

MATH 544 : Methods of Applied Mathematics II
lecture Notes I

Spring 2012

Introduction:

(2)

Necessary condition for extrema

- Derivative of a differentiable funcn at x_0 is defined by

$$f'(x_0) = \lim_{\varepsilon \rightarrow 0} \frac{f(x_0 + \varepsilon) - f(x_0)}{\varepsilon}$$

provided that the limit exists.

- We shall try to give a similar definition for functionals. For this purpose we need the "closeness" notion in A or distance between two neighbouring points in A . This leads us to normed linear spaces

A linear space is a normed space if it satisfies some further conditions. Let $\| \cdot \|$ define the norm in a linear space V , then $\| \cdot \| : V \rightarrow \mathbb{R}$ with

- i) $\|y\| = 0 \iff y = 0$
 - ii) $\|\alpha y\| = |\alpha| \|y\|$
 - iii) $\|y_1 + y_2\| \leq \|y_1\| + \|y_2\|$
- $\forall y \in V, \alpha \in \mathbb{R}$
 $\forall y_1, y_2 \in V$

$f(x_0 + \varepsilon), f(x)$
 $\forall x$ in U_δ
def
 $|x - x_0| < \delta$



(18)

a norm on a linear space puts a geometrical structure on V

Examples:

1) $V = \mathbb{R}^n$, $\|y\| = \sqrt{y_1^2 + y_2^2 + \dots + y_n^2}$
 $y = (y_1, y_2, \dots, y_n)$

2) A norm on a linear space is NOT unique
For an example we can give another norm

$\|y\| = \max \{|y_1|, |y_2|, \dots, |y_n|\}$ also defines
a norm on $\mathbb{R}^n$

3) $V = C[a, b]$ is a normed linear space
if we define

$$\|y\|_M = \max_{x \in [a, b]} |y(x)|$$

4) $\|y\|_1 = \int_a^b |y(x)| dx$ defines also a
norm on $C[a, b]$

5) $V = C^1[a, b]$ then

$$\|y\|_W = \max_{x \in [a, b]} |y| + \max_{x \in [a, b]} |y'|$$

called a weak norm. It plays an important
role in calculus of variations. This norm
is also defined in $C^1[a, b]$ $n \geq 2$

Distance between the points of A . Let $y_1, y_2 \in V$.
 The distance between y_1 and y_2 is given by

$$\|y_1 - y_2\|$$

as an example

$$i) \|y_1 - y_2\|_M = \max_{x \in [a, b]} |y_1(x) - y_2(x)|$$

$$ii) \|y_1 - y_2\|_W = \max_{x \in [a, b]} |y_1(x) - y_2(x)|$$

$$+ \max_{x \in [a, b]} |y_1'(x) - y_2'(x)|$$

Two functions y_1 and $y_2 \in V$ are close
 to each other if $\|y_1 - y_2\|$ is smaller

Derivates of Functionals

$J: A \rightarrow \mathbb{R}$ with a norm $\|\cdot\|$

Then J has a local minimum at $y_0 \in A$
 provided $J(y) \geq J(y_0)$ for all $y \in A$
 with $\|y - y_0\| < d$ for some d .

A : ^{set} subspace of a normed linear space V .
 $y_0 \in A$, increment of y_0 , let $h \in A$
 $\Rightarrow y_0 + \varepsilon h$, ε is a real number (sufficiently
 small) $\delta y_0 = \varepsilon h$ increment of y_0 or variation of y_0

Total change of J .

$$\Delta J = J(y_0 + \varepsilon h) - J(y_0).$$

Definition The first variation of a functional J at y_0 in the direction of h is defined as

$$\delta J(y_0, h) = \left. \frac{d}{d\varepsilon} J(y_0 + \varepsilon h) \right|_{\varepsilon=0}$$

provided that the derivative exists.
(admissible variation of J at y_0).

Necessary condition: $J: A \rightarrow \mathbb{R}$, here A is a subset of a normed linear space V . Let $y_0 \in A$ provide a local minimum for J relative to a norm $\| \cdot \|$ in V . We expect that

$$\delta J(y_0) = 0$$

or

$$\left. \frac{d}{d\varepsilon} J(y_0 + \varepsilon h) \right|_{\varepsilon=0} = 0$$

Calculus of Variations

①

Theorem: Let $J: A \rightarrow \mathbb{R}$ be a given functional, $A \subseteq V$, where V is a normed linear space. If $y_0 \in A$ provides a local minimum value for the functional J relative to the norm $\| \cdot \|$, then $\delta J(y_0, h) = 0$ (first variation of J) for all admissible functions h (variation of y , $\delta y = h$).

Example: Consider

$$J(y) = \int_0^1 (1 + y'^2) dx$$

where $y \in C^1[0,1]$ with $y(0) = 0$, $y(1) = 1$. Let $y_0(x) = x$. We shall show that $y_0(x)$ is the function which makes the functional minimum.

All admissible functions $y(x) = y_0(x) + \varepsilon h(x)$ where $h(0) = h(1) = 0$. Then

$$\begin{aligned} J[y_0 + \varepsilon h(x)] &= \int_0^1 [1 + (1 + \varepsilon h')^2] dx \\ &= \int_0^1 (2 + 2\varepsilon h' + \varepsilon^2 h'^2) dx \\ &= 2 + 2\varepsilon (h(1) - h(0)) + \varepsilon^2 \int_0^1 (h')^2 dx \end{aligned}$$

$$\Rightarrow J(y_0 + \varepsilon h(x)) - J(y_0) = \varepsilon^2 \int_0^1 (h')^2 dx$$

for all admissible functions $h(x)$, $x \in [0,1]$.

Hence

$$J(y_0 + \varepsilon h(x)) \geq J(y_0)$$

for all $h(x)$ and $x \in [0, 1]$. Then $J(y_0)$ is the minimum value of the functional. If we check the Theorem.

$$\frac{J(y_0 + \varepsilon h) - J(y_0)}{\varepsilon} = \varepsilon \left(\int_0^1 (h')^2 dx \right)$$

Then

$$\lim_{\varepsilon \rightarrow 0} \frac{J(y_0 + \varepsilon h) - J(y_0)}{\varepsilon} = 0, \quad \forall h$$

$$\begin{aligned} \text{or} \quad \delta J &= \left. \frac{d}{d\varepsilon} J(y_0 + \varepsilon h) \right|_{\varepsilon=0} = 0. \\ &= \delta J(y_0, h) \end{aligned}$$

Remark: Solutions to $\delta J(y_0, h) = 0$ are called (local) extremals or stationary points and are the candidates for maximum and minimum for J . We say that J is stationary at y_0 in the direction of h .

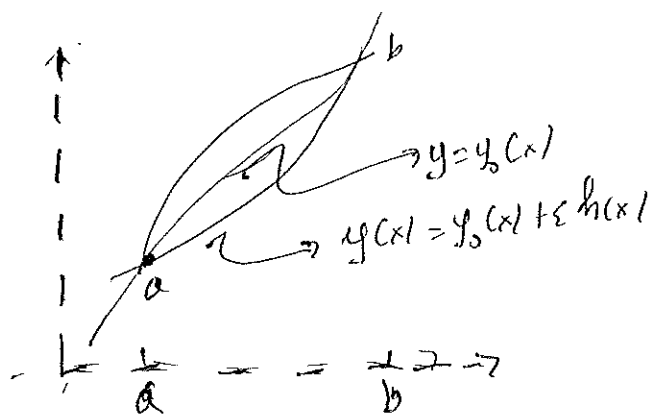
The Euler Equation

1) L containing x, y, y' : Let

$$J(y) = \int_a^b L(x, y, y') dx$$

where $y \in C^2[a, b]$ with $y_0(a) = y_0$, $y(b) = y_1$.

L is called "the Lagrange function", a twice differentiable function on $[a, b] \times \mathbb{R}^2$, and $\|\cdot\|$ is some norm in $C^2[a, b]$. We shall now check necessary condition for a local extremum of J with fixed end points.



Let $y_0(x)$ be the function corresponding to the extremum of J . All admissible functions (curves) are $y(x) = y_0(x) + \epsilon h(x)$

where $h(a) = h(b) = 0$ (fixed end points).
Omitting the subscript "0" we take

$$J(y + \epsilon h) = \int_a^b L(x, y + \epsilon h, y' + \epsilon h') dx$$

Since the necessary condition is $\left. \frac{d}{d\epsilon} J(y + \epsilon h) \right|_{\epsilon=0} = 0$ then we first find

$$\frac{dJ}{d\epsilon}(y + \epsilon h) = \int_a^b \frac{\partial L}{\partial \epsilon} dx$$

but

$$\frac{\partial L}{\partial \bar{y}} = \frac{\partial L}{\partial y} h + \frac{\partial L}{\partial y'} h'$$

where $\bar{y}(x) = y(x) + \varepsilon h(x)$. Then

$$\left. \frac{d}{d\varepsilon} J(y + \varepsilon h) \right|_{\varepsilon=0} = \int_a^b \left. \frac{\partial L}{\partial \varepsilon} \right|_{\varepsilon=0} dx = 0$$

on the other hand

$$\left. \frac{\partial L}{\partial \varepsilon} \right|_{\varepsilon=0} = \frac{\partial L}{\partial y} h + \frac{\partial L}{\partial y'} h'$$

Then the necessary condition reduces to

$$\int_a^b \left(\frac{\partial L}{\partial y} h + \frac{\partial L}{\partial y'} h' \right) dx = 0 \quad \text{for all admissible } h(x)$$

We can write the second term as

$$\frac{\partial L}{\partial y'} h' = \frac{d}{dx} \left(\frac{\partial L}{\partial y'} h \right) - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) h$$

Then

$$\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) \right] h dx + \int_a^b \frac{d}{dx} \left(\frac{\partial L}{\partial y'} h(x) \right) dx = 0$$

(5)

or

$$\int_a^b E(L) h(x) dx + \left. \frac{\partial L}{\partial y'} \right|_{x=b} h(b) - \left. \frac{\partial L}{\partial y'} \right|_{x=a} h(a) = 0$$

since $h(a) = h(b) = 0$ then we obtain

$$\int_a^b E(L) h(x) dx = 0$$

where

$$E(L) = \frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right)$$

Here L is called the "Euler operator"

$$E = \frac{\partial}{\partial y} - \frac{d}{dx} \frac{d}{dx}$$

Lemma: let $f(x)$ be a continuous function on $[a, b]$ and $h \in C^2[a, b]$. Let $\int_a^b f(x) h(x) dx = 0$ then

$$f(x) = 0 \quad \forall x \in [a, b].$$

proof: let $x_0 \in (a, b)$ and $f(x_0) > 0$. Since $f(x)$ is continuous at x_0 there exists an interval (x_1, x_2) such that $x_0 \in (x_1, x_2)$ where $f(x) > 0$ $\forall x \in (x_1, x_2)$. Choose $h(x) = (x - x_1)^3 (x_2 - x)^3$ in (x_1, x_2) and zero at other pts.

