi} + \frac{1}{y^2} \zeta_{\xi}$$

$$x^2 u_{xy} - y u_{yy} + u_x - 4u = \frac{1}{y} \zeta_{\xi\xi} - \frac{x^2}{y} \zeta_{\xi\eta} \\ - \frac{1}{y} \zeta_{\xi\xi} - \frac{1}{y} \zeta_{\xi} - \frac{1}{x^2} \zeta_{\xi} + \zeta_{\eta} \\ - 4u = 0$$

$$\frac{x^2}{y} \zeta_{\xi\eta} = -\left(\frac{1}{y} + \frac{1}{x^2}\right) \zeta_{\xi} + \zeta_{\eta} - 4\zeta.$$

$$x=y, \quad e^{\xi-\frac{1}{y}} = 1/y. \quad \text{Canonical form of the PDE}$$

$$(ii) \quad y^2 u_{xx} - u_{yy} = 0,$$

$$a=y^2, \quad b=0, \quad c=-1 \Rightarrow \Delta \geq y^2$$

$$U = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$$

we find that

$$\xi = x + \frac{1}{2}y^2, \quad \eta = x - \frac{1}{2}y^2$$

$\Rightarrow$ the canonical form of the equation is

$$4(\xi - \eta) \zeta_{\xi\eta} + \zeta_{\eta} - \zeta_{\xi} = 0$$

$$\text{or} \quad \zeta_{\xi\eta} = \frac{\zeta_{\xi} - \zeta_{\eta}}{4(\xi - \eta)}$$

The most general form

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

or with respect to the standard notation

$$p = u_x, \quad q = u_y, \quad r = u_{xx}, \quad s = u_{xy}, \quad t = u_{yy}$$

we have

$$F(x, y, u, p, q, r, s, t) = 0$$

Special cases:

a) Quasi-Linear PDEs

$$A(x, y, u, p, q) r + 2B(x, y, u, p, q) s + C(x, y, u, p, q) t = f(x, y, u, p, q)$$

b) Half-Linear

$$A(x, y) r + 2B(x, y) s + C(x, y) t = f(x, y, u, p, q)$$

c) Linear $f = f(x, y) u + g(x, y)$

Characteristic curves

Let $a(x,y)$, $b(x,y)$, and $c(x,y)$ be continuous functions in a domain $D \subseteq \mathbb{R}^2$, $f(x,y,u,u_x,u_y)$ be a function of five variables x,y,u,u_x,u_y . Let a half linear Pde be given by

$$a(x,y)r + 2b(x,y)s + c(x,y)t = f(x,y,u,u_x,u_y)$$

with the Cauchy data on a curve

$$\gamma = \{ (x,y) \in \mathbb{R}^2 \mid x = x(t), y = y(t), t \in I \subseteq \mathbb{R} \}$$

$$u = u_0(t) = u_0(x(t), y(t)) \quad \text{on } \gamma$$

$$\text{in } \mathbb{R}^3 : \quad C = \{ (x,y,z) \in \mathbb{R}^3 \mid x = x(t), y = y(t), z = u_0(t), t \in I \subseteq \mathbb{R} \}$$

where

$$\pi: C \rightarrow \gamma$$

Here π is the perpendicular projection along z -axis on the xy -plane. The Cauchy data is (u_0, p_0, q_0) on γ but one of them is not independent.

Two is enough.

$$u'_0(t) = p_0 x' + q_0 y' \quad (*)$$

$$(dz = p dx + q dy \quad \text{everywhere in } D)$$

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Let $(x_0, y_0) \in \gamma$ be a point on the initial curve at $t = t_0$. In a local neighborhood of this point let us expand $u(x, y)$ as Taylor's series

$$\begin{aligned} u(x, y) = & u(x_0, y_0) + (x - x_0) p_0 + (y - y_0) q_0 \\ & + \frac{1}{2} (x - x_0)^2 r_0 + (x - x_0)(y - y_0) s_0 + \frac{1}{2} (y - y_0)^2 t_0 \\ & + \dots \end{aligned}$$

To have this expansion possible we have to find all Taylor coefficients

$$u_0, p_0, q_0, r_0, s_0, t_0, \dots$$

at the point (x_0, y_0) on γ . We know

u_0, p_0, q_0 (Two of them are given we calculate third from (*)). To calculate r_0, s_0, t_0 we use

$$p'_0 = r_0 x' + s_0 y'$$

$$q'_0 = s_0 x' + t_0 y'$$

and writing the pde on γ

$$a r_0 + 2b s_0 + c t_0 = f_0$$

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We find assuming $y' \neq 0$

$$s_0 = \frac{p_0' - r_0}{y'}, \quad t_0 = \frac{q_0' - s_0 x'}{y'} = \frac{q_0'}{y'} - \frac{p_0' x'}{y'^2} + r_0 \frac{x'^2}{y'^2}$$

Inserting these into the pde on γ

$$a r_0 + 2b \frac{p_0' - r_0 x'}{y'} + c \left(\frac{q_0'}{y'} - \frac{p_0' x'}{y'^2} + r_0 \frac{x'^2}{y'^2} \right) = f_0$$

or

$$[a y'^2 - 2b x' y' + c x'^2] r_0 = f_0 y'^2 - c q_0' y' + c p_0' x' - 2b p_0' y' \quad (**)$$

Two distinct cases:

a) If $a y'^2 - 2b x' y' + c x'^2 \neq 0$ then r_0, s_0, t_0 can be calculated uniquely. It is straightforward to find higher order Taylor coefficients

$$u_{xxx}|_{\gamma}, u_{xy}|_{\gamma}, u_{xyy}|_{\gamma}, u_{yyy}|_0, \dots$$

uniquely. Hence $u(x, y)$ is analytic in the neighborhood of (x_0, y_0) on the initial curve. The solution exists and unique

b) If $ay'^2 - 2bx'y' + cx'^2 = 0$. We have two subcases

b₁) If the RHS of (**) is not equal to zero we get a contradiction, hence the solution does not exist.

b₂) If the RHS of (**) is equal to zero identically the solution exists but not unique. There exist infinitely many solutions. The reason is that in (**) r_0 is left arbitrary. We obtain all other Taylor coefficients in terms of r_0 which is arbitrary. Hence solution exists but contains a free parameter r_0 . For each value of r_0 we have a solution. This is equivalent to saying that there exist infinitely many solutions.

