

SOLUTIONS

MATH 544: METHODS OF APPLIED MATHEMATICS II

First Midterm Exam

March 20, 2019; 13.40-15.30

QUESTIONS: Choose any three of the following four problems

[35] 1. (a) Prove that if the Lagrange function $L(x, y, y')$ is a total derivative, $L = \frac{d\Omega}{dx}$, then the Euler-Lagrange equation is satisfied identically (for all Ω).
 (b) Prove that if the Euler-Lagrange equation is satisfied identically (for all function y) then the Lagrange function is null, i.e., $L = \frac{d\Omega}{dx}$.

[35] 2. (a) Let $J[y] = \int_a^b [L(x, y, y') + \alpha b] dx$ be the functional where the end point b is moving on a curve $y = g(x)$. and α is any real number. If $J[y]$ has an extremum value then show that the following should hold

$$\frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) = 0, \quad (1)$$

$$\frac{L|_b + 2\alpha b}{g'(b) - y'(b)} + \frac{\partial L}{\partial y'}|_b = 0 \quad (2)$$

(b) If b is not preassigned, show that the stationary functions corresponding to the problem $\delta \int_0^b [y'^2 + 4(y - b)] dx = 0$, with $y(0) = 2$ and $y(l) = b^2$ are of the form $y = x^2 - 2(x/b) + 2$, where b is one of two real roots of the equation $2b^4 - 2b^3 - 1 = 0$.

[35] 3. An initial and boundary value problem is given as follows:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \quad t > 0, \quad 0 < x < L \\ u(x, 0) &= f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq L \\ u(0, t) &= u(L, t) = 0, \quad t \geq 0, \end{aligned}$$

where f and g are given functions. Under what conditions this problem is a well posed problem?

[35] 4. Solve the following initial value problems

(a)

$$u_{xx} + 2u_{xy} - 8u_{yy} = 0, \quad (x, y) \in \mathbb{R}^2, \quad (3)$$

$$u(x, x) = \sigma_1(x), \quad u_y(x, x) = \sigma_2(x) \quad (4)$$

where $\sigma_1(x)$ and $\sigma_2(x)$ are some differentiable functions.
(b)

$$u_{xx} + 2u_{xy} - 8u_{yy} = 0, \quad (x, y) \in \mathbb{R}^2, \quad (5)$$

$$u(x, -2x) = \rho_1(x), \quad u_y(x, -2x) = \rho_2(x) \quad (6)$$

where $\rho_1(x)$ and $\rho_2(x)$ are some differentiable functions.

$$a) \quad L = \frac{d}{dx} \Omega \quad . \quad \text{Here } \Omega \text{ is a}$$

twice differentiable function of x and y .

It can not depend on y' . Then

$$L = \frac{d}{dx} \Omega(x, y) = \Omega_x + \Omega_y y'$$

$$E(L) = \frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) = \Omega_{xy} + \Omega_{yy} y' - \frac{d}{dx} \Omega_y$$

$$= \Omega_{xy} + \cancel{\Omega_{yy} y'} - \Omega_{yx} - \cancel{\Omega_{yy} y'}$$

$$= \Omega_{xy} - \Omega_{yx}$$

If the second partial derivatives of Ω are

$$\text{continuous everywhere} \Rightarrow \Omega_{xy} = \Omega_{yx} \Rightarrow E(L) = 0$$

b) $E(L) = 0$ for all y . Let $L \in L(x, y, y')$

$$E(L) = \frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) = L_y - \frac{d}{dx} L_{y'} = 0$$

$$= L_y - L_{y'x} - L_{y'y} y' - L_{y'y'} y'' = 0$$

$$\Rightarrow L_{y'y'} = 0 \quad \Rightarrow L = A(x, y) y' + B(x, y)$$

$$E(L) = \cancel{A_y y'} + B_y - A_x - \cancel{A_y y'}$$

1.2

$$= B_y - A_x = 0$$

let $B = \Omega_x$, $A = \Omega_y$.