$$\text{Then } \int_a^b f(x) h(x) dx = \int_{x_1}^{x_2} f(x) h(x) dx > 0$$

contradiction $\Rightarrow f(x) = 0$
 $\forall x \in (x_1, x_2)$

repeating this process at all pts in $I = [a, b]$ we get $f(x) = 0$

(6)

using the lemma we get the Euler-Lagrange Equation

$$E(L) = \frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) = 0$$

Examples:

$$(1) \quad J(y) = \int_0^1 (y'^2 + 3y + 2x) dx, \quad y(0) = 0, \quad y(1) = 1$$

$$E(L) = 0 \Rightarrow y'' = 3/2 \Rightarrow y(x) = \frac{3}{4}x^2 + \frac{1}{4}x$$

$$(2) \quad J(y) = \int_a^b \sqrt{1+y'^2} dx, \quad y \in C^2[a, b]$$

$$y(a) = y_0, \quad y(b) = y_1$$

$$L = \sqrt{1+y'^2}, \quad E(L) = 0 \Rightarrow \frac{y'}{\sqrt{1+y'^2}} = \text{constant}$$

$$\Rightarrow y(x) = \frac{y_1 - y_0}{b-a} x + \frac{by_0 - ay_1}{b-a}$$

$$(3) \quad J(y) = \int_0^1 x y y' dx, \quad y \in C^2(0, 1), \quad y(0) = 0, \quad y(1) = 1$$

$$2xyy' = (xy^2)' - y^2 \Rightarrow xyy' = \frac{1}{2} (xy^2)' - \frac{1}{2} y^2$$

$$\Rightarrow J(y) = \frac{1}{2} \int_0^1 (xy^2)' dx - \frac{1}{2} \int_0^1 y^2 dx$$

$$= \frac{1}{2} xy^2 \Big|_0^1 - \frac{1}{2} \int_0^1 y^2 dx = \frac{1}{2} - \frac{1}{2} \int_0^1 y^2 dx$$

$$\Rightarrow J(y) \leq 1/2$$

maximum value of J is $1/2$. For this value $y(x) = 0$, but this is not consistent with the boundary value $y(1) = 1$. Hence there exists no extremum of this variational problem. (7)

Let us solve this problem by using the Euler-Lagrange equation where $L = xy y'$

$$E(L) = xy' - \frac{d}{dx} (xy) = xy' - y - xy' = -y = 0$$

but $y=0$ is not consistent with the boundary condition. Hence we get the same conclusion.

(4) Null Lagrangians: Lagrangian functions leading or providing no equations for y . Example $J(y) = \int_0^1 y y' dx$, $y \in C^2[0,1]$, $y(0)=0$, $y(1)=1$

$$E(L) = y' - \frac{d}{dx} y = 0 \quad \text{identically satisfied.}$$

providing no condition on y .

On the other hand J has a fixed value

$$J(y) = \frac{1}{2} y^2 \Big|_0^1 = \frac{1}{2} \quad \text{a single value (constant).}$$

Remark: let

$$J(y) = \int_a^b L(x, y, y') dx$$

$$\Delta J = J(y_0 + \varepsilon h) - J(y_0)$$

where $y_0(x)$ is the function providing extremum value for J (or making J stationary). Using the Taylor series

$$J(y_0 + \varepsilon h) = J(y_0) + \varepsilon \delta J(y_0, h) + \frac{1}{2} \varepsilon^2 \delta^2 J(y_0, h) + \dots$$

$$\Rightarrow \Delta J = \varepsilon \delta J(y_0, h) + \frac{1}{2} \varepsilon^2 \delta^2 J(y_0, h) + \dots$$

where δJ is the first variation, $\delta^2 J$ is the second variation. The first variation of J provides the necessary condition (Euler-Lagrange equations) and the second variation provides the type of the extremum value (if it exists).

$$\delta J = \left. \frac{d}{d\varepsilon} J \right|_{\varepsilon=0}$$

$$\delta^2 J = \left. \frac{d^2}{d\varepsilon^2} J \right|_{\varepsilon=0}$$

(9)

$$(5) \quad J(y) = \int_0^1 (1+x) y'^2 dx, \quad y \in C^2[0,1],$$

$$y(0) = 0, \quad y(1) = 1$$

show that $y_0(x) = \frac{\ln(1+x)}{\ln 2}$

provides a minimum for J .

let $y(x) = y_0(x) + \varepsilon h(x)$, $h(0) = h(1) = 0$.

$$\Rightarrow y'(x) = \frac{1}{\ln 2} \cdot \frac{1}{1+x} + \varepsilon h'$$

$$(1+x) y'^2 = (1+x) \left[\frac{1}{\ln 2 (1+x)} + \frac{2\varepsilon h'}{\ln 2 (1+x)} + \varepsilon^2 h'^2 \right]$$

$$\Rightarrow J(y) = \frac{1}{\ln 2} \int_0^1 \frac{dx}{1+x} + \frac{2}{\ln 2} \int_0^1 h' dx + \varepsilon^2 \int_0^1 h'^2 dx$$

$$= 1 + \varepsilon^2 \int_0^1 h'^2 dx$$

$J(y) \geq 1$ equality is possible only if $\varepsilon = 0$

necency condition: $\frac{dJ}{d\varepsilon} \Big|_{\varepsilon=0} = 0$

second variation: $\frac{d^2 J}{d\varepsilon^2} = 2 \int_0^1 h'^2 dx > 0$

Hence $J(y_0) = 1$ is the minimum value

(6) let $J(y) = \int_0^1 (x^2 - y^2 + y'^2) dx$

$y \in C^1[0,1]$. let $y_0 = x$, $h(x) = x^2$
Find the first variation of J , $\delta J = ?$

(7) let $J(y) = \int_a^b (\sin^3 x + y^4) dx$

$$J(y) - J(y_0) = \int_a^b (y^4 - y_0^4) dx$$

$$\|J(y) - J(y_0)\| \leq \int_a^b \|y^4 - y_0^4\| dx$$

$$= \int_a^b (|y - y_0|^4 + 2|y_0||y - y_0|^3) dx$$

$$= \int_a^b (\delta^4 + 2|y_0|\delta^3) dx$$

$$= \delta^4(b-a) + 2c\delta^3 = \varepsilon$$

where $c = \int_a^b |y_0| dx$

Hence

$$|J(y) - J(y_0)| < \varepsilon$$

where

$$\varepsilon = \delta^4(b-a) + 2c\delta^3 \quad (\text{Given } \varepsilon \exists \text{ a } \delta)$$

proving the continuity in the strict sense

$$(8) \quad J(y) = \int_a^b (y'^2/x^3) dx$$

$$L = y'^2/x^3$$

Euler-Lagrange equations.

$$E(L) = -\frac{d}{dx} \left(\frac{2y'}{x^3} \right) = 0 \Rightarrow y' = Ax^3,$$

where A is an arbitrary constant. Then the solution

$$y(x) = \frac{A}{4} x^4 + B$$

where B is another arbitrary constant. A and B are determined when two boundary conditions are given.

$$(9) \quad \text{let } J(y) = \int_a^b f(x, y) \sqrt{1+y'^2} dx$$

Find the Euler-Lagrange equations and solve them

$$\left[y'' + \phi_y (1+y'^2)^2 - (\phi_x + \phi_y y') (1+y'^2) = 0 \right]$$

(10) First Integrals:

$$a) \text{ If } L = L(x, y) \Rightarrow L_y = 0$$

$$b) \text{ If } L = L(x, y') \Rightarrow \frac{\partial L}{\partial y'} = \text{constant}$$

$$c) \text{ If } L = L(y, y') \Rightarrow L - y' \frac{\partial L}{\partial y'} = \text{constant.}$$

$$\text{let } E = L - y' \frac{\partial L}{\partial y'}$$

$$\frac{d}{dx} E = \frac{\partial L}{\partial x} + \frac{\partial L}{\partial y} y' + \frac{\partial L}{\partial y'} y'' - y'' \frac{\partial L}{\partial y'} - y' \frac{d}{dx} \frac{\partial L}{\partial y'}$$

$$= 0 + y' E(L) = 0$$

$$\Rightarrow E = \text{constant.}$$

Natural Boundary Conditions:

If the boundary conditions, one or two are not assigned then we may not use $h(a) = 0$ or $h(b) = 0$ in the derivation of the E-L equations.

$$\int_a^b E(L) h(x) dx + \left. \frac{\partial L}{\partial y'} \right|_{x=b} h(b) - \left. \frac{\partial L}{\partial y'} \right|_{x=a} h(a) = 0$$

Let $y(a) = y_0$ but $y(b)$ is not specified.
Then $h(a) = 0$ but $h(b) \neq 0$ (arbitrary)

(13)

$$\int_a^b E(L) h(x) dx + \left. \frac{\partial L}{\partial y'} \right|_{x=b} h(b) = 0$$

$h(b)$ is arbitrary and it takes zero value as well for this case $E(L) = 0$ which leads to

$$\left. \frac{\partial L}{\partial y'} \right|_{x=b} h(b) = 0 \quad \text{when } h(b) \neq 0$$

$$\Rightarrow \left. \frac{\partial L}{\partial y'} \right|_{x=b} = 0$$

This is a boundary condition which is called the "Natural boundary condition".
Hence the optimization problem is

$$E(L) = 0, \quad y(a) = y_0, \quad \left. \frac{\partial L}{\partial y'} \right|_{x=b} = 0$$

- Similarly if $y(b) = y_1$ but $y(a)$ is not specified we get

$$E(L) = 0, \quad y(b) = y_1, \quad \left. \frac{\partial L}{\partial y'} \right|_{x=a} = 0$$

- If both are not specified

$$E(L) = 0, \quad \left. \frac{\partial L}{\partial y'} \right|_{x=a} = 0, \quad \left. \frac{\partial L}{\partial y'} \right|_{x=b} = 0$$

Example: let $J(y) = \int_0^1 (y'^2 + y^2) dx$.