The condition $ay'^2 - 2bx'y' + cx'^2 = 0$ is indeed the characteristic curve of the Pde. Let γ be defined by $\phi(x, y) = c$ (constant), hence

$$\phi_x x' + \phi_y y' = 0$$

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Since $y' \neq 0 \Rightarrow x' = \frac{\phi_y}{\phi_x} y'$

Then the condition becomes

$$a \phi_x^2 + 2b \phi_x \phi_y + c \phi_y^2 = 0.$$

($A=0$) in the classification of 2nd order half linear pdes. This equation kills one of the coefficients of the transformed pde, $A=0$. Such curves are called the "Characteristic curves" of the pde. Hyperbolic type of pdes have two, parabolic type of pdes have one characteristic curve. The above analysis tells us that if the Cauchy data is given on the characteristic curve of the pdes either there exists no solution or there are infinitely many solutions.

Example : $u_{xx} + 2u_{xy} - 3u_{yy} = 0$

$$\gamma: x = \tau, y = 0, \quad -a < \tau < a.$$

$$u(\tau, 0) = h(\tau).$$

$$u_n = u_y = \sigma(\tau) \quad (u_n = \langle n, \text{grad } u \rangle).$$

Solution of DE :

$$(D + 3D')(D - D')u = 0$$

$$\Rightarrow u(x, y) = F(3x - y) + G(x + y)$$

where F and G are arbitrary functions. We shall determine these functions from the Cauchy data h and σ .

$$u(\tau, 0) = F(3\tau) + G(\tau) = h(\tau).$$

$$u_y(\tau, 0) = -F_{\xi} \Big|_{y=0} + G_{\tau} = \sigma, \quad \xi = 3x - y$$

$$= -\frac{1}{3} F_{\tau} + G_{\tau} = \sigma.$$

$$\Rightarrow -\frac{1}{3} F(3\tau) + G(\tau) = \beta(\tau) = \int^{\tau} \sigma(\tau') d\tau'$$

$$\Rightarrow G(\tau) = \frac{1}{4} (3\beta(\tau) + h(\tau))$$

$$F(3\tau) = \frac{3}{4} (h(\tau) - \beta(\tau))$$

Hence

$$\begin{aligned}
 u(x,y) &= \frac{1}{4} \left[h\left(\frac{3x-y}{3}\right) + 3h(x+y) \right] \\
 &\quad + \frac{1}{4} \left[\beta(x+y) - \beta\left(\frac{3x-y}{3}\right) \right] \\
 &= \frac{1}{4} \left[h\left(\frac{3x-y}{3}\right) + 3h(x+y) \right] \\
 &\quad + \frac{1}{4} \int_{\frac{3x-y}{3}}^{x+y} \sigma(c') dc'
 \end{aligned}$$

unique solution:

Example: The same equation with different data.

$$\gamma: x=\tau, y=3\tau$$

This is a characteristic curve of the equation

$$a=b=1, c=-3$$

$$ay'^2 - 2bx'y' + cx'^2 = y'^2 - 2x'y' - 3x'^2 = 0$$

$$(y' + x')(y' - 3x') = 0$$

$$i) x' + y' = 0 \Rightarrow \gamma: x=\tau, y=-\tau.$$

$$ii) y' - 3x' = 0 \Rightarrow \gamma: x=\tau, y=3\tau. \quad \checkmark$$

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on γ

$$u(z, 3z) = h(z)$$

$$u_n(z, 3z) = H(z)$$

$$\hat{t} = \frac{1}{\sqrt{10}} (1, 3) \quad , \quad \hat{n} = \frac{1}{\sqrt{10}} (-3, 1)$$

$$u_n = \hat{n} \cdot \text{grad } u = \frac{1}{\sqrt{10}} (-3u_x + u_y)$$

$$u(x, y) = F(3x-y) + G(x+y)$$

$$i) \quad u(z, 3z) = F(0) + G(4z) = h(z)$$

$$ii) \quad u_n|_{\gamma} = \frac{1}{\sqrt{10}} [-3(3F_{\xi} + G_{\eta}) - F_{\xi} + G_{\eta}]|_{\gamma}$$

$$\xi = 3x-y, \quad \eta = x+y$$

$$\frac{1}{\sqrt{10}} [-10 F_{\xi}(0) - 2 G_{\eta}(4z)] = H(z)$$

$$-10 F_{\xi}(0) - 2 G_{\eta}(4z) = \sqrt{10} H(z)$$

$$F(0) + G(4z) = h(z)$$

$$\frac{d}{dz} G(4z) = h' \Rightarrow 4 G_{\eta}(4z) = h'$$

$$-10 F_{\xi}(0) - \frac{1}{2} h'(z) = \sqrt{10} H(z)$$

$$\text{If} \quad h''(z) = -2\sqrt{10} H'(z)$$

There exist solutions but infinitely many

$$u(x,y) = F(3x-y) - F(0) + h\left(\frac{x+y}{4}\right).$$

where F is arbitrary. If

$$h'' \neq -2\sqrt{10} h'(\tau)$$

There exist no solution.

