MATH 544: METHODS OF APPLIED MATHEMATICS II
Final Exam
May 5, 2016; 15.30-17.30
Solutions

QUESTIONS: Choose any three of the following four problems
[35] **1**. An initial value problem is given as follows:

$$\begin{aligned} &u_t = au_{xx}, \quad (a>0), t>0, \quad x\in[0,1],\\ &u(0,t) = u(1,t) = 0, \quad t\geq 0,\\ &u(x,0) = x - x^2, \quad x\in[0,1] \end{aligned}$$

Is this problem well posed?

Solution: This problem was solved in class. Here $L = 1$ and $f(x) = x - x^2$. With this information we find that

$$u(x,t) = \Sigma_{n=1}^{\infty} a_n\, e^{-\lambda_n^2 at}\, \sin(\lambda_n x), \tag{1}$$
$$f(x) = x - x^2 = \Sigma_{n=1}^{\infty} a_n\, \sin(\lambda_n x) \tag{2}$$

we find that ($\lambda_n = n\pi$)

$$a_n = \int_0^1 (x - x^2)\sin(\lambda_n x) = \frac{8}{(n\pi)^3} \quad \text{for all n=odd} \tag{3}$$

Hence both series are uniformly convergent hence solution exist and unique. Stability of the solution is verified very easily with this a_n. Hence this is a well posed initial and boundary value problem.

[35] **2**. Let a 2-dimensional dynamical system be given by

$$\begin{aligned} \dot{x} &= 2y^2 + 2x - 8y,\\ \dot{y} &= y^2 + x - 2y \end{aligned}$$

Find the equilibrium points and classify them. Discuss also the stability of this system against linear perturbations.

Solution: The equilibrium solution is $(0,0)$. Let $x(t) = \bar{x}(t) + O(\bar{x}^2, \bar{y}^2)$ and $y(t) = \bar{y}(t) + O(\bar{x}^2, \bar{y}^2)$ The stability analysis of this point against linear perturbations is given by the solutions of the coupled equations

$$\dot{\bar{x}} = 2\bar{x} - 8\bar{y}, \tag{4}$$
$$\dot{\bar{y}} = 2\bar{x} - 2\bar{y}, \tag{5}$$

This equation can be written as $\dot{u} = Au$ where

$$u = \begin{pmatrix} \bar{x}(t) \\ \bar{y}(t) \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -8 \\ 1 & -2 \end{pmatrix} \tag{6}$$

The eigenvalues of A are $\lambda_1 = 2i$ and $\lambda_2 = -2i$ and the corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 2(1+i) \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2(1-i) \\ 1 \end{pmatrix} \tag{7}$$

Then the solution is

$$u(t) = c_1\, v_1\, e^{2it} + c_2\, v_2\, e^{-2it} \tag{8}$$

Letting $c_1 = \alpha + i\,\beta$ and $c_2 = \alpha - i\beta$ we get

$$\bar{x}(t) = 4(\alpha - \beta)\cos 2t - 4(\alpha + \beta)\sin 2t, \tag{9}$$

$$\bar{y}(t) = 2\alpha \cos 2t - 2\beta \sin 2t \tag{10}$$

where α and β are constant. Stability type the equilibrium point $(0, 0)$ is **a center and hence stable**. Phase portrait can be easily seen from the identity

$$\bar{x}^2 + 8\bar{y}^2 - 4\bar{x}\,\bar{y} = 16(\alpha^2 + \beta^2) \tag{11}$$

which are co-centric ellipses.

[35] **3**. Let $J(y) = \int_0^L (y'')^2 dx$ with constraint $\int_0^L y^2\, dx = 1$ and with boundary conditions $y(0) = y''(0) = 0, \quad y(L) = y''(L) = 0$
(a). Find the function extremizing the functional J.
(b). Consider the same problem without the constraint.
Solution:Using the EL operator

$$E = \frac{\partial}{\partial y} - \frac{d}{dx}\frac{\partial}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial}{\partial y''}$$

for $L = y'' + \lambda y^2$ where λ is any real number we obtain

$$y'''' + \lambda y = 0$$

Due to the boundary conditions it is easy to prove (multiplying the above equation by y and integrating over $(0, L)$, and integrating by part two times one obtains)