(14)

$y(0) = 1$, but $y(1)$ is not specified.
 $L = y'^2 + y^2$, $\frac{\partial L}{\partial y'} = 2y' \Rightarrow$ natural b.c is $y'(1) = 0$.

$$\Rightarrow E(L) = 2y \cdot 2y' = 0 \Rightarrow y'' + y = 0$$

$$y(x) = A e^x + B e^{-x}$$

$$y(0) = A + B = 1$$

$$y'(1) = A e - B e^{-1} = 0 \Rightarrow B = e^2 A$$

$$A(1 + e^2) = 1 \Rightarrow A = \frac{1}{1 + e^2}$$

$$y(x) = \frac{1}{1 + e^2} (e^x + e^{2-x})$$

Variable end point:

let one of the end point is not fixed and moving on a curve $y = g(x)$.

$$J(y) = \int_a^b L(x, y, y') dx$$

a is fixed but b is not, & it is varying. Hence

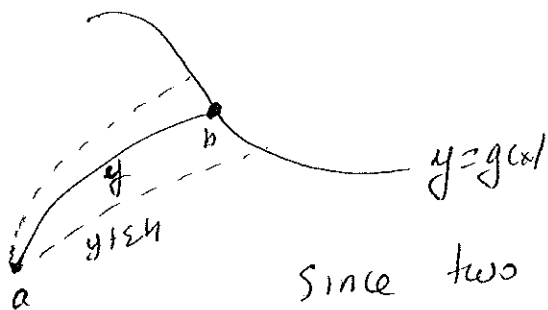
$$\delta J = \frac{\partial J}{\partial b} \delta b + \int_a^b \frac{\partial L}{\partial \varepsilon} \Big|_{\varepsilon=0} dx$$

(15)

$$= L \Big|_{x=b} \delta b + \int_a^b E(L) h(x) dx + \frac{\partial L}{\partial y'} \Big|_{x=b} h(b) - \frac{\partial L}{\partial y'} \Big|_{x=a} h(a)$$

since a is fixed $h(a)=0 \Rightarrow$

$$\delta J = L \Big|_{x=b} \delta b + \frac{\partial L}{\partial y'} \Big|_{x=b} h(b) + \int_a^b E(L) h(x) dx$$



Since two curves intersect at b

$$y(b) = g(b)$$

Varying $b \rightarrow b + \Delta b$ we have

$$y \rightarrow y(b + \Delta b) + \varepsilon h(b + \Delta b)$$

$$g \rightarrow g(b + \Delta b)$$

$$y(b + \Delta b) + \varepsilon h(b + \Delta b) = g(b + \Delta b)$$

$$y(b + \Delta b) - y(b) + \varepsilon h(b + \Delta b) = g(b + \Delta b) - g(b)$$

$$y'(b) \Delta b + \varepsilon h(b) = g'(b) \Delta b$$

Then

$$\varepsilon h(b) = [g'(b) - y'(b)] \Delta b$$

or

$$\Delta b = \frac{\varepsilon h(b)}{g'(b) - y'(b)}$$

where $\Delta b = \varepsilon \delta b$ hence.

$$\delta b = \frac{h(b)}{g'(b) - y'(b)}$$

$$\Rightarrow \delta J = \left(\frac{L_b}{g'(b) - y'(b)} + \frac{\partial L}{\partial y'} \bigg|_{x=b} \right) h(b)$$

$$+ \int_a^b E(L) h(x) dx = 0$$

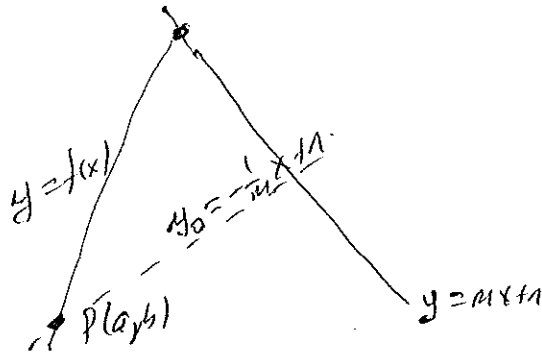
$\Rightarrow$ we get

$$E(L) = 0$$

$$\frac{L|_{x=b}}{g'(b) - y'(b)} + \frac{\partial L}{\partial y'} \bigg|_{x=b} = 0$$

Example : Find the shortest distance from a point to a line. (17)

let $y = mx + n$ be the line and $P(a, b)$ be the point.



First we determine the curve $y = f(x)$. It is the curve gives shortest distance

$$J = \int_a^b \sqrt{1 + y'^2} dx$$

Here c is the point moving on the straight line $y = mx + n$.

$$E(L) = 0 \Rightarrow y = Ax + B$$

where A is the slope of y .

$$L_c + (g' - y') \frac{\partial L}{\partial y'} \Big|_{x=c} = 0$$

$$\sqrt{1 + A^2} + (m - A) \frac{A}{\sqrt{1 + A^2}} = 0$$

$$1 + A^2 + (m - A)A = 0 \Rightarrow mA = -1$$

The slopes of $y = f(x)$ and $y = mx + n$ are perpendicular. (Lines should be perpendicular)

Lagrangian having higher derivatives of y

Theorem: Let $J: A \rightarrow \mathbb{R}$ where A is a set of functions in $C^{2n}[a,b]$ satisfying 2n boundary conditions

$$y(a) = A_1, y'(a) = A_2, \dots, y^{(n-1)}(a) = A_n.$$

$$y(b) = B_1, y'(b) = B_2, \dots, y^{(n-1)}(b) = B_n.$$

Then if y provides a local extremum to the functional $J(y) = \int_a^b L(x, y, y', \dots, y^{(n)}) dx$

then y must satisfy the equation (Euler-Lagrange equation) $E(L) = 0$ where

$$E(L) = \frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} + \frac{d^2}{dx^2} \left(\frac{\partial L}{\partial y''} \right) + \dots + (-1)^n \frac{d^n}{dx^n} \left(\frac{\partial L}{\partial y^{(n)}} \right) = 0$$

which is an ODE of order $2n$.

Example: $J(y) = \int_0^1 (y y' + y''^2) dx$

with $y(0) = 0, y'(0) = 1, y(1) = 2, y'(1) = 4$

Solution, $E(L) = \frac{d^2}{dx^2} y'' = 0 \Rightarrow$

$$y(x) = Ax^3 + Bx^2 + Cx + D$$

$$\begin{aligned} y(0) = 0 &\Rightarrow D = 0 \\ y'(0) = 1 &\Rightarrow C = 1 \end{aligned} \quad \text{other two given} \quad \begin{aligned} A + B &= 1 \\ 3A + 2B &= 3 \end{aligned} \Rightarrow B = 0, A = 1$$

$$y(x) = x^3 + x$$

Example : prove theorem for $n=2$.

$$\begin{aligned}
 \left. \frac{\partial L}{\partial \varepsilon} \right|_{\varepsilon=0} &= \int_a^b \frac{\partial L}{\partial y} + \frac{\partial L}{\partial y'} h' + \frac{\partial L}{\partial y''} h'' \\
 &= \frac{\partial L}{\partial y} h + \frac{d}{dx} \left(\frac{\partial L}{\partial y'} h \right) - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) h \\
 &\quad + \frac{d}{dx} \left(\frac{\partial L}{\partial y''} h' \right) - \frac{d}{dx} \frac{\partial L}{\partial y''} h' \\
 &= \frac{\partial L}{\partial y} h - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) h + \frac{d}{dx} \left(\frac{\partial L}{\partial y'} h \right) \\
 &\quad + \frac{d}{dx} \left(\frac{\partial L}{\partial y''} h' \right) - \frac{d}{dx} \left(\frac{d}{dx} \frac{\partial L}{\partial y''} h \right) + \frac{d^2}{dx^2} \left(\frac{\partial L}{\partial y''} \right) h \\
 &= E(L) h + \frac{d}{dx} \left[\frac{\partial L}{\partial y'} h + \frac{\partial L}{\partial y''} h' - \frac{d}{dx} \frac{\partial L}{\partial y''} h \right]
 \end{aligned}$$

where

$$E(L) = \frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial L}{\partial y''} \right)$$

Then

$$\delta J = \int_a^b E(L) h(x) dx + \left. \varphi \right|_{x=b} - \left. \varphi \right|_{x=a} = 0$$

where

$$Q = \left(\frac{\partial L}{\partial y'} - \frac{d}{dx} \frac{\partial L}{\partial y''} \right) h(x) + \frac{\partial L}{\partial y''} h'(x).$$

$$\begin{aligned} Q|_{x=b} - Q|_{x=a} &= \left[\frac{\partial L}{\partial y'} - \frac{d}{dx} \left(\frac{\partial L}{\partial y''} \right) \right]_b h(b) - \left[\frac{\partial L}{\partial y'} - \frac{d}{dx} \left(\frac{\partial L}{\partial y''} \right) \right]_a h(a) \\ &\quad + \frac{\partial L}{\partial y''} \Big|_{x=b} h'(b) - \frac{\partial L}{\partial y''} \Big|_{x=a} h'(a). \end{aligned}$$

a) If all the B.C.s are specified.

$$h(a) = h(b) = 0, \quad h'(a) = h'(b) = 0$$

$$\Rightarrow E(L) = 0 \quad (\text{Euler Lagrange Equations})$$

b) If some of the B.C.s are not specified one can find the corresponding N.B.C. very easily. As an example

$y'(b)$ is not specified. $\Rightarrow$ the problem becomes

$$\begin{aligned} E(L) = 0 \quad & \left. \begin{array}{l} y(a) = A_1, \quad y'(a) = A_2 \\ y(b) = B_1, \end{array} \right\} \begin{array}{l} h(a) = h(b) = 0 \\ h'(a) = 0 \text{ but} \\ h'(b) \neq 0 \end{array} \\ & \Rightarrow \frac{\partial L}{\partial y''} \Big|_{x=b} = 0 \quad \text{NBC.} \end{aligned}$$

Natural Transition Conditions

Lagrange functions which are not continuous at some pts.

If the Lagrange function L is such that $\frac{\partial L}{\partial y}$ or $\frac{\partial L}{\partial y'}$ (or both) are discontinuous at some point or more points in $[a, b]$. Let $x = c \in (a, b)$ be such a point of discontinuity. We split the integral

$$J(y) = \int_a^{c-} L \, dx + \int_{c+}^b L \, dx$$

$$\Rightarrow \delta J(y) = \int_a^{c-} E(L) h(x) \, dx + \int_{c+}^b E(L) h(x) \, dx$$

$$+ \left. \frac{\partial L}{\partial y'} \right|_{x=b} h(b) - \left. \frac{\partial L}{\partial y'} \right|_{x=c} h(c)$$

$$- \left[\left. \frac{\partial L}{\partial y'} \right|_{c+} - \left. \frac{\partial L}{\partial y'} \right|_{c-} \right] h(c) = 0$$

Here we assumed that $h(x)$ is continuous in $[a, b]$

$$h(c_+) = h(c_-) = h(c)$$

and y is continuous at $x=c$

which forces h to be also continuous at $x=c$.