• Another way of obtaining this condition.

$$\gamma: x=\tau, y=3\tau, u=h(\tau).$$

$$u_n = \frac{1}{\sqrt{10}} (-3p_0 + q_0) = H(\tau).$$

$$u' = p_0 + 3q_0 = h'$$

$$-3p_0 + q_0 = \sqrt{10} H(\tau).$$

$$p_0 = \frac{1}{10} [h' - 3\sqrt{10} H(\tau)]$$

$$q_0 = \frac{1}{10} [3h' + \sqrt{10} H(\tau)]$$

$$p_0' = r_0 + 3s_0 = \frac{1}{10} [h'' - 3\sqrt{10} H']$$

$$q_0' = s_0 + 3t_0 = \frac{1}{10} [3h'' + \sqrt{10} H']$$

$$r_0 + 2s_0 + 3t_0 = 0$$

$$r_0 = -3s_0 + \frac{1}{10} [h'' - 3\sqrt{10} H']$$

$$t_0 = -\frac{1}{3}s_0 + \frac{1}{30} [3h'' + \sqrt{10} H']$$

$$-3s_0 + \frac{1}{10} [h'' - 3\sqrt{10} H'] + 2s_0$$

$$+ s_0 - \frac{1}{10} [3h'' + \sqrt{10} H'] = 0$$

$$h'' - 3\sqrt{10} H' - 3h'' - \sqrt{10} H' = 0$$

$$-2h'' - 4\sqrt{10} H' = 0$$

$$\text{or } \underline{h'' = -2\sqrt{10} H'}$$

Classification of higher dimensional PDEs.

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$$\sum_{i,j=1}^n A_{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^n B_i \frac{\partial u}{\partial x^i} + C u = D$$

Where A_{ij}, B_i ($i, j = 1, 2, \dots, n$), C and D are functions of the independent variables x^k , ($k = 1, 2, \dots, n$). When $n = 2$ we have the matrix A ,

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Consider the eigenvalues of A .

$$\det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ b & c - \lambda \end{vmatrix}$$

$$= (a - \lambda)(c - \lambda) - b^2$$

$$= ac - b^2 - \lambda(a + c) + \lambda^2$$

$$\lambda_1, \lambda_2 = \frac{(a + c) \pm \sqrt{(a + c)^2 - 4(ac - b^2)}}{2}$$

$$= \frac{(a + c) \pm \sqrt{(a - c)^2 + 4b^2}}{2}$$

$$\lambda_1 \cdot \lambda_2 = ac - b^2 = -\Delta, \quad \lambda_1 + \lambda_2 = a + c$$

i) $\Delta > 0$: Hyperbolic type
 $\lambda_1 \cdot \lambda_2 < 0$
 $\text{Sign}(\lambda_1) = -\text{Sign}(\lambda_2)$

ii) $\Delta = 0$ parabolic
 $\lambda_1 \cdot \lambda_2 = 0$
 One of the eigenvalues is zero

iii) $\Delta < 0$ elliptic type
 $\lambda_1 \cdot \lambda_2 > 0$
 $\text{Sign}(\lambda_1) = \text{Sign}(\lambda_2)$

Hence it is possible to classify the PDE with respect to the eigenvalues of the matrix A (coefficient matrix of the second derivatives).
 Eigenvalues of A are found from.

$$\det(A - \lambda I) = 0$$

Since $A^T = A$ then the eigenvalues are all real numbers. Under the transformations we have the following numbers are conserved.

of positive eigenvalues.

of negative eigenvalues.

and hence

Signature = # positive eigenvalue - # of negative eigenvalues.

Definition: Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A .

a) At a point $P(x^1, x^2, \dots, x^n)$ if all eigenvalues are different from zero have the same sign then the PDE is elliptic at P .

b) At a point $P(x^1, x^2, \dots, x^n)$ eigenvalues of A are all different from zero and all, except one, have the same sign then the PDE is called hyperbolic at P .

c) At a point $P(x^1, x^2, \dots, x^n)$ eigenvalues of A are all different from zero and if at least two of them has different sign then the others then PDE is called "ultra hyperbolic" at P .

(27)

d) If at a point $P(x^1, x^2, \dots, x^n)$ any one of the eigenvalue is zero then the PDE is called "parabolic" at P .

If a PDE, in a region $\Omega \subset \mathbb{R}^n$ at all points, is hyperbolic, elliptic, parabolic, ... we say that the PDE is hyperbolic, elliptic, parabolic in Ω .

Hence if we identify the coefficient matrix A we find transformation which reduces A to its diagonal form

$$\xi_i = \sum_{j=1}^n b_{ij} x_j \quad \xi = b x$$

$$\frac{\partial}{\partial x_i} = \sum_{k=1}^n \frac{\partial \xi_k}{\partial x_i} \cdot \frac{\partial}{\partial \xi_k} = \sum_{k=1}^n b_{ki} \frac{\partial}{\partial \xi_k}$$

$\Rightarrow$

$$\sum_{i,j} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{i,j} A_{ij} \left(\sum_{k=1}^n b_{ki} \frac{\partial}{\partial \xi_k} \right) \left(\sum_{m=1}^n b_{mj} \frac{\partial}{\partial \xi_m} \right) u$$