$\Rightarrow E(L) = 0$. Then we obtain

$$L = \Omega_x + \Omega_y y' = \frac{d}{dx} \Omega(x, y)$$

$$\underline{L = \frac{d}{dx} \Omega}$$

Solution of (2)

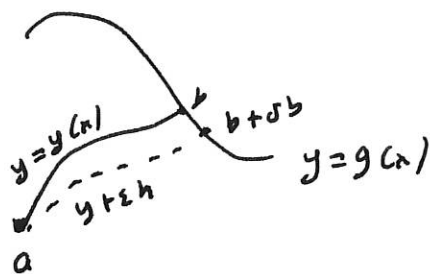
(2)

$$J[y] = \int_a^b [L(x, y, y') + \alpha b] dx$$

$$\delta J = \frac{\partial J}{\partial b} \delta b + \int_a^b \left. \frac{\partial L}{\partial \xi} \right|_{\xi=0} dx + \alpha b \delta b$$

$$= L \Big|_{x=b} \delta b + \int_a^b E(L) h(x) dx + \left. \frac{\partial L}{\partial y'} \right|_{x=b} h(b) + 2\alpha b \delta b$$

(Since a fixed $\Rightarrow h(a) = 0$)



Since two curves intersect at $b \Rightarrow y(b) = g(b)$

varying $b \rightarrow b + \Delta b$ we get

$$y \rightarrow y(b + \Delta b) + \epsilon h(b + \Delta b)$$

$$g \rightarrow g(b + \Delta b)$$

$$\Rightarrow y(b + \Delta b) + \epsilon h(b + \Delta b) = g(b + \Delta b)$$

$$\Rightarrow y'(b) \Delta b + \epsilon h(b) = g'(b) \Delta b$$

$$\epsilon h(b) = [g'(b) - y'(b)] \Delta b$$

$$\Delta b = \epsilon \delta b$$

$$\Rightarrow \delta b = \frac{h(b)}{g'(b) - y'(b)}$$

$$\Rightarrow \int_a^b E(L) h(x) dx + \left[\frac{L|_b + 2\alpha b}{g'(b) - y'(b)} + \left. \frac{\partial L}{\partial y'} \right|_b \right] h(b) = 0$$

(12)

$h(x)$ is a twice differentiable function in $[a, b]$ and $h(a) = 0$, otherwise arbitrary using this we can show that

$$i) \text{ leting } h(b) = 0 \Rightarrow \int_a^b E(L) h(x) dx = 0$$

Using the Lemma we have studied in class

$$E(L) = \frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) = 0$$

ii) Then we get

$$\left[\frac{L|_b + 2\alpha b}{g'(b) - y'(b)} + \frac{\partial L}{\partial y'} \right]_b h(b) = 0$$

Since $h(b)$ is arbitrary it may take non zero values, hence we get

$$\frac{L|_b + 2\alpha b}{g'(b) - y'(b)} + \frac{\partial L}{\partial y'} \bigg|_b = 0$$

$$b) \quad L = y'^2 + 4y, \quad \alpha = -4$$

$$E(L) = 4 - 2y'' = 0 \Rightarrow y'' = 2$$

$$y(x) = x^2 + Px + Q$$

$$y(b) = 2 \Rightarrow Q = 2$$

$$y(b) = b^2 \Rightarrow P = -\frac{2}{b}$$

$$y(x) = x^2 - \frac{2}{b}x + 2$$

$$y'(x) = 2x - \frac{2}{b}, \quad g'(x) = 2x$$

(23)

$$g'(x) - y'(x) = 2x - 2x + \frac{2}{b} = \frac{2}{b}$$

$$L|_b = y(b)^2 + 4y(b) = 4(b - \frac{1}{b})^2 + 4b^2$$

$$L|_b + 2\alpha b = 4(b - \frac{1}{b})^2 + 4b^2 - 8b.$$

$$\frac{L|_b + 2\alpha b}{g'(b) - y'(b)} + \frac{\partial L}{\partial y'} \Big|_b = \frac{4(b - \frac{1}{b})^2 + 4b^2 - 8b}{2/b} + 2(1 - \frac{1}{b^2}) = 0$$

$$4(b - \frac{1}{b})^2 + 4b^2 - 8b + 8(1 - \frac{1}{b^2}) = 0$$

$$4b^2 - \cancel{8} + 4\frac{1}{b^2} + 4b^2 - 8b + \cancel{8} - \frac{8}{b^2} = 0$$

$$8b^2 - \frac{4}{b^2} - 8b = 0$$

$$8b^4 - 8b^3 - 4 = 0$$

$$\boxed{2b^4 - 2b^3 - 1 = 0}$$

Solution of (3): Done in class and also in

Lecture Notes 2

Solution of (4)

4.1

$$u_{xx} + 2u_{xy} - 8u_{yy} = 0$$

$$a=1, b=1, c=-8, \quad b^2 - ac = 1 + 8 = 9 > 0$$

Hyperbolic type.