Euler Lagrange eqns.

$$E(L) = 0 \quad \text{for } x \in (a, c) \cup (c, b).$$

$$y(c_+) = y(c_-)$$

$$\left. \frac{\partial L}{\partial y'} \right|_{c_+} = \left. \frac{\partial L}{\partial y'} \right|_{c_-}$$

Example: (i) Let $J(y) = \int_0^1 [T y'^2 + \rho \omega^2 y^2] dx$

T, ρ, ω are constants. Euler-Lagrange equations.

$$E(L) = 0 \Rightarrow \frac{d}{dx} \left(T \frac{dy}{dx} \right) + \rho \omega^2 y = 0$$

when T, ω, ρ are constants. the general solution is

$$y(x) = C_1 \cos \alpha x + C_2 \sin \alpha x, \quad \alpha^2 = \frac{\rho \omega^2}{T}$$

i) if $y(0) = 0, y(1) = 1 \Rightarrow$

$$y(x) = \frac{\sin \alpha x}{\sin \alpha} \quad \alpha = \pi, 2\pi, \dots$$

ii) $y(1) = 1$ but $y(0)$ is not assigned

$$\frac{\partial L}{\partial y'} = 2T y' \Rightarrow y'(0) = 0$$

Solution $y(x) = \frac{\cos \alpha x}{\cos \alpha}$, $\alpha = \pi/2, 3\pi/2, \dots$

iii) if both BCs are not specified.

$$\Rightarrow y'(0) = 0, \quad y'(1) = 0.$$

$$y'(0) = \alpha c_2 = 0 \quad c_2 = 0$$

$$y'(1) = -\alpha c_1 \sin \alpha = 0, \quad \alpha = n\pi.$$

$$\Rightarrow y(x) = c_1 \cos(n\pi x)$$

Example 2 : The same functional

$$J(y) = \int_0^1 [T y'^2 - \rho \omega^2 y^2] dx$$

$$y(0) = 0, \quad y(1) = 1$$

$$\text{If } \begin{array}{ll} T = T_1, \quad \rho = \rho_1 & \text{for } 0 \leq x \leq c = I_1 \\ T = T_2, \quad \rho = \rho_2 & \text{for } c < x \leq 1 = I_2 \end{array}$$

where T_1, T_2, ρ_1 , and ρ_2 are positive real constants.

$$y(x) = \begin{cases} c_1 \cos \alpha_1 x + c_2 \sin \alpha_1 x & x \in I_1 \\ d_1 \cos \alpha_2 x + d_2 \sin \alpha_2 x & x \in I_2 \end{cases}$$

$$\alpha_1^2 = \rho_1 \frac{\omega^2}{T_1}, \quad \alpha_2^2 = \rho_2 \frac{\omega^2}{T_2}$$

$$y(b) = 0 \Rightarrow c_1 = 0$$

BCs

$$y(1) = 1 \Rightarrow d_1 \cos \alpha_2 + d_2 \sin \alpha_2 = 1$$

Continuity
conditions

$$y(c_+) = y(c_-)$$

$$c_2 \sin \alpha_1 c = d_1 \cos \alpha_2 c + d_2 \sin \alpha_2 c$$

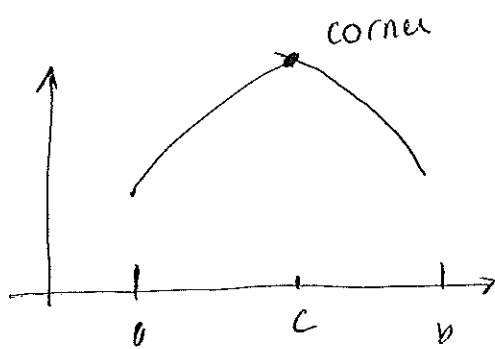
$$\left. \frac{\partial L}{\partial y'} \right|_{c_-} = \left. \frac{\partial L}{\partial y'} \right|_{c_+} \Rightarrow T_1 y'(c_-) = T_2 y'(c_+)$$

$$T_1 \alpha_1 c_2 \cos \alpha_1 c = T_2 (-\alpha_2 d_1 \sin \alpha_2 c + d_2 \alpha_2 \cos \alpha_2 c)$$

Four conditions and four eqns hence

c_1, c_2, d_1, d_2 can be solved exactly.

Discontinuous solutions ($L = L(y, y')$)



y' is not continuous
at $x=c$. Let

$$y' \Big|_{c-} = m, \quad y' \Big|_{c+} = n.$$

lemma: $\frac{\partial L}{\partial y'}$ and $L - y' \frac{\partial L}{\partial y'}$ are
continuous at the corner point. i.e

$$\frac{\partial L}{\partial y'} \Big|_{c-} = \frac{\partial L}{\partial y'} \Big|_{c+}$$

$$E \Big|_{c-} = E \Big|_{c+}, \quad \text{where } E = L - y' \frac{\partial L}{\partial y'}$$

Example: $J = \int (y'^2 - 1)^2 dx$

EL equation: y' constant $\Rightarrow y = ax + b$

suppose that a corner exist at $x=c$.
with slopes m and n on either side

Then

$$a) \quad \frac{\partial L}{\partial y'} \Big|_{c-} = \frac{\partial L}{\partial y'} \Big|_{c+} \Rightarrow y' (y'^2 - 1) \Big|_{c-} = y' (y'^2 - 1) \Big|_{c+}$$

$$m(m^2-1) = n(n^2-1).$$

$$\Rightarrow (m-n)(m^2+n^2+mn-1) = 0$$

$$m \neq n \Rightarrow m^2+mn+n^2 = 1.$$

$$b) \quad y(c_-) = y(c_+)$$

$$m\cancel{c} + m_1 = \cancel{m}c + n_1$$

$$(m-n)c = n_1 - m_1$$

$$\begin{aligned} c) \quad E|_{c_-} &= (m^2-1)^2 - m[4m(m^2-1)] \\ &= (m^2-1)^2 - 4m^2(m^2-1) \\ &= (m^2-1)(m^2-1-4m^2) \\ &= -(m^2-1)(3m^2+1) \end{aligned}$$

$$E|_{c_+} = -(n^2-1)(3n^2+1)$$

$$\Rightarrow (m^2-1)(3m^2+1) = (n^2-1)(3n^2+1)$$

$$3m^4 - 2m^2 = 3n^4 - 2n^2$$

$$3(m^2-n^2)(m^2+n^2) - 2(m^2-n^2) = 0$$

$$(m^2-n^2)[3m^2+3n^2-2] = 0$$

$$\text{since } m \neq n \Rightarrow m = -n$$

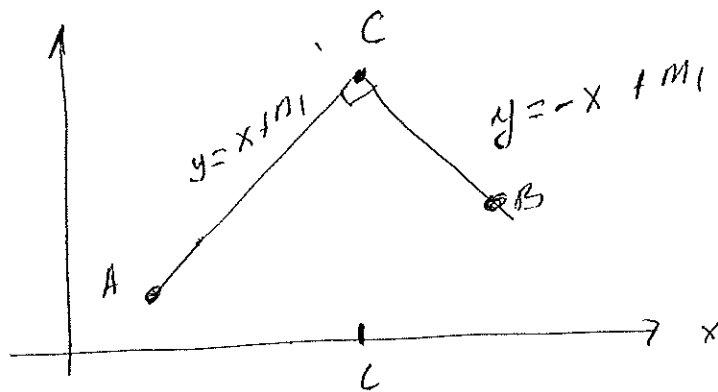
from (a) $\Rightarrow n^2 = 1 \Rightarrow n = \pm 1$.

- i) $m = -1, n = 1$
 ii) $m = 1, n = -1$

Hence

i) $-2c = n_1 - m_1$

ii) $2c = n_1 - m_1$



$$n_1 - m_1 = -2c.$$

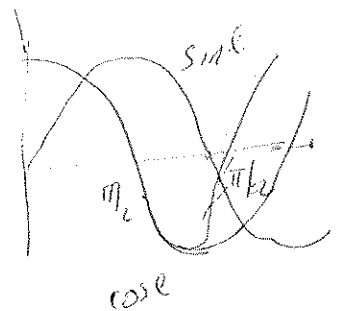
1) If l is not preassigned, show that the stationary functions corresponding to the problem

$$\delta \int_0^l y'^2 dx = 0$$

$$y(0) = 2, \quad y(l) = \sin l$$

are of the form $y(x) = 2 + 2x \cos l$ where l satisfies the transcendental equation $2 + 2l \cos l - \sin l = 0$. Also show that the smallest positive value l is between $\pi/2$ and $3\pi/4$.

$$l = \frac{\sin l - 2}{2 \cos l}$$



• Geodesic on the sphere
arclength parameter on the sphere
satisfies.

$$ds^2 = dx^2 + dy^2 + dz^2 = a^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

Then we have the functional to be
minimised

$$I = a \int_A^B \left[1 + \left(\frac{d\phi}{d\theta} \right)^2 \sin^2 \theta \right]^{1/2} d\theta.$$

EL eqns. $\frac{\partial L}{\partial \phi} - \frac{d}{d\theta} \left(\frac{\partial L}{\partial \phi'} \right) = 0$

with $L = [1 + \phi'^2 \sin^2 \theta]^{1/2}$.

Evidently we have

$$\frac{\partial L}{\partial \phi'} = \text{const.}$$

$$\frac{\phi' \sin^2 \theta}{\sqrt{1 + \phi'^2 \sin^2 \theta}} = \sin \alpha, \quad \alpha = \text{const}$$

solving for ϕ' we get

$$\phi' = \frac{\sin \alpha}{\sin \theta \sqrt{\sin^2 \theta - \sin^2 \alpha}}$$

(19)

letting $\phi \text{ and } \theta = \frac{1}{u}$ we obtain

$$\phi + \beta = - \int \frac{\tan \alpha \, du}{(1 - u^2 \tan^2 \alpha)^{1/2}}$$

Integration gives

$$\cos(\phi + \beta) = \frac{\tan \alpha}{\tan \theta}$$

where β is another constant. Transforming the last equation to the cartesian coordinate.

$$\cos(\phi + \beta) = \tan \alpha \cos \theta$$

we get

$$x \cos \beta - y \sin \beta = z \tan \alpha$$

This is the equation of a plane through the centre of the sphere. The minimizing curve is obtained by the intersection of the sphere and the above plane which is known as the great circles (geodesics).

6. Inversion: Given DE find the corresponding Lagrangian function

EL equation $E(L) = 0$, $L(x, y, y')$

$$L_y - \frac{d}{dx} L_{y'} = 0 \Rightarrow L_y - L_{y'x} - L_{yy'} y' - L_{y'y'} y'' = 0$$

$$L_y - L_{y'x} - L_{yy'} y' - L_{y'y'} y'' = 0$$

$$\cancel{L_{yy'}} y' - L_{y'y'} x - \cancel{L_{yy'}} y' - \cancel{L_{yy'}} y' y' - \cancel{L_{yy'}} y' y'' = 0$$

$\searrow F$

let $y'' = F(x, y, y')$ and let $L_{y'y'} = u(x, y, y')$

$$u_x + u_y y' + u_{y'} F \neq 0 \quad \text{at } F_y = 0$$

Given F solve u then using

$$L_{y'y'} = u$$

solve L so that $E(L) = 0$. Solution is not unique

Example: $m \ddot{y} = -k y - a \dot{y} \Rightarrow F = -\frac{k}{m} y - \frac{a}{m} \dot{y}$

$$u_x + u_y y' - u_{y'} \left(\frac{k}{m} y + \frac{a}{m} y' \right) \neq 0 \quad \left(-\frac{a}{m} \right) = 0$$

$$u = m f(x)$$

$$m f_x - a f = 0$$

$$f = f_0 e^{\frac{a}{m} x} \quad \checkmark$$

$$L_{y'y'} = f_0 e^{\frac{a}{m}x}$$

$$L = \frac{1}{2} f_0 e^{\frac{a}{m}x} y'^2 + \alpha y' + \beta$$

$$E(L) = 0$$

$$d_y y' + \beta_y - \frac{d}{dx} \left(f_0 e^{\frac{a}{m}x} y' + \alpha \right) = 0$$

$$\cancel{d_y y'} + \beta_y - \frac{a}{m} f_0 e^{\frac{a}{m}x} y' - f_0 e^{\frac{a}{m}x} \left(-\frac{k}{m} y - \frac{a}{m} y' \right) - \alpha_x - \cancel{d_y y'} = 0$$

$$\beta_y - \alpha_x + f_0 \frac{k}{m} y e^{\frac{a}{m}x} = 0$$

Choose $\alpha = 0 \Rightarrow \beta = -f_0 \frac{k}{2m} y^2 e^{\frac{a}{m}x}$

$$L = \frac{1}{2} f_0 e^{\frac{a}{m}x} \left(y'^2 - \frac{k}{m} y^2 \right)$$

Example: Find the Lagrangian L such that the EL equation coincides with Emden-Fowler eq

$$\ddot{y} + \frac{2}{t} \dot{y} + y^5 = 0$$

some well-known problems

(32)