(28)

$$= \sum_{i,j} \sum_k \sum_m b_{ki} A_{ij} b_{mj} \frac{\partial^2 \mathcal{U}}{\partial \xi_k \partial \xi_m}$$

$$\text{let } \sum_k \sum_m b_{ki} A_{ij} b_{mj} = (A_d)_{ij}$$

$$\Rightarrow \sum_{k,j} A_{kj} \frac{\partial^2 \mathcal{U}}{\partial x_i \partial x_j} = \sum_{k,m} (A_d)_{km} \frac{\partial^2 \mathcal{U}}{\partial \xi^k \partial \xi^m}$$

$$(A_d)_{km} = (b A b^T)_{km}$$

where b^T is the transpose of b . and A_d is the diagonal of A

$$A_d = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \dots & \lambda_n \end{pmatrix}$$

Hence

i) Find b mapping $A \rightarrow A_d$

$$A_d = b A b^T$$

ii) use $\xi = b x$ then find the

new equation. Here b is an orthogonal matrix (A is symmetric) $\Rightarrow b^T = b^{-1}$

(29)

Example $u_{xx} + 2u_{xy} + 2u_{yy} + 2u_{yz} + 3u_z - u = 0$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$b = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

$$b A b^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\lambda_1 = \lambda_2 = 1$$

$$\lambda_3 = -1$$

hyperbolic type.

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\xi_1 = x^1, \quad \xi_2 = -x^1 + x^2, \quad \xi_3 = -x^1 + x^2 + x^3$$

$$\Rightarrow u_{\xi_1 \xi_1} + u_{\xi_2 \xi_2} - u_{\xi_3 \xi_3} + 3u_{\xi_3} - u = 0$$

Hyperbolic type in three dimensions

$$u_{xx} + 3u_{yy} + 84u_{2z} + 23u_{yz} + 16u_{2x} + 2u_{xy}$$

$$\rightarrow u_{x'x'} + 2u_{y'y'} + 2u_{z'z'}$$

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CHARACTERISTICS

[2.8

If we put $\delta_1 = \alpha - \beta$, $\delta_2 = \alpha + \beta$, this becomes

$$\begin{aligned} & 4\alpha^2 - 4\beta^2 + 4\alpha\delta_3 \\ &= (2\alpha + \delta_3)^2 - 4\beta^2 - \delta_3^2 \\ &= (\delta_1 + \delta_2 + \delta_3)^2 - (\delta_1 - \delta_2)^2 - \delta_3^2 \\ &= \delta_1'^2 - \delta_2'^2 - \delta_3'^2, \end{aligned}$$

where

$$\delta_1' = \delta_1 + \delta_2 + \delta_3, \quad \delta_2' = \delta_1 - \delta_2, \quad \delta_3' = \delta_3.$$

Then

$$x = x' + y', \quad y = x' - y', \quad z = x' + z'.$$

In terms of these new variables,

$$L(u) = \frac{\partial^2 u}{\partial x'^2} - \frac{\partial^2 u}{\partial y'^2} - \frac{\partial^2 u}{\partial z'^2}.$$

Exercises

1. Find the types of the following differential equations, and reduce them to normal form:

- (i) $y^2 r - t = 0$,
- (ii) $y^2 r + 2ys + t = p$,
- (iii) $y^2 r + t = 0$,
- (iv) $r - 2s + 3t - q - u = 0$,
- (v) $(1 + x^2)^2 r - t = 0$,
- (vi) $x^2 r - 2xys + y^2 t = 0$,
- (vii) $x^2 r + 2xys + y^2 t = 0$.

2. Reduce the following equations to normal form:

$$\begin{aligned} \text{(i)} \quad & \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial y^2} + 3 \frac{\partial^2 u}{\partial z^2} + 4 \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + 2 \frac{\partial^2 u}{\partial x \partial z} + 2 \frac{\partial^2 u}{\partial x \partial t} \\ & + 4 \frac{\partial^2 u}{\partial y \partial z} + 4 \frac{\partial^2 u}{\partial y \partial t} + 6 \frac{\partial^2 u}{\partial z \partial t} = 0. \end{aligned}$$

$$\text{(ii)} \quad \frac{\partial x^2}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + 2 \frac{\partial^2 u}{\partial x \partial z} + 2 \frac{\partial^2 u}{\partial x \partial t} + 2 \frac{\partial^2 u}{\partial z \partial t} = 0.$$

$$\text{(iii)} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 u}{\partial x \partial t} = 0.$$

3. Prove that $q^2 r - 2pqs + p^2 t = 0$

has an intermediate integral $p = qf(u)$, where f is an arbitrary function. Hence show that

$$y + xf(u) = g(u),$$

where g is a second arbitrary function.

CHARACTERISTICS

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4. Transform the equation

$$x^2 \frac{\partial^2 u}{\partial x^2} = y^2 \frac{\partial^2 u}{\partial y^2}$$

into one with characteristic variables. Hence show that

$$u = f(xy) + xg\left(\frac{y}{x}\right),$$

where f and g are arbitrary functions.

5. Find the characteristics of $r - yt = 0$. Reduce this equation to normal form when $y > 0$.

6. Find the characteristics of $(1+x^2)r - (1+y^2)t = 0$, and reduce the equation to normal form.

7. Prove that $x^2r + 2xys + y^2t = 0$ has an intermediate integral

$$px + qy - u + f\left(\frac{y}{x}\right) = 0,$$

where f is an arbitrary function. Hence show that

$$u = f\left(\frac{y}{x}\right) + xg\left(\frac{y}{x}\right).$$

where g is also an arbitrary function.