Characteristic curves:

$$a y'^2 - 2b x' y' + c x'^2 = 0 \Rightarrow y'^2 - 2x' y' - 8x'^2 = 0$$

$$(y' - 4x')(y' + 2x') = 0$$

$$\text{i) } y' - 4x' = 0 \rightarrow y = 4x + C_1, \quad C_1 = \text{const.}$$

$$\text{ii) } y' + 2x' = 0 \rightarrow y = -2x + C_2, \quad C_2 = \text{const.}$$

- If the data is ^{or} one of these curves then either there exist no solution or infinitely many solutions.
- If the data is not any of these curves then there exist unique solution of this initial value problem.

$$\text{a) } \gamma = \{ (x, y) \in \mathbb{R}^2 \mid y = x, \quad x \in I \}$$

is not a characteristic curve, hence solution should exist and unique

Differential equation can be written as

$$(\partial_x^2 + 2\partial_x\partial_y - 8\partial_y^2)u = 0$$

$$(\partial_x + 4\partial_y)(\partial_x - 2\partial_y)u = 0$$

Hence $u = u_1 + u_2$ where

$$(\partial_x + 4\partial_y)u_1 = 0 \Rightarrow u_1(x, y) = F(y - 4x)$$

$$(\partial_x - 2\partial_y)u_2 = 0 \Rightarrow u_2(x, y) = G(y + 2x)$$

where F and G are arbitrary functions. Then we have the general solution.

$$\underline{u(x, y) = F(y - 4x) + G(y + 2x)}$$

if the data is : $u(x, x) = \sigma_1(x)$, $u_y(x, x) = \sigma_2(x)$

$$\Rightarrow F(-3x) + G(3x) = \sigma_1(x)$$

$$F_\xi(-3x) + G_\eta(3x) = \sigma_2(x)$$

$$\xi = y - 4x, \quad \eta = y + 2x$$

$$\Rightarrow F(-3x) + G(3x) = \sigma_1(x)$$

$$-\frac{1}{3}F_\xi(-3x) + \frac{1}{3}G_\eta(3x) = \sigma_2(x)$$

$$\Rightarrow F(-3x) + G(3x) = \sigma_1(x)$$

$$-F(-3x) + G(3x) = 3 \int^x \sigma_2(x') dx'$$

$$\Rightarrow F(x) = \frac{1}{2} \sigma_1(-x/3) - \frac{3}{2} \int^{-x/3} \sigma_2(x') dx'$$

$$G(x) = \frac{1}{2} \sigma_1(x/3) + \frac{3}{2} \int^{x/3} \sigma_2(x') dx'$$

$$\Rightarrow u(x, y) = \frac{1}{2} \left[\sigma_1\left(\frac{y+2x}{3}\right) + \sigma_1\left(\frac{4x-y}{3}\right) \right] + \frac{3}{2} \int_{\frac{4x-y}{3}}^{\frac{y+2x}{3}} \sigma_2(x') dx'$$

If σ_1 is twice differentiable and σ_2 is differentiable everywhere then the solution of the initial problem exists and is unique. Solution is given above

b) If the data is $u(x, -2x) = f_1(x)$ and $u_y(x, -2x) = f_2(x)$ where f_1 and f_2 are given functions, then the initial curve $\gamma = \{ (x, y) \in \mathbb{R}^2 \mid y = -2x, x \in I \}$ is one of the characteristic curves of the differential equation. Hence either there exists no

no solution or infinitely many solutions.

4-4

using the data we get. Since the differential equation is the same as part (c) we have

$$u(x, y) = F(y - 4x) + G(y + 2x)$$

using the data

$$F(-6x) + G(0) = f_1(x)$$

$$-\frac{1}{6} F_x(-6x) + G_y(0) = f_2(x)$$

These equations give

$$-\frac{1}{6} f_1' + G_y(0) = f_2(x)$$

$$\text{or } f_1'' + 6 f_2 = 0$$

- If the data $\{f_1, f_2\}$ does not satisfy this constraint there exist no solution
- If the data $\{f_1, f_2\}$ satisfies this constraint $G(x, y)$ is left arbitrary and

$$F(x) = -G(0) + f_1(-x/6)$$

There are infinitely many solutions.