(1) The shortest line:

$$J(y) = \int_a^b \sqrt{1+y'^2} dx$$

$$E(L) = -\left(\frac{y'}{\sqrt{1+y'^2}}\right)' = 0 \Rightarrow y' = \text{constant}$$

$y = Ax + B$ straight line connecting
the points $P(a, y_0)$ to $Q(b, y_1)$.

$$\Rightarrow \frac{y - y_0}{y_1 - y_0} = \frac{x - a}{b - a}$$

$$y = \frac{y_1 - y_0}{a - b} x + \frac{ay_1 - by_0}{a - b}$$

(2) The brachistochrone: For this problem:

$$L = \sqrt{\frac{1+y'^2}{y}} \quad E(L) = 0 \Rightarrow$$

$$\frac{y'^2}{\sqrt{y(1+y'^2)}} - \frac{\sqrt{1+y'^2}}{\sqrt{y}} = C \quad \text{a constant}$$

$$\text{or} \quad \frac{1}{y(1+y'^2)} = C^2 \quad \text{another constant}$$

$$y' = \sqrt{\frac{2a-y}{y}} \quad C^2 = 1/2a$$

Integration gives

$$\frac{dy}{dx} = \sqrt{\frac{2a-y}{y}} \Rightarrow dx = dy \sqrt{\frac{y}{2a-y}}$$

$$\Rightarrow x - x_0 = \int \sqrt{\frac{y}{2a-y}} dy$$

$$\text{Let } y = a(1 - \cos \theta) \Rightarrow$$

$$x - x_0 = 2a \int \sin^2 \theta / 2 d\theta$$

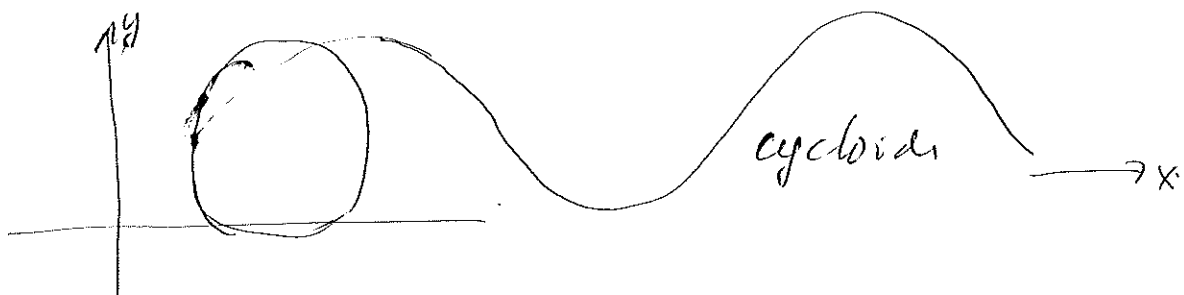
$$= a(\theta - \sin \theta)$$

$\Rightarrow$ The curve is (brachistochrone)

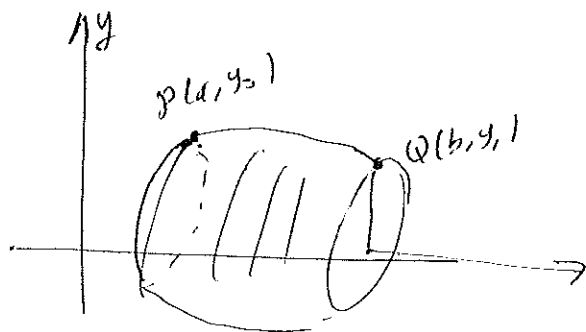
$$x = a(\theta - \sin \theta) + x_0$$

$$y = a(1 - \cos \theta)$$

There are cycloids: generated by a fixed point on the circumference of a circle of radius a which rolls down on the positive side of $y=0$



- 3) The minimum surface: Finding a surface of revolution passing through the points (a, y_0) and (b, y_1) having minimum surface area.



$$S(y) = \int_a^b y \, ds$$

$$= \int_a^b y \sqrt{1+y'^2} \, dx$$

$$E = y \sqrt{1+y'^2}, \quad E(L) = 0 \Rightarrow$$

$$\sqrt{1+y'^2} - \frac{d}{dx} \frac{y y'}{\sqrt{1+y'^2}} = 0 \Rightarrow$$

$$\frac{y}{\sqrt{1+y'^2}} = b \quad \text{constant}$$

$$\frac{y^2}{b^2} = 1+y'^2 \Rightarrow y' = \pm \sqrt{\frac{y^2}{b^2} - 1}$$

$$\frac{dy}{\sqrt{y^2/b^2 - 1}} = \pm dx \Rightarrow b \cosh^{-1}(y/b) = \pm (x - x_0)$$

$$\Rightarrow y = b \cosh \frac{x - x_0}{b}$$

This curve is called "catenary"

The constant b and x_0 must be picked so that the curve passes through the points $P(a, y_0)$ and $Q(b, y_1)$ with y_0, y_1 below the x -axis (not intersecting x -axis).

Multiple dependent Variables:

let $y: I \rightarrow \mathbb{R}^n$, $y = (y_1(x), y_2(x), \dots, y_n(x))$, $x \in I$

Hence $L = L(x, y_1, y_2, \dots, y_n; y_1', y_2', \dots, y_n')$.

Euler Lagrange equations are

$$E_{y_i}(L) = 0, \quad i = 1, 2, \dots, n.$$

$$E_{y_i} = \frac{\partial}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial L}{\partial y_i'} \right)$$

with the boundary conditions, let $y: I = [a, b]$

$$y_i(a) = A_i, \quad i = 1, 2, \dots, n.$$

$$y_i'(b) = B_i$$

all $y_i \in C^2[a, b]$.

proof: let y all admissible functions

$$y_i(x) = y_i^0(x) + \varepsilon h_i(x)$$

where $y_i^0(x)$ are the functions providing

$$J = \int_a^b L \, dx \quad \text{a minimum value}$$

$$\Rightarrow \delta J(y_i^0, h_i) = \int_a^b \frac{\partial L}{\partial \varepsilon} \Big|_{\varepsilon=0} dx = 0$$

$$\left. \frac{\partial L}{\partial \varepsilon} \right|_{\varepsilon=0} = \sum_{i=1}^n \left[\frac{\partial L}{\partial y_i} h_i + \frac{\partial L}{\partial y_i'} h_i' \right] \Rightarrow$$

$$\sum_{i=1}^n \int_a^b \left[\frac{\partial L}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial L}{\partial y_i'} \right) \right] h_i(x) dx + \sum_{i=1}^n \frac{\partial L}{\partial y_i'} h_i(x) \bigg|_a^b = 0$$

where for fixed variations all

$$h_i(a) = h_i(b) = 0 \quad \forall i = 1, 2, \dots, n.$$

$$\Rightarrow \sum_{i=1}^n \int_a^b E_{y_i}(L) h_i(x) dx = 0$$

using the similar argument that we used for $n=1$ case we obtain that

$$E_{y_i}(L) = 0 \quad \forall i = 1, 2, \dots, n.$$

Example: Show that

$$E = L - \sum_{i=1}^n y_i' \frac{\partial L}{\partial y_i'} = \text{constant}.$$

Example: $J(y) = \int_0^{\pi/4} [4y_1^2 + 4y_2^2 + y_1' y_2'] dx$

$$y_1(0) = 1, \quad y_1(\pi/4) = 0, \quad y_2(0) = 0, \quad y_2(\pi/4) = 1.$$

Solution: $L = 4y_1^2 + 4y_2^2 + y_1' y_2'$

$$E_{y_1}(L) = 8y_1 - \frac{d}{dx}(y_2') = 0$$

$$y_2'' - 8y_1 = 0$$

$$E_{y_2}(L) = 8y_2 - \frac{d}{dx}(y_1') = 0$$

$$y_1'' - 8y_2 = 0$$

$$\frac{2\sqrt{2}}{8}$$

$$\Rightarrow y_1^{(iv)} - 8y_1 = 0$$

$$y_1(x) = A_1 \cos 2\sqrt{2}x + A_2 \sin 2\sqrt{2}x + B_1 \cosh 2\sqrt{2}x + B_2 \sinh 2\sqrt{2}x$$

$$y_2(x) = \frac{1}{8} [-8A_1 \cos 2\sqrt{2}x + 8A_2 \sin 2\sqrt{2}x + 8B_1 \cosh 2\sqrt{2}x + 8B_2 \sinh 2\sqrt{2}x]$$

$$= [-A_1 \cos 2\sqrt{2}x - A_2 \sin 2\sqrt{2}x + B_1 \cosh 2\sqrt{2}x + B_2 \sinh 2\sqrt{2}x]$$

$$y_1(0) = 1 \Rightarrow A_1 + B_1 = 1$$

$$y_2(0) = 0 \Rightarrow -A_1 + B_1 = 0$$

$$A_1 = B_1 = 1/2$$

$$y_1(\pi/4) = A_1 \cos \frac{\sqrt{2}\pi}{2} + A_2 \sin \frac{\sqrt{2}\pi}{2} + B_1 \cosh \frac{\sqrt{2}\pi}{2} + B_2 \sinh \frac{\sqrt{2}\pi}{2} = 0$$

$$y_2(\pi/4) = -A_1 \cos \frac{\sqrt{2}\pi}{2} - A_2 \sin \frac{\sqrt{2}\pi}{2} + B_1 \cosh \frac{\sqrt{2}\pi}{2} + B_2 \sinh \frac{\sqrt{2}\pi}{2} = 1$$

$$B_1 \cosh \frac{\sqrt{2}\pi}{2} + B_2 \sinh \frac{\sqrt{2}\pi}{2} = 1/2$$

$$\underline{B_2 \sinh \frac{\sqrt{2}\pi}{2} = 1/2 - 1/2 \cosh \frac{\sqrt{2}\pi}{2}}$$

$$\frac{1}{2} \cos \frac{\sqrt{2}\pi}{2} + A_2 \sin \frac{\sqrt{2}\pi}{2} + \frac{1}{2} \cosh \frac{\sqrt{2}\pi}{2} + \frac{1}{2} - \frac{1}{2} \cosh \frac{\sqrt{2}\pi}{2} = 0$$

$$\underline{A_2 \sin \frac{\sqrt{2}\pi}{2} = -\frac{1}{2} - \frac{1}{2} \cos \frac{\sqrt{2}\pi}{2}}$$

We calculated all unknowns A_1, A_2, B_1 , and B_2

Constraint Systems and Lagrange multipliers (Isoperimetric problems).