8. Show that $qr + (uq - p)s - upt = 0$ has an intermediate integral

$$p + qu = f(u),$$

where f is an arbitrary function. Hence solve the equation.

9. Prove that the two families of characteristic strips of

$$s = F(x, y, u, p, q)$$

are given by

$$(i) \quad dx = 0, \quad du - qdy = 0, \quad dp - Fdy = 0,$$

and

$$(ii) \quad dy = 0, \quad du - pdx = 0, \quad dq - Fdx = 0.$$

Hence solve the equation $s = pq$.

Well posed Problems

If a boundary value problem satisfies the following three conditions:

i) There exists a solution

ii) The solution is unique

iii) The solution is stable; i.e., the

solution depends continuously on the data, then this boundary value problem is called a "wellposed", otherwise it is an ill-posed problem

Example: $u_{tt} + u_{xx} = 0$, $t > 0$, $x \in \mathbb{R}$

a) zero initial conditions: $u(x, 0) = 0$, $u_t(x, 0) = 0$, $x \in \mathbb{R}$. It is not difficult to show that $u(x, t) = 0$ $\forall t > 0$ and $\forall x \in \mathbb{R}$

b) Small initial data: $u(x, 0) = 0$, $u_t(x, 0) = \varepsilon \sin(\frac{x}{\varepsilon})$

where ε is sufficiently small. This represents a small change in the initial data. The solution of this initial value problem is

$$u(x, t) = \varepsilon^2 \sin(\frac{x}{\varepsilon}) \sinh(t/\varepsilon)$$

A small change in the initial data leads to an arbitrarily large (exponentially) solution as $t \rightarrow \infty$ (takes larger values). This problem is not well posed.

$$i) \text{ as } t \rightarrow \infty \quad u(x,t) \sim \varepsilon^2 e^{t/\varepsilon}$$

$$ii) \text{ as } \varepsilon \rightarrow 0 \quad |\sin(x/\varepsilon)| < 1 \Rightarrow$$

$$|u(x,t)| \rightarrow \varepsilon^2 e^{t/\varepsilon}$$

Hence the solution is unbounded for all $t > 0$

iii) At $\varepsilon = 0$ we have zero data. For zero data, from part (a), we should have zero solution but in (ii) we get an unbounded solution, a contradiction.

Hence this is an ill-posed problem.

Wave Equation and generalizations:

R. B. Guenther and John W. Lee, "Partial Differential Equations of Mathematical Physics and Integral Equations".

1) Wave equation and d'Alembert's solution

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$(A) \quad u(x, 0) = f(x), \quad x \in \mathbb{R}$$

$$u_t(x, 0) = g(x), \quad x \in \mathbb{R}$$

The solution of this initial value problem is

$$u(x, t) = \frac{1}{2} [f(x+ct) + f(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(x') dx'$$

where f and g are given twice differentiable functions. This solution is known as the d'Alembert solution. It is clear that the initial value problem given above is, hence, a wellposed problem.

~~Wave equation in a finite interval with homogeneous boundary conditions.~~

proof:

It is clear that, if f is twice differentiable and g is differentiable then solution exists and unique which is clear from the d'Alembert solution.

stability of the solution: Dependence of the solution continuously on the data.

let (f_1, g_1) and (f_2, g_2) be the data for u_1 and u_2 . let

$$|f_1 - f_2| < \varepsilon, \quad |g_1 - g_2| < \varepsilon$$

$\forall x \in [0, L]$ and ε is a small positive real number. Then using the d'Alembert solution

$$\begin{aligned} u_1(x, t) - u_2(x, t) &= \frac{1}{2} (f_1(x+ct) - f_2(x+ct)) \\ &\quad + \frac{1}{2} (f_1(x-ct) - f_2(x-ct)) \\ &\quad + \frac{1}{2c} \int_{x-ct}^{x+ct} (g_1(x') - g_2(x')) dx' \\ |u_1 - u_2| &< \varepsilon + \frac{1}{2c} \int_{x-ct}^{x+ct} |g_1 - g_2| dx' = \varepsilon + \varepsilon T. \\ &\quad \forall t \leq T. \end{aligned}$$

$$\Rightarrow |u_1 - u_2| \leq \varepsilon(1+T)$$

$|u_1 - u_2|$ can be made as close together as desired over any finite time interval by choosing the initial data for these two solutions close enough. In other words the solution is "stable"

Theorem: Let $f(x)$ be twice differentiable and $g(x)$ be differentiable for $x \in \mathbb{R}$. Then the initial value problem ^(A) is well posed.

Solution exists and unique and stable.

This initial value problem was on the whole real line ($x \in \mathbb{R}$). Let us check whether the initial value problem on a finite interval is well posed or not.

2) Wave equation in a finite interval with homogeneous boundary conditions

Using the orthogonality of the functions
 $\sin(\lambda_n x)$, $\cos(\lambda_n x)$, $n=1, 2, \dots$

we get

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{2}{c\lambda_n L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

with $\lambda_n = \frac{n\pi}{L}$, $n=1, 2, \dots$ we have the
 formal solution

$$u(x, t) = \sum_{n=1}^{\infty} [a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t)] \sin(\lambda_n x)$$