$$\int_0^L (y'')^2 dx + \lambda \int_0^L y^2 dx = 0$$

Hence $\lambda < 0$. For that case we have the complete solution (letting $\lambda = -\mu^4$)

$$y(x) = a_1 \cosh \mu x + a_2 \sinh \mu x + a_3 \cos \mu x + a_4 \sin \mu x$$

where $a_i, \quad i = 1 - 4$ are arbitrary constants to be determined by the initial conditions. Using the initial conditions we obtain

$$y = a_4 \sin \mu_n x, \quad \mu_n = \frac{n\pi}{L}$$

Here n is an any integer. Hence $\lambda = {\mu_n}^4$. There remains only the constraint to be satisfied

$$\int_0^L y^2 dx = 1$$

which gives $a_4^2 = \frac{2}{L}$. Hence

$$y(x) = \pm\sqrt{\frac{2}{L}} \sin \mu_n x$$

b. When there is no constraint then we may start with $\lambda = 0$. Then $y = a_1 + a_2 x + a_2 x^2 + a_4 x^3$. Initial conditions give $a_i = 0$ for all $i = 1 - 4$, hence $y(x) = 0$ for this case.

[35] **4**. Consider the following initial+boundary value problem

$$u_{tt} + a^2 u_{xxxx} = 0, \quad t > 0, \quad 0 < x < L \tag{12}$$

$$u(0,t) = u_{xx}(0,t) = 0 \quad u(L,t) = u_{xx}(L,t) = 0, \quad t \geq 0 \tag{13}$$

$$u(x,0) = f(x), \quad u_t(x,0) = 0 \quad 0 \leq x \leq L \tag{14}$$

a. Find the formal solution of the above problem
b. Find restrictions on the function $f(x)$ so that the formal solution in part **a** is a solution of the problem.
c. Prove the uniqueness of the solutions of the above problem. (Hint: solve

this question without considering the first two parts **a** and **b** and use the energy integral .

$$E(t) = \frac{1}{2}\int_0^L [(u_t)^2 + a^2(u_{xx})^2]dx$$

(a). Letting $u = X(x)T(t)$ we obtain

$$\frac{T''}{T} + a^2\,\frac{X''''}{X} = 0$$

Hence we obtain

$$T'' = -a^2\lambda T \tag{15}$$
$$X'''' - \lambda X = 0 \tag{16}$$

Letting $\lambda = \mu^4$ and using the same reasoning we made in the first problem we get (using the boundary conditions)

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos a^2\,{\mu_n}^2 t\ \ \sin \mu_n x \tag{17}$$

with $\mu_n = \frac{n\pi}{L}$ and

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \mu_n x, \tag{18}$$
$$a_n = \frac{2}{L}\int_0^L f(x)\sin \mu_n x\,dx \tag{19}$$

(b). We have two sums above. The first one is the representation of $u(x,t)$ which must be differentaiable four times with respect x variable. Hence $|{\mu_n}^4\,a_n| \leq \frac{C}{n^2}$. To have such a Fourier coeffient f must satisfy

$$f(0) = f(L) = f''(0) = f''(L) = f''''(0) = f''''(L) = 0$$

and f must have continuous 6th derivatve. Then it is easy to show that

$$|{\mu_n}^4\,a_n| \leq \frac{C}{{\mu_n}^2}$$

where $C = 2max\,|\frac{d^6 f}{dx^6}|, \quad x \in (0, L)$. These conditions guarente convergence of twice t derivative of u and also the convergence of the second summation representating f.

(c). Assume that there are two solutions u_1 and u_2 satisfying the same boundary conditions. Let $w = u_1 - u_2$. Then w satisfies the same differential equations given above with the zero initial conditions ($f = 0$ for w). Write the energy functional for the w. Taking the derivative of $E(t)$ with respect to t and using the differential equation and the initial+boundary conditions we can prove that $E(t) = E(0) = 0$. This leads to $w_t = 0$ and $w_{xx} =$ for all $t > 0$ and $x \in (0, L)$. These equations have unique solution $w = 0$. Hence $u_1 = u_2$.