Two cases

I. let y_1 and $y_2 \in C^2[a, b]$ and

$$J(y_1, y_2) = \int_a^b L(x, y_1, y_2, \dot{y}_1, \dot{y}_2, \dots) dx$$

with a constraint

$$\phi(y_1, y_2) = 0$$

We suppose that y_1 and y_2 are prescribed at the end points. This means that the boundary conditions are given

$$y_i(a) = A_i, \quad y_i(b) = B_i \quad (i=1, 2)$$

To extremize J we proceed as usual.

$$\delta J = \int_a^b [E_{y_1}(L) h_1(x) + E_{y_2}(L) h_2(x)] dx = 0$$

On the other hand

$$\left. \frac{d\phi}{d\varepsilon} \right|_{\varepsilon=0} = \phi_{y_1} h_1 + \phi_{y_2} h_2 = 0$$

Hence h_1 and h_2 are not independent.

They can not be assigned arbitrarily in $[a, b]$.

but

$$h_1(a) = h_1(b) = 0, \quad h_2(a) = h_2(b) = 0$$

Hence we have

$$\int_a^b [E_{y_1}(x) h_1 + E_{y_2}(x) h_2] dx = 0$$

$$\phi_{y_1} h_1 + \phi_{y_2} h_2 = 0.$$

Multiply the second one by λ (Lagrange multiplier) and integrating in $[a, b]$ we

get

$$\int_a^b [\lambda \phi_{y_1} h_1 + \lambda \phi_{y_2} h_2] dx = 0$$

subtracting we get

$$\int_a^b [(E_{y_1}(x) - \lambda \phi_{y_1}) h_1 + (E_{y_2}(x) - \lambda \phi_{y_2}) h_2] dx = 0$$

Now choosing λ so that the first term vanishes.

$$E_{y_1}(L) - \lambda \phi_{y_1} = 0$$

Then in the remaining integral since h_1 can be considered ~~under~~ arbitrary we get

$$E_{y_2}(L) - \lambda \phi_{y_2} = 0.$$

II. Another type of constraint may be given as integral. Let $y \in C^2[a, b]$ and

$$J(y) = \int_a^b L(x, y, y') dx$$

with a constraint

$$W(y) = \int_a^b G(x, y, y') dx = \text{constant}$$

Extremizing the problem: All admissible functions have two arbitrary functions

$$y(x) = y_0(x) + \varepsilon_1 h_1(x) + \varepsilon_2 h_2(x)$$

One variation is not enough for this variational problem with constraint. We have

$$J(y_0 + \varepsilon, h_1 + \varepsilon_1 h_1) = \int_a^b L(x, y_0 + \varepsilon, h_1 + \varepsilon_1 h_1, y_0' + \varepsilon, h_1' + \varepsilon_1 h_1') dx$$

$$W(y_0 + \varepsilon, h_1 + \varepsilon_1 h_1) = \int_a^b G(x, y_0 + \varepsilon, h_1 + \varepsilon_1 h_1, y_0' + \varepsilon, h_1' + \varepsilon_1 h_1') dx$$

As we do it in calculus, we introduce here the Lagrange multiplier λ so that the first variation of the function $L + \lambda G$ wrt to ε_1 and wrt ε_2 vanish, i.e.

$$\int_a^b [E(L) + \lambda E(G)] h_1 dx = 0$$

and

$$\int_a^b [E(L) + \lambda E(G)] h_2 dx = 0$$

It is not possible to say that $E(L + \lambda G) = 0$ immediately.

One may choose λ so that the first integral vanishes identically

$$\int_a^b E(L) h_1 dx + \lambda \int_a^b E(G) h_2 dx = 0$$

Hence the second integral given, since h_2 can be considered as independent (arbitrary).

$$E_y(L + \lambda G) = 0.$$

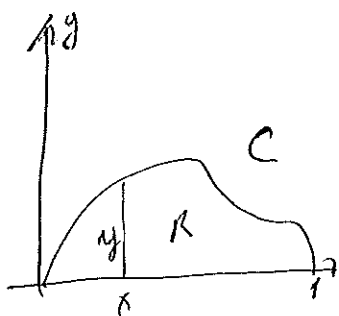
Remark: h_1 and h_2 cannot be considered independent due to $W(\xi_1, \xi_2) = \text{const.}$ but we may consider one of them to be independent. Hence as a result we have

$$E(L + \lambda G) = 0, \quad W = C$$

with some prescribed boundary conditions
 $y(a) = y_0, \quad y(b) = y_1$

(44)

Example: Let C be a curve with fixed length l . Consider the region R between the curve and x -axis, assuming C intersects only at the origin and $x=1$. Show that the circle is the curve ~~being~~ ^{so that} R has the maximum area. [Assume that C is smooth]



Proof:

$$\text{Area of } R = J = \int_0^1 y \, dx$$

$$\text{length of } C = W = \int_0^1 \sqrt{1+y'^2} \, dx = l.$$

$$L = y + \lambda \sqrt{1+y'^2}$$

$$E(L) = 1 - \lambda \frac{d}{dx} \frac{y'}{\sqrt{1+y'^2}} = 0.$$

$$\Rightarrow \kappa - \kappa_0 = \frac{\lambda y'}{\sqrt{1+y'^2}}$$

$$y' = \pm \frac{(x-x_0)}{\sqrt{\lambda^2 - (x-x_0)^2}}$$

$$y - y_0 = \sqrt{\lambda^2 - (x-x_0)^2}$$

$$\Rightarrow (x-x_0)^2 + (y-y_0)^2 = \lambda^2$$

(circle with
radius λ)
center (x_0, y_0))

since

$$y' = \pm \frac{(x-x_0)}{\sqrt{\lambda^2 - (x-x_0)^2}}$$

$$1 + y'^2 = 1 + \frac{(x-x_0)^2}{\lambda^2 - (x-x_0)^2} = \frac{\lambda^2}{\lambda^2 - (x-x_0)^2}$$

$$W = \int_0^1 \frac{\lambda \, dx}{\sqrt{\lambda^2 - (x-x_0)^2}} = \lambda \sin^{-1} \left(\frac{x-x_0}{\lambda} \right) \Big|_0^1 = \ell.$$

$$\lambda \left[\sin^{-1} \left(\frac{1-x_0}{\lambda} \right) + \sin^{-1} \left(\frac{x_0}{\lambda} \right) \right] = \ell.$$

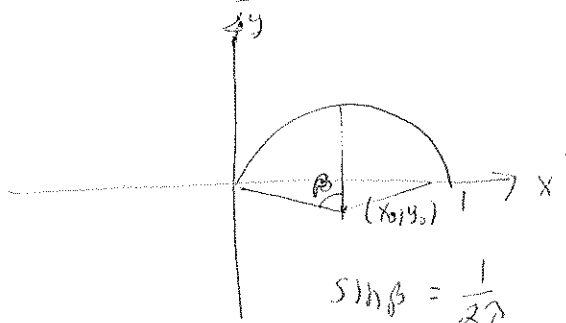
$$x=0, y=0$$

$$x_0^2 + y_0^2 = \lambda^2$$

$$x=1, y=0$$

$$(1-x_0)^2 + y_0^2 = \lambda^2$$

$$\boxed{\begin{aligned} x_0 = 1/2 \quad & 1/4 + y_0^2 = \lambda^2 \\ & 2\lambda \sin^{-1} (1/2\lambda) = \ell. \end{aligned}}$$



$$\beta = \sin^{-1} \left(\frac{1}{2\lambda} \right)$$

$$2\beta\lambda = \ell$$

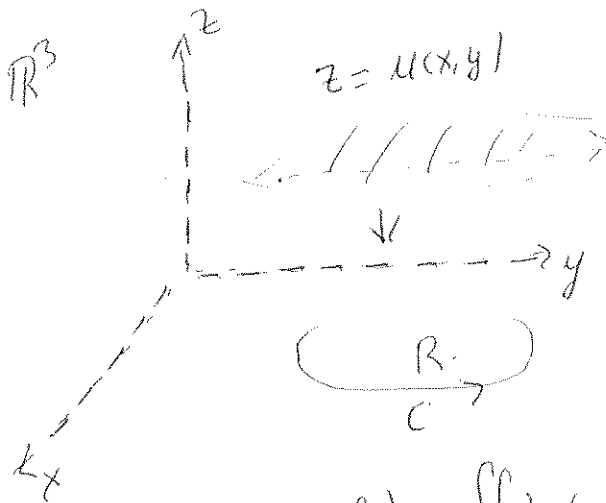
$$\Rightarrow 2\lambda \sin^{-1} \left(\frac{1}{2\lambda} \right) = \ell$$

$$\sin \beta = \frac{1}{2\lambda}$$

More ∞ dependent Variables

$$J(u) = \iint_R L(x, y, u(x, y), u_x(x, y), u_y(x, y)) dx dy.$$

$$u: \mathbb{R}^2 \rightarrow \mathbb{R} \quad \text{and} \quad u \in C^2(\mathbb{R}).$$



All admissible functions

$$u(x, y) + \varepsilon h(x, y)$$

where $h(x, y) \in C^2(\mathbb{R})$

Minimizing the functional J

$$J(u + \varepsilon h) = \iint_R L(x, y, u + \varepsilon h, u_x + \varepsilon h_x, u_y + \varepsilon h_y) dx dy$$

$$\delta J(u_0, h) = \iint_R \left. \frac{\partial L}{\partial \varepsilon} \right|_{\varepsilon=0} dx dy = 0$$

$$\left. \frac{\partial L}{\partial \varepsilon} \right|_{\varepsilon=0} = \frac{\partial L}{\partial u} h + \frac{\partial L}{\partial u_x} h_x + \frac{\partial L}{\partial u_y} h_y$$

$$= \left[\frac{\partial L}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial L}{\partial u_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial L}{\partial u_y} \right) \right] h(x, y) + \frac{\partial}{\partial x} \left(\frac{\partial L}{\partial u_x} h \right) + \frac{\partial}{\partial y} \left(\frac{\partial L}{\partial u_y} h \right)$$

Then

$$\iint_R E(L) h(x,y) dx dy + \iint_R \left[\frac{\partial}{\partial x} \left(\frac{\partial L}{\partial u_x} h \right) + \frac{\partial}{\partial y} \left(\frac{\partial L}{\partial u_y} h \right) \right] dx dy = 0$$

Using the Green's Theorem.

$$\begin{aligned} \iint_R \left[\frac{\partial}{\partial x} \left(\frac{\partial L}{\partial u_x} h \right) + \frac{\partial}{\partial y} \left(\frac{\partial L}{\partial u_y} h \right) \right] dx dy \\ = \oint_C \left(\frac{\partial L}{\partial u_x} dy - \frac{\partial L}{\partial u_y} dx \right) h(x,y) \end{aligned}$$

We assume that $\frac{\partial L}{\partial u_x}$ and $\frac{\partial L}{\partial u_y}$ are differentiable functions in R . For admissible functions (for fixed boundary C) $h|_C = 0$ therefore closed line integral vanishes. Then we get

$$\iint_R E(L) h(x,y) dx dy = 0$$

See the remark at the end of next page

Here $h(x,y) \in C^2(\mathbb{R})$, and vanish on the boundary curve C otherwise arbitrary, then from the last integral we get

$$E(L) = 0.$$

with

$$E = \frac{\partial}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial L}{\partial u_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial L}{\partial u_y} \right).$$

Remark: Let $f(x,y) \in C^1(x,y)$ and h be arbitrary function in $C^2(\mathbb{R})$ with $h=0$ on C .