For the existence of the solution, the above
 formal solution should be at least twice
 differentiable. This requires the uniform convergence
 of the above infinite sum. Assuming the uniform convergence

$$u_{xx} = - \sum' (a_n \lambda_n^2 \cos(\lambda_n t) + b_n \lambda_n^2 \sin(\lambda_n t)) \sin \lambda_n x$$

$$\left| \sum' (a_n \lambda_n^2 \cos(\lambda_n t) + b_n \lambda_n^2 \sin(\lambda_n t)) \sin \lambda_n x \right|$$

$$\leq \sum \lambda_n^2 (|a_n| + |b_n|)$$

$$u_{tt} - c^2 u_{xx} = 0, \quad t \geq 0, \quad 0 \leq x \leq L$$

$$(B) \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq L$$

$$u(0, t) = u(L, t) = 0, \quad t \geq 0.$$

Solution: By using the separation of variables and superposition principle

$$u(x, t) = \sum_{n=1}^N u_n(x, t)$$

where

$$u_n(x, t) = [a_n \cos(\lambda_n c t) + b_n \sin(\lambda_n c t)] \sin(\lambda_n x)$$

$$n = 1, 2, \dots, N, \quad \lambda_n = \frac{n\pi}{L}$$

$$\forall x \in [0, L], \quad t \geq 0$$

$$u(x, 0) = \sum_{n=1}^N a_n \sin(\lambda_n x)$$

$$u_t(x, 0) = \sum_{n=1}^N c b_n \lambda_n \sin(\lambda_n x)$$

Letting $N \rightarrow \infty$ then we have

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x), \quad 0 \leq x \leq L$$

$$g(x) = \sum_{n=1}^{\infty} b_n c \lambda_n \sin(\lambda_n x), \quad 0 \leq x \leq L$$

Hence for uniform convergence we must have

(39)

$$|\lambda_n^2 a_n| \leq \frac{C_1}{n^2}, \quad |\lambda_n^2 b_n| \leq \frac{C_2}{n^2}$$

Where C_1 and C_2 are some positive real numbers. Then, if a_n and b_n satisfy the above condition then the "formal" solution is really the solution of the initial value problem. We can differentiate, at least twice, and hence satisfy the wave equation. The above conditions on a_n and b_n bring condition on the Cauchy data f and g . Using integration by part four times we get

$$a_n = \frac{2}{L} \left[-\frac{L}{n\pi} f(x) \cos(\lambda_n x) + \frac{1}{\lambda_n} \int_0^L f'(x) \cos \lambda_n x dx \right]$$

$$f(0) = f(L) = 0 \quad (1)$$

$$a_n = \frac{2}{L \lambda_n} \int_0^L f'(x) \cos \lambda_n x dx$$

$$= \frac{2}{L \lambda_n} \left[\frac{1}{\lambda_n} \sin(\lambda_n x) f'(x) - \frac{1}{\lambda_n} \int_0^L f'' \sin \lambda_n x dx \right]$$

$$= -\frac{2}{L \lambda_n^2} \int_0^L f'' \sin \lambda_n x dx$$

No new condition

(10)

$$a_n = -\frac{2}{L\lambda_n^2} \left[-\frac{1}{\lambda_n} \cos \lambda_n x f''(x) \right]_0^L + \frac{1}{\lambda_n} \int_0^L f'''(x) \cos \lambda_n x dx$$

Let

$$f''(0) = f''(L) = 0 \quad (2)$$

$$a_n = -\frac{2}{L\lambda_n^3} \int_0^L f'''(x) \cos \lambda_n x dx$$

$$= -\frac{2}{L\lambda_n^3} \left[\cancel{\frac{1}{\lambda_n} \sin \lambda_n x f'''(x)} \right]_0^L - \frac{1}{\lambda_n} \int_0^L f^{(iv)}(x) \sin \lambda_n x dx$$

$$a_n = \frac{2}{L\lambda_n^4} \int_0^L f^{(iv)}(x) \sin(\lambda_n x) dx$$

$$\lambda_n^2 |a_n| \leq \frac{2}{L\lambda_n^2} \int_0^L |f^{(iv)}(x)| dx = \frac{2L}{\pi^2} \cdot \frac{1}{n^2} \cdot \int_0^L |f^{(iv)}(x)| dx$$

$$C_1 = \frac{2L}{\pi^2} \int_0^L |f^{(iv)}(x)| dx$$

$f(0) = f(L) = 0$, $f''(0) = f''(L) = 0$ and $f^{(iv)}$
is integrable in $(0, L)$

Similarly

$$b_n = \frac{2}{c \lambda_n L} \int_0^L g(x) \sin(\lambda_n x) dx$$

$$= \frac{2}{c L \lambda_n} \left[-\frac{1}{\lambda_n} g(x) \cos(\lambda_n x) \Big|_0^L + \frac{1}{\lambda_n} \int_0^L g'(x) \cos(\lambda_n x) dx \right]$$

$$g(0) = g(L) = 0$$

$$b_n = \frac{2}{c L \lambda_n^2} \int_0^L g'(x) \cos(\lambda_n x) dx$$

$$= \frac{2}{c L \lambda_n^2} \left[\frac{1}{\lambda_n} \sin \lambda_n x g'(x) \Big|_0^L - \frac{1}{\lambda_n} \int_0^L g'' \sin \lambda_n x dx \right]$$

$$= -\frac{2}{c L \lambda_n^3} \int_0^L g'' \sin \lambda_n x dx$$

$$= -\frac{2}{c L \lambda_n^3} \left[-\frac{1}{\lambda_n} g'' \cos \lambda_n x \Big|_0^L + \frac{1}{\lambda_n} \int_0^L g'''(x) \cos \lambda_n x dx \right]$$