Let

$$\iint_{\mathbb{R}} h f \, dx dy = 0$$

Then $f(x,y) = 0$, $\forall (x,y) \in \mathbb{R}$. ~~101~~

This lemma is similar to the one we used for one dependent variable case

Example 1: $J(u) = \iint_R [u_x^2 + u_y^2] dx dy.$

$$E(u) = -2 \frac{\partial}{\partial x} u_x - 2 \frac{\partial}{\partial y} u_y = 0$$

or $u_{xx} + u_{yy} = 0 \Rightarrow \nabla^2 u = 0.$

with some BCs we can solve $u.$

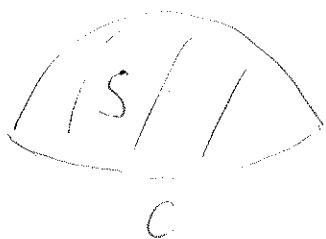
Example: 2 $J(u) = \iint_R \sqrt{1 + u_x^2 + u_y^2} dx dy.$

(minimizing area).

$$\frac{\partial}{\partial x} \frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}} + \frac{\partial}{\partial y} \frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}} = 0$$

$$\Rightarrow (1 + u_y^2) u_{xx} + (1 + u_x^2) u_{yy} - 2 u_x u_y u_{xy} = 0$$

This is the equation of minimal surface



Sufficient conditions: Jacobi Equations

$$\text{let } J(y) = \int_a^b L(x, y, y') dx \quad \text{with}$$

$$E(L) = L_y - \frac{d}{dx} L_{y'} = 0 \quad \forall x \in (a, b)$$

Differentiation E-L equation with x

$$\frac{d}{dx} E(L) = \frac{d}{dx} \left(L_y - \frac{d}{dx} L_{y'} \right) = 0$$

$$= L_{yx} + L_{yy} y' + L_{yy'} y'' - \frac{d}{dx} (L_{y'x} + L_{y'y} y' + L_{y'y'} y'')$$

$$L_{xy} - \frac{d}{dx} L_{xy'} \neq (L_{yy} - \frac{d}{dx} L_{yy'}) y'$$

$$L_{yy'} y'' - \frac{d}{dx} L_{y'y'} y'' = 0$$

$$L_{xy} - \frac{d}{dx} L_{xy'} + y' L_{yy} - \left(\frac{d}{dx} L_{yy'} \right) y' - \cancel{L_{yy'} y''} + \cancel{L_{yy'} y''} - \frac{d}{dx} L_{y'y'} y'' = 0$$

If $L_u = 0$ then

$$(L_{yy} - \frac{d}{dx} L_{yy'}) y' - \frac{d}{dx} (L_{yy'} \frac{dy'}{dx}) = 0$$

Definition: Any function u satisfying

$$(L_{yy} - \frac{d}{dx} L_{yy'}) u - \frac{d}{dx} (L_{yy'} \frac{du}{dx}) = 0 \quad (*)$$

is called the Jacobi field and this equation is called the Jacobi equation.

In order to understand the type of extremum point for the variational problem we study the second variation of $J(y)$.

$$\begin{aligned} \delta_2 J = J''(0) &= \int_0^b [L_{yy} h^2 + 2L_{yy'} h h' + L_{y'y'} h'^2] dx \\ &= \int_0^b [(L_{yy} - \frac{d}{dx} L_{yy'}) h^2 - h \frac{d}{dx} (L_{yy'} h')] dx \end{aligned}$$

Since $h(x) \in C^2(a, b)$ and $h(a)$ and $h(b)$ vanish at the end points ~~we can choose it to satisfy the Jacobi equation~~ ~~then~~ and using the Jacobi equation we get

$$\begin{aligned}
\delta_2 J &= \int_a^b \left[\frac{h^2}{u} \frac{d}{dx} (L_{yy'} \frac{du}{dx}) - h \frac{d}{dx} (L_{yy'} h') \right] dx \\
&= \int_a^b \left(\frac{h}{u} \right) \left[h \frac{d}{dx} (L_{yy'} u) - u \frac{d}{dx} (L_{yy'} h) \right] dx \\
&= \int_a^b \left(\frac{h}{u} \right) \frac{d}{dx} \left[(hu' - uh') L_{yy'} \right] dx \\
&= - \int_a^b \left(\frac{h}{u} \right) \frac{d}{dx} \left[\left(\frac{h}{u} \right)' u^2 L_{yy'} \right] dx
\end{aligned}$$

Integration by parts and using $h(a) = h(b) = 0$
we get

$$\delta_2 J = \int_a^b \left[\left(\frac{h}{u} \right)' \right]^2 u^2 L_{yy'} dx$$

Hence the sign of $\delta_2 J$ is equal to the
sign of $L_{yy'}$

$$\text{sign}(\delta_2 J) = \text{sign}(L_{yy'})$$

Since

$$J(y_0 + \varepsilon h) = J(y_0) + \varepsilon \delta J + \frac{\varepsilon^2}{2} \delta^2 J + \dots$$

$$\Delta J = \frac{\varepsilon^2}{2} \delta^2 J + \dots = \frac{\varepsilon^2}{2} J''(y_0) + \dots$$

Hence $\text{Sign}(\Delta J) = \text{sign}(Ly'y')$

This leads to the determination of the character of the extremum point $y_0(x)$.

Problems

1) Find the geodesics of right circular cylinder

2) Let $J(y) = \int_0^1 (x^2 - y^2 + y'^2) dx$. Calculate ΔJ , δJ , $\delta^2 J$ when $y = x$ and $\delta y = \varepsilon x^2$

3) Determine the stationary function $y(x)$ (extremum of J) for the problem

$$\delta \left[\int_0^1 y'^2 dx + [y(1)]^2 \right] = 0$$

$$y(0) = 1$$

4) Obtain the E-L equation and the associated natural boundary conditions for the problem

$$\delta \left[\int_a^b L(x, y, y') dx - \beta y(b) + \alpha y(a) \right] = 0$$

where α and β are given constants but $y(a)$ and $y(b)$ are not prescribed

5) a) show that the extremals of the problem

$$\delta \int_a^b [p(x) y'^2 - q(x) y^2] dx = 0, \quad \int_a^b r(x) y^2 dx = 1$$

where $y(a)$, and $y(b)$ are prescribed, are solutions of the equation

$$\frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) + (q + \lambda r) y = 0$$

where λ is a constant

b) show that the associated boundary conditions are of the form

$$\left[p(x) \frac{dy}{dx} \delta y \right]_a^b = 0$$

6) specialize Problem 5 in the following special case:

$$\delta \int_0^\pi y'^2 dx = 0, \quad \int_0^\pi y^2 dx = 1.$$

$$y(0) = 0, \quad y(\pi) = 0$$

7) show that if the constraint $\int_0^\pi y^2 dx = 1$ is omitted in problem (6), the only stationary function is the trivial case

$$y = 0$$

(56)

- (8) If $L(x, y, y')$ depends on b .
 Show that the "transversality" condition
 (variable end condition) must be replaced
 by the condition

$$\left[L + (y' - y') \frac{\partial L}{\partial y'} \right] \Big|_{x=b} + \int_0^b \frac{\partial L}{\partial b} dx = 0$$

- (9) If b is not prescribed, show that
 the stationary functions corresponding to the
 problem

$$\delta \int_0^b y'^2 dx = 0, \quad y(0) = 2$$

$$y(b) = \sin b$$

are of the form $y = 2 + 2x \cos b$.
 where b satisfies the "transcendental
 equation" $2 + 2l \cos l - \sin l = 0$. Also
 verify that the smallest positive value
 of l is between $\pi/2$ and $3\pi/4$.

- (9) Use the calculus of variables to find the
 shortest distance between the line $y = x$
 and the parabola $y^2 = x - 1$.

(10) let $u = u(x, y, z)$ and $J(u) = \iiint_R (u_x^2 + u_y^2 + u_z^2) dx dy dz$ (57)

where R is a region in $\mathbb{R}^3$. If u is given on the boundary of R find E-L so that

$$\delta J(u) = 0.$$

ASSIGNED EXERCISES OF MATH544 *February, 2001*

CALCULUS OF VARIATIONS

1. Find the shortest distance between two points A and B on the plane using the polar coordinates.
2. Find the shortest distance between two points A and B on a sphere.
3. Find the shortest distance between two points A and B on a right cylinder.

In the above problems (2) and (3) you may write the expressions for the length on the sphere and on the cylinder then use the calculus variations directly. On the other hand you may use the length formula in $\mathbb{R}^3$ with the constraints $x^2 + y^2 = 1$ and $x^2 + y^2 + z^2 = 1$ respectively.

4. Let $J(y) = \int_a^b L(x, y, y') dx$ be a functional. Let y be continuous in $[a, b]$. Assume that the Euler Lagrange equation holds on the interval (a, b) . Let y be differentiable at all $x \in [a, b]$ except one point $x_0 \in (a, b)$. Assume that y' tends to m or n as x tends to x_0 from the left or from the right.

4.a. Prove that $\frac{\partial L}{\partial y'}$ and $L - y' \frac{\partial L}{\partial y'}$ must be continuous functions of x at all points in $[a, b]$ including the corner points like x_0 .

- 4.b. Let $J(y) = \int_a^b (y'^2 - 1)^2 dx$. Suppose a corner exists at $x_0 \in [a, b]$. Minimize this functional (prove that there exist m and n where the functional $J(y)$ has a minimum value).

5. Determine the function extremizing (sometimes called the stationary function associated with) the functional $J(y) = \int_0^1 y'^2 f(x) dx$ when $y(0) = 0$ and $y(1) = 1$, where $f(x) = -1$ for $0 \leq x < \frac{1}{4}$ and $f(x) = +1$ for $\frac{1}{4} < x \leq 1$

6. The Euler-Lagrange (EL) equation is in general an ODE of the second order. We have (in the following we use the notation $p = y'$)

$$L_{px} + L_{py} y' + L_{pp} y'' = L_y$$

In the derivation of the EL we usually do not assume (in advance) the existence of the second derivative of y . It can easily be shown, however, that y'' exists at least at those points where $L_{pp} \neq 0$ (prove this). Consider the functional $J(y) = \int_{-1}^1 y^2 (p - 2x)^2 dx$ with $y(-1) = 0$ and $y(1) = 1$. Find the function which minimizes the functional under the given boundary conditions. Prove that y'' does exist at $x = 0$.

7. Let $J(y) = \int_0^\pi y''^2 dx$ with the constraint $\int_0^\pi y^2 dx = 1$ and with the boundary conditions $y(0) = y''(0) = 0$, $y(\pi) = y''(\pi) = 0$.

7.a. Find the function extremizing the functional $J(y)$

7.b. Consider the above problem without the constraint.

8. If l is not preassigned, show that the stationary functions corresponding to the problem $\delta \int_0^l y'^2 dx = 0$ with $y(0) = 2$ and $y(l) = \sin l$ are of the form $y = 2 + 2x \cos l$, where l satisfies the transcendental equation $2 + 2l \cos l - \sin l = 0$. Also verify that the smallest positive value of l is between $\pi/2$ and $3\pi/4$.

9. If l is not preassigned, show that the stationary functions corresponding to the problem $\delta \int_0^l [y'^2 + 4(y - l)] dx = 0$, with $y(0) = 2$ and $y(l) = l^2$ are of the form $y = x^2 - 2(x/l) + 2$, where l is one of two real roots of the equation $2l^4 - 2l^3 - 1 = 0$.

10. Find the shortest distance between the line $y = x$ and the parabola $y^2 = x - 1$.

11. Find the Lagrangian $L(x, y, y')$ such that Euler-Lagrange equation coincides with the (Emden-Fowler) equation $y'' + \frac{2}{x} y' + y^5 = 0$.

12. Determine the natural boundary condition at $x = b$ for the variational problem defined by $J(y) = \int_a^b L(x, y, y') dx + G(y(b))$, $y \in C^2[a, b]$, $y(a) = y_0$, where G is a given differentiable function. As an application of this problem let $L = y'^2$, $y \in [0, 1]$, $y(0) = 1$, $y(1)$ is unspecified and $G = y^2$. Find y extremizing this problem.

13. Give a proof that if $\int_a^b h'(x) f(x) dx = 0$ for all $h(x)$ satisfying $h(a) = h(b) = 0$, $h(x)$ is continuous and $h'(x)$ is piecewise continuous in $[a, b]$ where $f(x)$ is piecewise continuous in $[a, b]$.

14. *Legendre Condition.* In the theory of extrema of functions of single variable, a necessary condition for a minimum, besides $f'(x) = 0$, is that $f'' \geq 0$ (if it exists). A condition somewhat analogous to this holds for functionals. Let us suppose that there is an admissible function y for which $J(y) = \int_a^b L(x, y, y') dx$ is a minimum. Then $f(\epsilon) = J(y + \epsilon h)$ has a minimum at $\epsilon = 0$; accordingly $f'(0) = 0$ (from which follows the Euler-Lagrange equation $E_y(L) = 0$ and $f''(0) \geq 0$, assuming its existence. Hence for all h ,

$$f''(0) = \int_a^b [L_{yy} h^2 + 2L_{yy'} h h' + L_{y'y'} h'^2] dx \geq 0$$

Choose a special variation

$$h = \begin{cases} 0 & a \leq x \leq \xi - \epsilon \\ 1 + (x - \xi)/\epsilon & \xi - \epsilon \leq x \leq \xi \\ 1 - (x - \xi)/\epsilon & \xi \leq x \leq \xi + \epsilon \\ 0 & \xi + \epsilon \leq x \leq b \end{cases} \quad (1)$$

If we substitute this function in the above expression and let $\epsilon \rightarrow 0$, then the term

$$\frac{1}{\epsilon^2} \int_{\xi-\epsilon}^{\xi+\epsilon} L_{y'y'} dx$$

will dominate the left hand side of the above inequality and determine its sign. Thus the sign of $L_{y'y'}$ determines the sign of $f''(0)$, and for minimum, the Legendre condition $L_{y'y'} \geq 0$ must hold. We have the Legendre test:

Theorem 1. *The Legendre test. If*
(i) Euler-Lagrange equation is satisfied,

(i) the range of integration is sufficiently small,
 (iii) the sign of $L_{y'y'}$ is constant throughout this range,
 then $J(y)$ is a minimum or a maximum value of J according as the sign of $L_{y'y'}$ is positive or negative.

In case L contains more dependent functions, the Legendre condition is that the matrix $L_{y'_i y'_j}$ be positive or negative definite, that is

$$\sum_{(i,j)} \lambda_i \lambda_j L_{y'_i y'_j} \geq 0 \text{ or } \leq 0$$

for all λ .

The following part is prepared by using the book *An Introduction to the Calculus of Variations* by Charles Fox, Oxford Press (1950). In particular the second chapter on the second variations.

15. *The second variation:* Let $y = y(x)$ be the path of integration for which the integral $J(y) = \int_a^b L(x, y, y') dx$ is minimum or maximum (sometimes called the integral is stationary). Consider all admissible functions $y + \varepsilon h$ where both y and h belong to $C^2[a, b]$. Then

$$J(y + \varepsilon h) = J(y) + \frac{\varepsilon^2}{2} J_2 + O(\varepsilon^3)$$

where

$$J_2 = \int_a^b \left[h^2 \frac{\partial^2 L}{\partial y'^2} + 2 h h' \frac{\partial^2 L}{\partial y \partial y'} + h'^2 \frac{\partial^2 L}{\partial y^2} \right] dx$$

If $J(y)$ is minimum (or maximum) the sign of J_2 must be positive (or negative) without depending upon the choice of $h(x)$, for all sufficiently small ε . For simplicity we use the following notation: $L_0 = \frac{\partial L}{\partial y}$, $L_1 = \frac{\partial L}{\partial y'}$, $L_{00} = \frac{\partial^2 L}{\partial y^2}$, $L_{01} = \frac{\partial^2 L}{\partial y \partial y'}$, $L_{11} = \frac{\partial^2 L}{\partial y'^2}$. Then J_2 takes the form

$$J_2 = \int_a^b [L_{00} h^2 + 2L_{01} h h' + L_{11} h'^2] dx$$

We have the following Lemma. Proof is straightforward (done in the class)

Lemma 2. Let $h(a) = h(b) = 0$, then

$$J_2 = \int_a^b \left\{ h^2 \left(L_{00} - \frac{d}{dx} L_{01} \right) - h \frac{d}{dx} (h' L_{11}) \right\} dx$$

Definition. (Jacobi equation and Jacobi field). Let $u \in C^2[a, b]$. Then the following equation is called the Jacobi equation.

$$\left\{ L_{00} - \frac{d}{dx} (L_{01}) \right\} u - \frac{d}{dx} \left(L_{11} \frac{du}{dx} \right) = 0$$

the function u satisfying this second order ODE is called the Jacobi field

Lemma 3. If $h(a) = h(b) = 0$ and is a Jacobi field then

$$J_2 = \int_a^b L_{11} \left[h' - h \frac{u'}{u} \right]^2 dx,$$

Proof. Taking the term $L_{00} - \frac{d}{dx} (L_{01})$ from the Jacobi equation and inserting into the expression for J_2 given in the first Lemma one obtains the required result.

If the term $\{h' - h(\frac{u'}{u})\} \neq 0$ and L_{11} has constant sign for all points of the extremal arc AB , where $A = (a, y(a))$ and $B = (b, y(b))$, then J_2 must have a sign which is independent of the choice of h . Now in the extremal case

$$J(y + \varepsilon h) - J(y) = \frac{\varepsilon^2}{2} J_2 + O(\varepsilon^3)$$

Hence J takes its minimum value (or maximum value) if L_{11} is positive (or negative) at all points on the extremal curve $y(x)$ with $x \in [a, b]$. This is essentially the Legendre test stated above.

If, however, L_{11} , does not keep its sign constant at all $x \in [a, b]$ on the curve $y(x)$, then the value $J(y)$ is neither a minimum nor a maximum.

If the term $\{h' - h(\frac{u'}{u})\}$ vanishes at all points of the extremal curve $y(x)$. Then it is clear that $h(x) = \alpha u(x)$, where α is an arbitrary constant. Along

the extremal curve $y(x)$ the first variation vanishes. If in addition $h(x) = \alpha u(x)$ is chosen then the second variation, J_2 vanishes as well. The sign of $\delta J = J(y + \varepsilon h) - J(y)$ will depend on the third variation $\delta J = \frac{\varepsilon^3}{6} J_3$, where

$$J_3 = \int_a^b [h^3 L_{000} + 3h^2 h' L_{001} + 3h h'^2 L_{011} + h'^3 L_{111}] dx$$

Since the sign of δJ depends on that of ε there can be no maximum or minimum value of J unless J_3 vanishes, in which case the sign of δJ will depend on that of J_4 , the fourth variation. In order to avoid this difficulty Jacobi proposed a test. This test provides whether the $h(x) = \alpha u(x)$ at all points of the extremal curve $y(x)$.

Definition: Let $u(x)$ be a solution of the Jacobi equation. Let $u(a) = 0$. This means that the Jacobi field vanishes at the point $A = (a, y(a))$. The all other points on the extremal curve $y(x)$ at which $u(x)$ vanishes are called the *conjugate points* to the point A .

The jacobi field (a function satisfying the Jacobi equation with $u(a) = 0$) may be given as the linear sum of the fundamental solutions u_1 and u_2 of the Jacobi equation.

$$u(x) = a_1 u_1(x) + a_2 u_2(x)$$

where a_1 and a_2 are constants. Since $u(a) = 0$, then

$$\frac{u_1(a)}{u_2(a)} = -\frac{a_1}{a_2}$$

Hence if x is the abscissa of the conjugate point to A ($u(x) = 0$) then

$$\frac{u_1(x)}{u_2(x)} = -\frac{a_1}{a_2}$$

It shows that this ratio is the same for all conjugate points. This is the way obtaining the conjugate points. We now give a nice way of determining the fundamental solutions u_1 and u_2 . Since the function $y(x)$, the solution of

the Euler-Lagrange equation is second order ODE then it contains two independent constants of integration $y(x, c_1, c_2)$. It is easy to prove the following Lemma.

Lemma 4. $u_i = \frac{\partial y}{\partial c_i}, i = 1, 2.$

Hence the conjugate points are easily determined from the following formula. Let $Y = (\frac{\partial y}{\partial c_1})/(\frac{\partial y}{\partial c_2})$ then

$$Y(x) = Y(a)$$

Let A' be the first conjugate point to A when moving on the extremal curve $y(x)$ in the direction from $A = (a, y(a))$ to $B = (b, y(b))$. There are three possibilities

- (i) B lies between A and A' ,
- (ii) B coincides with A' ,
- (iii) B lies beyond A' .

In all three cases $u(x)$ vanishes at A , but in case (i) it can not vanish again at the point B , in case (ii) it vanishes again at B , and in case (iii) it vanishes again at some point of the arc AB lying between A and B . In case (i) we have $u(a) = h(a) = 0$ but $u(b) \neq 0, h(b) = 0$. Hence $u(x)$ can not be proportional to $h(x)$ at all points of the extremal arc AB . Therefore we must have $(h' - h \frac{u'}{u})^2 > 0$ at all points of AB , except, possibly, at finite number of points where $h(x)$ and $h'(x)$ vanish simultaneously. Then we have the following theorem *Jacobi test*.

Theorem 5. *Let $y = y(x)$ be the curve connecting the points A and B for which the functional $J(y) = \int_a^b L(x, y, y')dx$ takes its extremum value. Let A and A' be the conjugate points on this curve. If (i) B lies between A and A' and (ii) L_{11} has constant sign for all points on the arc AB , then $J(y)$ is a minimum (maximum) when $L_{11} > 0$ ($L_{11} < 0$)*

For all the other cases we need higher variations for extremum tests.