~~$g'(0) = g'(L) = 0$~~

$$= -\frac{2}{c L \lambda_n^4} \left[\int_0^L g'''(x) \cos \lambda_n x dx - g'' \cos \lambda_n x \Big|_0^L \right]$$

$$\lambda_n^2 |b_n| \leq \frac{2L}{c \pi^2} \frac{1}{n^2} \left[\int_0^L |g'''(x)| dx + |g''(0)| + |g''(L)| \right]$$

$$\Rightarrow C_2 = \frac{2L}{c \pi^2} \left[\int_0^L |g'''| dx + |g''(0)| + |g''(L)| \right]$$

$g(0) = g(L) = 0$, ~~$g''(0) = g''(L) = 0$~~ and g''' is integrable in $(0, L)$. (42)

With there a_n and b_n the infinite series of the RHS of the following equations are also uniformly convergent

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x)$$

$$g(x) = \sum b_n c_n \sin(\lambda_n x)$$

$$a_n \sim \frac{1}{n^4}, \quad b_n \sim \frac{1}{n^4}, \quad \lambda_n b_n \sim \frac{1}{n^3}.$$

Both of these ~~equ~~ series in the RHS are uniformly convergent.

Theorem. Let $f(x)$ have a continuous fourth derivative (or fourth derivative of f is integrable in $(0, L)$) on $0 \leq x \leq L$, $g(x)$ have a continuous third derivative on $0 \leq x \leq L$ (or third derivative of g is integrable in $(0, L)$), and $f(0) = f(L) = f''(0) = f''(L) = 0$, $g(0) = g(L) = 0$.

Then the boundary value problem B for the wave equation has a unique solution and

it is given by

$$u(x,t) = \sum' (a_n \sin(\lambda_n t) + b_n \cos(\lambda_n t)) \sin(\lambda_n x)$$

with

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \lambda_n x \, dx$$

$n=1,2,\dots$

$$b_n = \frac{2}{c \lambda_n L} \int_0^L g(x) \sin \lambda_n x \, dx$$

Theorem: The initial value problem (B) is well posed.

proof: let

$u_1(x,t)$ corresponds to data (f_1, g_1)

$u_2(x,t)$ " " " (f_2, g_2)

$$|u_1(x,t) - u_2(x,t)| \leq \sum_{n=1}^{\infty} (|a_n^1 - a_n^2| + |b_n^1 - b_n^2|)$$

for the convergence of the infinite sum (from page 39)

$$|a_n^1 - a_n^2| < \frac{2}{L \lambda_n^2} \int_0^L |f_1'' - f_2''| \, dx$$

$$|b_n^1 - b_n^2| < \frac{2}{c L \lambda_n^2} \int_0^L |g_1'(x) - g_2'(x)| \, dx$$

$$|u_1 - u_2| < (\tilde{C}_1 + \tilde{C}_2) \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} (\tilde{C}_1 + \tilde{C}_2)$$

where

$$\tilde{C}_1 = \frac{2L}{\pi^2} \int_0^L |f_1'' - f_2''| dx$$

$$\tilde{C}_2 = \frac{2L}{c\pi^2} \int_0^L |g_1' - g_2'| dx$$

or

$$|u_1 - u_2| < \frac{L}{3} \left(\int_0^L |f_1'' - f_2''| dx + \frac{1}{c} \int_0^L |g_1' - g_2'| dx \right)$$

This tells us that a small change in the data corresponds to the small change in the solution, i.e., letting

$$|f_1'' - f_2''| < \varepsilon, \quad |g_1' - g_2'| < \varepsilon c \quad \forall x \in [0, L]$$

$$\Rightarrow |u_1 - u_2| < \frac{2L}{3} \varepsilon$$

For the proof of the stability theorem is as follows

(45)

Let the initial data f and g
in the initial boundary value problem (B)
satisfying $f(0)=f(1)=f'(0)=f'(1)=0$ and
 $g(0)=g(1)$, f is twice differentiable and
 $g(x)$ is differentiable in $(0,1)$ then the
initial and boundary value problem (B) is stable

Exercise

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$$(1) \quad u_{tt} = c^2 u_{xx} \quad 0 < x < L, \quad t > 0$$

$$u(0, t) = u_x(L, t) = 0 \quad t > 0$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x)$$

$$0 \leq x \leq L$$

$$(2) \quad u_{tt} = c^2 u_{xx} \quad 0 < x < L, \quad t > 0$$

$$u_x(0, t) = u_x(L, t) = 0, \quad t > 0$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x)$$

$$0 \leq x \leq L$$

4.3 Inhomogeneous Initial, BV problem

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$$u_{tt} + k u_t = c^2 u_{xx} + F(x, t)$$

vibration of a damped string

k : damping constant

F : external force (density)

an appropriate initial, boundary value problem for the damped wave equation is

$$u_{tt} + k u_t = c^2 u_{xx} + F(x, t), \quad 0 < x < L, \quad t > 0$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x) \quad 0 \leq x \leq L$$

$$u(0, t) = r(t), \quad u(L, t) = s(t) \quad t \geq 0$$

compatibility conditions

$$f(0) = r(0), \quad f(L) = s(0)$$

$$g(0) = r'(0), \quad g(L) = s'(L)$$

We expect that this problem is
well-posed.

Uniqueness: Let u_1 and u_2 both solve

the above initial + BV problem. Define $u = u_1 - u_2 \Rightarrow$

$$u = u_1 - u_2 \quad u_{tt} + k u_t = c^2 u_{xx} \quad 0 < x < L, \quad t > 0$$

$$u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad 0 \leq x \leq L$$

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0.$$

$$k > 0$$

Theorem: There exists at most one solution

to the ^{above} inhomogeneous initial + BV problem which is the trivial solution.

proof: let

$$E(t) = \frac{1}{2} \int_0^L (\rho_0 u_t^2 + \tau u_x^2) dx, \quad E(0) = 0$$

where $\tau = \rho_0 c^2$, eqn.

$$\rho_0 u_{tt} + \rho_0 k u_t = \tau u_{xx}$$

$$\dot{E} = \int_0^L [\cancel{\rho_0} u_t (\tau u_{xx} - \rho_0 k u_t) + \tau u_x u_{xt}] dx$$

$$= \int_0^L [\tau (u_{xx} u_t + u_x u_{xt}) - \rho_0 k u_t^2] dx$$

$$= \tau u_x u_t \Big|_0^L - \rho_0 k \int_0^L u_t^2 dx$$

$$\frac{d\bar{E}}{dt} = -\rho_0 k \int_0^L u_t^2 dx \leq 0$$

$$\frac{d\bar{E}}{dt} \leq 0 \Rightarrow E(t) \leq E(0)$$

$$\forall t \in [0, \infty] \quad \text{but}$$

$$E(t) > 0$$

$$\text{Contradiction} \Rightarrow E(t) = 0 \quad \forall t \geq 0$$

$$u_t = 0, \quad u_x = 0 \quad \forall x, t$$

$$u(x, t) = 0 \quad \forall (x, t)$$

Hence, initial value problem with non vanishing data has a unique solution.

(10)

let's consider inhomogeneous eqn. with
zero data

$$u_t + c^2 u_{xx} + F(x,t) = 0 \quad 0 < x < L, \quad t > 0$$

$$u(x,0) = 0, \quad u_t(x,0) = 0 \quad 0 < x < L$$

$$u(0,t) = 0, \quad u(L,t) = 0 \quad t > 0$$

Series expansion:

$$u(x,t) = \sum u_n(t) \sin(\lambda_n x), \quad \lambda_n = \frac{n\pi}{L}$$

We assume

$$F(x,t) = \sum F_n(t) \sin(\lambda_n x) \quad 0 \leq x \leq L$$

$$F_n(t) = \frac{2}{L} \int_0^L F(x,t) \sin \lambda_n x \, dx$$

$$\Rightarrow \quad u_n''(t) + c^2 \lambda_n^2 u_n(t) = f_n(t)$$

with initial conditions

$$u_n(0) = 0, \quad u_n'(0) = 0$$

Solution :

$$u_n(t) = \frac{1}{c\lambda_n} \int_0^t f_n(\tau) \sin[c\lambda_n(t-\tau)] d\tau$$

"Formal solution"

Theorem . Let $F(x,t)$ have continuous third order partial derivatives for $0 \leq x \leq L$ and $t > 0$. Assume that $F(0,t) = F(L,t) = 0$
 $F_{xx}(0,t) = F_{xx}(L,t) = 0$ for all $t > 0$
Then the inhomogeneous, initial + BV
has a unique solution given by ---
where $\lambda_n = \frac{n\pi}{L}$.

HOMEWORK II (MATH 544): *February 22, 2012*
(For March 5, 2012)

1. Obtain the Euler-Lagrange equation and the associated natural boundary conditions for the problem $\delta J = 0$ where

$$J(y) = \int_a^b L(x, y, y') dx - \beta y(b) + \alpha y(a)$$

Here α and β are arbitrary constants and $y(a)$ and $y(b)$ are not prescribed.

2. Find the shortest distance between the line $y = x$ and the parabola $y^2 = x - 1$.

3. If b is not prescribed (moving end point), show that the stationary functions corresponding to the problem $\delta J = 0$ where

$$J(y) = \int_0^b [(y')^2 + 4(y - b)] dx$$

with $y(0) = 2$ and $y(b) = b^2$ are of the form $y = x^2 - 2(x/b) + 2$, where b is one of the real roots of the equation $2b^4 - 2b^3 - 1 = 0$.

4. Obtain the stationary functions of the problem $\delta J = 0$ where

$$J(y) = \int_0^\pi [\lambda^2 (y')^2 - w^2 y^2] dx$$

with $y(0) = 0$, where λ and w are constants. Determine whether J has minimum or maximum value for these stationary functions.

Solution of the 4. th problem

$$J(y) = \int_0^{\pi} (\lambda^2 y'^2 - \omega^2 y^2) dx$$

$$J(y_0 + \varepsilon h) = \int_0^{\pi} [\lambda^2 (y'_0 + \varepsilon h')^2 - \omega^2 (y_0 + \varepsilon h)^2] dx$$

$$\frac{d^2 J}{d\varepsilon^2} = 2 \int_0^{\pi} (\lambda^2 h'^2 - \omega^2 h^2) dx \quad h(0) = 0$$

use the Schwarz field u so that

$$\frac{u''}{u} = -\frac{\omega^2}{\lambda^2} \quad u(0) = u'(\pi) = 0$$

$\Rightarrow$

$$\frac{d^2 J}{d\varepsilon^2} = 2\lambda^2 \int_0^{\pi} \left[h'^2 + \frac{u''}{u} h^2 \right] dx$$

$$= 2\lambda^2 \int_0^{\pi} \left(h'^2 + \underbrace{\left(\frac{u'}{u} h^2 \right)'}_{+\frac{u''}{u^2} h^2} - 2 \frac{u'}{u} h h' \right) dx$$

$$= 2\lambda^2 \int_0^{\pi} \left(h'^2 - 2 \frac{u'}{u} h h' + \frac{u''}{u^2} h^2 \right) dx$$

$$= 2\lambda^2 \int_0^{\pi} u^2 \left(\left(\frac{h}{u} \right)' \right)^2 dx > 0$$

Hence J has a local minimum at $y = y_0(x)$